Solutions

Trench = \left[ T \right]

\( \text{(a)} \) \( P(n) \Rightarrow P(n+1) \) \text{ proof}

\( P(n+1) \) is \( 1 + 2 + \ldots + n + n+1 = \frac{(n+3)n}{2} \).

If \( P(n) \) is true, then the left hand side is \( \frac{(n+2)(n-1)}{2} + (n+1) = \frac{(n+2)(n-1) + 2n+2}{2} \)

\[ = \frac{n^2 + n - 2 + 2n + 2}{2} = \frac{n(n+3)}{2} \Rightarrow P(n+1) \]

\textbf{Answer: \( \bigcirc \)}

\( \text{(b)} \) We know that, in fact,

\[ 1 + 2 + \ldots + n = \frac{n(n+1)}{2} \text{ (prove)}. \]

If there existed \( n \) so that \( P(n) \) true, since \( (a) \) is true, \( P(n) \) would be true for all \( n \geq n_0 \). But \( \frac{n(n+1)}{2} + \frac{(n+2)(n-1)}{2} \) for large \( n \).
The inequality \( \frac{1}{n!} > \frac{8^n}{(2n)!} \) is true for \( n > n_0 \), \( n_0 \) to be determined.

For \( n < n_0 \) we have to check one by one.

Let's see when is the induction step true.

If \( \frac{1}{n!} > \frac{8^n}{(2n)!} \) \( \Rightarrow \frac{1}{(n+1)!} > \frac{8^{n+1}}{(2n+2)!} \).

Re-write as: \( \frac{(2n)!}{8^n \cdot n!} > 1 \) \( \Rightarrow \frac{(2n+2)!}{(n+1)! \cdot 8^{n+1}} > 1 \).

If the left hand side is true, then the right hand side is

\[
\frac{(2n)!}{8^n \cdot n!} \cdot \frac{(2n+1)(2n+2)}{8 \cdot (n+1)} > 1 \cdot \frac{2n+1}{4}
\]

which exceeds 1 as soon as \( n \geq 2 \).

This does not mean the statement is true for \( n \geq 2 \).
Verify

\[
\frac{(2n)!}{8^n n!} = \frac{(n+1)(n+2) \ldots (n+n)}{8^n}
\]

\[
\begin{align*}
n &= 1 & \text{false} \\
n &= 2 & \text{false} \\
n &= 5 & \text{false} \\
n &= 6 & \text{true} \\
n &> 6
\end{align*}
\]

Answer

\[
\begin{align*}
n &= 1, 2, 3, 4, 5 & \text{false} \\
n &> 6 & \text{true}
\end{align*}
\]
Let $a_1 = a_2 = 5$ and

$$a_{n+1} = a_n + 6a_{n-1}, \quad m \geq 2$$

Show that $a_n = 3^n - (-2)^n$ if $n \geq 1$.

In general, when we have

$$a_{n+1} = Aa_n + B a_{n-1}$$

we solve the characteristic eq

$$A^2 = A + B$$

1. If it has two distinct roots $d_1, d_2$
   (here $\lambda_1 = -2$, $\lambda_2 = +3$)
   then the solution is

$$a_n = c_1d_1^n + c_2d_2^n$$

where $c_1$, $c_2$ can be found from $a_1$, $a_2$.

2. If $d_1 = d_2 = d$, then use
   with $d_1^m \rightarrow d_1^m$ and $d_2^m \rightarrow nd_1^m$. 
The Fibonacci numbers can be found in many textbooks with many interesting properties.

The reason why
\[ F_{n+1} = F_n + F_{n-1} \]
\[ F_1 = F_2 = 1 \]
has a formula involving \( \sqrt{5} \) is given by the characteristic eq.
of the recurrence

\[ d^2 = d + 1 \implies d^2 - d - 1 = 0 \]
\[ d = \frac{1 \pm \sqrt{5}}{2} \]

\[ F_n = c_1 \left( \frac{1 - \sqrt{5}}{2} \right)^n + c_2 \left( \frac{1 + \sqrt{5}}{2} \right)^n \]

with the correct choice of \( c_1, c_2 \) giving the formulae in the text.
(you can still do it by induction)
Section 1.1

1. (a) \( \text{No} \) \( 1 \notin S \) in general \( N \neq S \)

