possible outcomes when sampling with replacement and \( P_r \) outcomes when sampling without replacement. For unordered samples, there are \( C_r \) outcomes when sampling without replacement. Each of the preceding outcomes is equally likely, provided that the experiment is performed in a fair manner.

**REMARK** Although not needed as often in the study of probability, it is interesting to count the number of possible samples of size \( r \) that can be selected out of \( n \) objects when the order is irrelevant and when sampling with replacement. For example, if a six-sided die is rolled 10 times (or 10 six-sided dice are rolled once), how many possible unordered outcomes are there? To count the number of possible outcomes, think of listing \( r \) 0's for the \( r \) objects that are to be selected. Then insert \((n - 1)1's\) to partition the \( r \) objects into \( n \) sets, the first set giving objects of the first kind, and so on. So if \( n = 6 \) and \( r = 10 \) in the die illustration, a possible outcome is

\[ 0010000100000, \]

which says there are two 1's, zero 2's, three 3's, one 4, three 5's, and one 6. In general, each outcome is a permutation of \( r \) 0's and \((n - 1)1's\). Each distinguishable permutation is equivalent to an unordered sample. The number of distinguishable permutations, and hence the number of unordered samples of size \( r \) that can be selected out of \( n \) objects when sampling with replacement, is

\[ n^{-1+r}C_r = \frac{(n - 1 + r)!}{r!(n - 1)!}. \]

**Exercises**

1.2-1. A boy found a bicycle lock for which the combination was unknown. The correct combination is a four-digit number, \( d_1d_2d_3d_4 \), where \( d_i, i = 1, 2, 3, 4, \) is selected from 1, 2, 3, 4, 5, 6, 7, and 8. How many different lock combinations are possible with such a lock?

1.2-2. In designing an experiment, the researcher can often choose many different levels of the various factors in order to try to find the best combination at which to operate. As an illustration, suppose the researcher is studying a certain chemical reaction and can choose four levels of temperature, five different pressures, and two different catalysts.

(a) To consider all possible combinations, how many experiments would need to be conducted?

(b) Often in preliminary experimentation, each factor is restricted to two levels. With the three factors noted, how many experiments would need to be run to cover all possible combinations with each of the three factors at two levels? (Note: This is often called a \( 2^3 \) design.)

1.2-3. How many different license plates are possible if a state uses

(a) Two letters followed by a four-digit integer (leading zeros are permissible and the letters and digits can be repeated)?

(b) Three letters followed by a three-digit integer? (In practice, it is possible that certain "spellings" are ruled out.)

1.2-4. The "eating club" is hosting a make-your-own sundae at which the following are provided:

<table>
<thead>
<tr>
<th>Ice Cream Flavors</th>
<th>Toppings</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chocolate</td>
<td>Caramel</td>
</tr>
<tr>
<td>Cookies 'n' cream</td>
<td>Hot fudge</td>
</tr>
<tr>
<td>Strawberry</td>
<td>Marshmallow</td>
</tr>
<tr>
<td>Vanilla</td>
<td>M&amp;M's</td>
</tr>
<tr>
<td></td>
<td>Nuts</td>
</tr>
<tr>
<td></td>
<td>Strawberries</td>
</tr>
</tbody>
</table>

(a) How many sundaes are possible using one flavor of ice cream and three different toppings?

(b) How many sundaes are possible using one flavor of ice cream and from zero to six toppings?

(c) How many different combinations of flavors of three scoops of ice cream are possible if it is permissible to make all three scoops the same flavor?
1.2-5. How many four-letter code words are possible using the letters in IOWA if
   (a) The letters may not be repeated?
   (b) The letters may be repeated?

1.2-6. Suppose that Novak Djokovic and Roger Federer are playing a tennis match in which the first player to win three sets wins the match. Using D and F for the winning player of a set, in how many ways could this tennis match end?

1.2-7. In a state lottery, four digits are drawn at random one at a time with replacement from 0 to 9. Suppose that you win if any permutation of your selected integers is drawn. Give the probability of winning if you select
   (a) 6, 7, 8, 9.
   (b) 6, 7, 8, 8.
   (c) 7, 7, 8, 8.
   (d) 7, 8, 8, 8.

1.2-8. How many different varieties of pizza can be made if you have the following choice: small, medium, or large size; thin 'n' crispy, hand-tossed, or pan crust; and 12 toppings (cheese is automatic), from which you may select from 0 to 12?

1.2-9. The World Series in baseball continues until either the American League team or the National League team wins four games. How many different orders are possible (e.g., ANNAAA means the American League team wins in six games) if the series goes
   (a) Four games?
   (b) Five games?
   (c) Six games?
   (d) Seven games?

