1. Models and coordinates for the hyperbolic plane

The models:

• Poincaré disc: \{ |z| < 1 \} with metric

\[
 ds^2 = \frac{4dzd\bar{z}}{(1-|z|^2)^2}
\]  

(1.1)

• Upper half plane: \{ \Im z > 0 \} with metric

\[
 ds^2 = \frac{dzd\bar{z}}{(\Im z)^2}
\]  

(1.2)

• Minkowski model: \{ (x_0, x_1, x_2) \in \mathbb{R}^3 : -x_0^2 + x_1^2 + x_2^2 = 1 \} with metric

induced from the Minkowski metric \(-dx_0^2 + dx_1^2 + dx_2^2\).

\[
 \tanh(u) = \frac{1}{2} \ln(1 + u) - \frac{1}{2} \ln(1 - u)
\]

1.1. Hyperbolic polar coordinates in the Poincaré disc.

(1.3) \[ \tanh(r/2) = |z|, \quad \tan(\phi) = y/x \]

Then

\[
 ds^2 = dw^2 + \sinh^2(w)d\phi^2
\]  

(1.4a)

\[
 x_0 = \cosh(r)
\]

(1.4b)

\[
 x_1 = \cos(\phi) \sinh(r)
\]

(1.4c)

\[
 x_2 = \sin(\phi) \sinh(r)
\]

gives the isometry to the hyperboloid \(-x_0^2 + x_1^2 + x_2^2 = -1\) with metric \(-dx_0^2 + dx_1^2 + dx_2^2\).

1.2. Upper half plane \rightarrow Poincaré. A Möbius transformation of \( \mathbb{C} \) is a map

\[
 T(z) = \frac{az+b}{cz+d}, \quad ad - bc = 1
\]

\( T \) is uniquely given by its values at 3 points. Let \( T(0) = 1, T(i) = 0, T(\infty) = -1 \).

If \( \Im z > 0 \), then \( |T(z)| < 1 \) so \( T \) maps the upper half plane \( \{ \Im z > 0 \} \) to the Poincaré disc \(|z| < 1\).

The conditions on \( T \) give \( a = (i/2)^{1/2}, b = d = -ia, c = -a \), and after simplifying we have

\[
 T(z) = \frac{i - z}{i + z}
\]

Let

\[
 w = T(z)
\]

Then in terms of real coordinates \( z = x + iy \), we have

\[
 w = \frac{(i - z)(i + z)}{|i + z|^2}
\]

\[
 = \frac{1 - x^2 - y^2 + 2ix}{x^2 + (1 + y)^2}
\]
and

\[(1.5a) \quad |w|^2 = \frac{1 - |z|^2 - 2\text{Re}(z)}{1 + |z|^2 + 2\text{Re}(z)} = \frac{x^2 + (1 - y)^2}{x^2 + (1 + y)^2} \]

\[(1.5b) \quad \text{Re}(w) = \frac{1 - |z|^2}{|i + z|^2} = \frac{1 - x^2 - y^2}{x^2 + (1 + y)^2} \]

\[(1.5c) \quad \text{Im}(w) = \frac{2\text{Re}(z)}{|i + z|^2} = \frac{2x}{x^2 + (1 + y)^2} \]

With \(w = T(z)\) we have

\[\frac{4dwd\bar{w}}{(1 - w\bar{w})^2} = \frac{dzd\bar{z}}{(\text{Im}z)^2} \]

Thus \(T\) is an isometric isomorphism from the the upper half plane with metric (1.2) to the Poincare disc with metric (1.1).

1.3. Poincaré \(\rightarrow\) upper half plane. Let \(z = T^{-1}(w)\). Then with \(a, b, c, d\) as above,

\[z = \frac{dw - b}{-cw + a} = \frac{1 - w}{1 + w} \]

and

\[\frac{dzd\bar{z}}{(\text{Im}z)^2} = \frac{4dwd\bar{w}}{(1 - z\bar{z})^2} \]

1.4. Poincaré \(\rightarrow\) Minkowski. Equation (1.3) gives

\[\sinh(r) = 2 \frac{|z|}{1 - |z|^2} \]

\[\cosh(r) = \frac{1 + |z|^2}{1 - |z|^2} \]

Therefore using \(\cos(\phi) = x/|z|, \sin(\phi) = y/|z|\) and (1.4) we are able to write the Minkowski coordinates \(x_0, x_1, x_2\) in terms of the Poincare coordinate \(z\) as

\[(1.6a) \quad x_0 = \frac{1 + |z|^2}{1 - |z|^2} \]

\[(1.6b) \quad x_1 = \frac{2x}{1 - |z|^2} \]

\[(1.6c) \quad x_2 = \frac{2y}{1 - |z|^2} \]

1.5. Upper half plane to Minkowski. We now compose the map \(T\) from the upper half plane to the Poincare disc with the map (1.6) from the Poincaré disc to the Minkowski model. Let \(w\) be the coordinate in the Poincaré disc and
Then (1.6) is

\[
\begin{align*}
  x_0 &= \frac{1 + |w|^2}{1 - |w|^2} \\
  x_1 &= \frac{2\Re(w)}{1 - |w|^2} \\
  x_2 &= \frac{2\Im(w)}{1 - |w|^2}
\end{align*}
\]

which using (1.5) gives

(1.7a) \hspace{1cm} x_0 = \frac{1 + x^2 + y^2}{2y}

(1.7b) \hspace{1cm} x_1 = \frac{(1 - x^2 - y^2)}{2y}

(1.7c) \hspace{1cm} x_2 = \frac{x}{y}

From this we get

(1.8) \hspace{1cm} x_0 + x_2 = \frac{(1 + x)^2 + y^2}{2y}

(1.9) \hspace{1cm} x_0 - x_2 = \frac{(1 - x)^2 + y^2}{2y}

(1.10) \hspace{1cm} x_0 + x_1 = \frac{1}{y}

(1.11) \hspace{1cm} x_0 - x_1 = \frac{x^2 + y^2}{y}