(b) Would imply that all naturals are squares of rationals, i.e. \( \sqrt{2} \in \mathbb{Q} \)
   False. Answer \( \text{NO} \)

(c) \( \sqrt{2} + 5 - \sqrt{2} = 5 \in \mathbb{Q} \)
   Yet \( \sqrt{2} \notin \mathbb{Q} \) and if \( 5 - \sqrt{2} = p \in \mathbb{Q} \)
   \( \implies \sqrt{2} = 5 - p \in \mathbb{Q} \) false.
   We gave an example of two irrationals with sum \( \in \mathbb{Q} \). Statement (c) false

(d) \( \sqrt{2} \cdot \frac{1}{\sqrt{2}} = 1 \in \mathbb{Q} \) yet
   \( \sqrt{2}, \frac{1}{\sqrt{2}} \notin \mathbb{Q} \). False

(e) \( \text{true} \). If \( n \) were even, \( n = 2k \)
   \( k \in \mathbb{N} \). Then \( n^2 = 4k^2 = 2(2k^2) \) even.
   So \( n^2 \) cannot be odd unless \( n \) odd as well.
(a) False. \[ S = (1, 2) \] bounded by \( M = 3 \) yet \( \sup S = 2 \notin S \).

(b) True. Since \( \forall s \in S, s > 0 \)

\[ \Rightarrow 0 \text{ is a lower bound for } S \]

\[ \Rightarrow 0 \leq \inf S \]

(c) True. Let \( b \in B \). Since \( B \subseteq S \)

\[ \Rightarrow b \in S \Rightarrow b \leq \sup S \]

Since \( \forall b \in B, b \leq \sup S \Rightarrow \sup S \]

is an upper bound of \( B \). \( \Rightarrow \sup B \leq \sup S \).

(4) in 1.1. How to guess the answer?

One way is to see that.

\[ 3j^2 + 3j = (j+1)^3 - j^3 - 1 \]

When we add them up from \( j = 1 \) to \( j = n \)

we obtain.

\[ 3 \sum_{j=1}^{n} j(j+1) = (n+1)^3 - 3 - n \]

\[ \text{Sum} = \frac{(n+1)n(n+2)}{3} \quad \text{prove it by induction} \]
(a) \( \frac{1}{n} \) is decreasing in \( n \)

then all elements \( \sup S \leq 1 \) achieved for \( n = 1 \). Since the set has a largest element (equal 1) it must be

\[ \sup S = \max S = 1 \quad (\text{for } n=1). \]

We shall show \( \inf_{n \geq 1} (\frac{1}{n}) = 0 \).

Since \( 0 < \frac{1}{n} \geq 4n > 1 \)

\[ \Rightarrow 0 \leq \inf S. \]

Let \( \epsilon > 0 \) Then \( \exists n \) such that

\[ \frac{1}{n} < \epsilon \quad \text{because the Archimedean property says there is } n \quad \text{such that} \]

\[ n > \frac{1}{\epsilon}. \]

This implies \( \boxed{0 = \inf S} \)

The set has no minimum because \( 0 \notin S \) as \( 0 \neq \frac{1}{n} \) for any possible \( n \geq 1 \).
5. If $a > 0$ there exists $m \in \mathbb{N}$ with $m > \frac{1}{a} > 0$.

$\Rightarrow a > \frac{1}{m} > 0$ contradicting the assumptions in the exercise. $\Rightarrow a \leq 0$

6. Both are true with reversed signs but are false as stated.

(a) Pick $a = 5$, $b = -5$

(b) Pick $a = 5$, $b = 6$.

7. In 1.3 The triangle inequality $|A + B| \leq |A| + |B|$. To prove $||a| - |b|| \leq |a + b|$ means to prove

- RHS $|a| - |b| \leq |a + b|$.
- LHS $-|a + b| \leq |a| - |b|$. 

\( \text{RHS} \quad |a| \leq |b| + |a+b|. \)

\[ |a + b + (-b)| \leq |a| + |b| \]
\[ \text{done.} \]

\( \text{LHS} \quad |b| \leq |a| + |a+b|. \)

\[ |a + [- (a+b)]| \leq |a + [- (a+b)]| \]
\[ \text{done.} \]

\( \text{2nd part. if we put } b \rightarrow -b \text{ we get} \)

\[ |a| - |b| \leq |a + (-b)| \]
\[ |a| - |b| \leq |a - b|. \]