1.2-10. Pascal's triangle gives a method for calculating the binomial coefficients; it begins as follows:

\[
\begin{array}{cccccc}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

The \( n \)th row of this triangle gives the coefficients for \((a + b)^{n-1}\). To find an entry in the table other than a 1 on the boundary, add the two nearest numbers in the row directly above. The equation

\[
\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1},
\]
called Pascal's equation, explains why Pascal's triangle works. Prove that this equation is correct.

1.2-11. Three students (S) and six faculty members (F) are on a panel discussing a new college policy.
   (a) In how many different ways can the nine participants be lined up at a table in the front of the auditorium?
   (b) How many lineups are possible, considering only the labels S and F?
   (c) For each of the nine participants, you are to decide whether the participant did a good job or a poor job stating his or her opinion of the new policy, that is, give each of the nine participants a grade of G or P. How many different "scorecards" are possible?

1.2-12. Prove

\[
\sum_{r=0}^{n} (-1)^r \binom{n}{r} = 0 \quad \text{and} \quad \sum_{r=0}^{n} \binom{n}{r} = 2^n.
\]

**Hint:** Consider \((1 - 1)^n\) and \((1 + 1)^n\), or use Pascal's equation and proof by induction.

1.2-13. A bridge hand is found by taking 13 cards at random and without replacement from a deck of 52 playing cards. Find the probability of drawing each of the following hands.
   (a) One in which there are 5 spades, 4 hearts, 3 diamonds, and 1 club.
   (b) One in which there are 5 spades, 4 hearts, 2 diamonds, and 2 clubs.
   (c) One in which there are 5 spades, 4 hearts, 1 diamond, and 3 clubs.
   (d) Suppose you are dealt 5 cards of one suit, 4 cards of another. Would the probability of having the other suits split 3 and 1 be greater than the probability of having them split 2 and 2?

1.2-14. A bag of 36 dum-dum pops (suckers) contains up to 10 flavors. That is, there are from 0 to 36 suckers of each of 10 flavors in the bag. How many different flavor combinations are possible?

1.2-15. Prove Equation 1.2-2. **Hint:** First select \( n_1 \) positions in \( \binom{n}{n_1} \) ways. Then select \( n_2 \) from the remaining \( n - n_1 \) positions in \( \binom{n - n_1}{n_2} \) ways, and so on. Finally, use the multiplication rule.

1.2-16. A box of candy hearts contains 52 hearts, of which 19 are white, 10 are tan, 7 are pink, 3 are purple, 5 are yellow, 2 are orange, and 6 are green. If you select nine pieces
of candy randomly from the box, without replacement, give the probability that
(a) Three of the hearts are white.
(b) Three are white, two are tan, one is pink, one is yellow, and two are green.

1.2-17. A poker hand is defined as drawing 5 cards at random without replacement from a deck of 52 playing cards. Find the probability of each of the following poker hands:

(a) Four of a kind (four cards of equal face value and one card of a different value).
(b) Full house (one pair and one triple of cards with equal face value).
(c) Three of a kind (three equal face values plus two cards of different values).
(d) Two pairs (two pairs of equal face value plus one card of a different value).
(e) One pair (one pair of equal face value plus three cards of different values).

1.3 CONDITIONAL PROBABILITY

We introduce the idea of conditional probability by means of an example.

Example 1.3-1

Suppose that we are given 20 tulip bulbs that are similar in appearance and told that 8 will bloom early, 12 will bloom late, 13 will be red, and 7 will be yellow, in accordance with the various combinations listed in Table 1.3-1. If one bulb is selected at random, the probability that it will produce a red tulip \( R \) is given by \( P(R) = 13/20 \), under the assumption that each bulb is “equally likely.” Suppose, however, that close examination of the bulb will reveal whether it will bloom early \( (E) \) or late \( (L) \). If we consider an outcome only if it results in a tulip bulb that will bloom early, only eight outcomes in the sample space are now of interest. Thus, under this limitation, it is natural to assign the probability 5/8 to \( R \); that is, \( P(R \mid E) = 5/8 \), where \( P(R \mid E) \) is read as the probability of \( R \) given that \( E \) has occurred. Note that

\[
P(R \mid E) = \frac{5}{8} = \frac{N(R \cap E)}{N(E)} = \frac{N(R \cap E) / 20}{N(E) / 20} = \frac{P(R \cap E)}{P(E)},
\]

where \( N(R \cap E) \) and \( N(E) \) are the numbers of outcomes in events \( R \cap E \) and \( E \), respectively.

This example illustrates a number of common situations. That is, in some random experiments, we are interested only in those outcomes which are elements of a subset \( B \) of the sample space \( S \). This means, for our purposes, that the sample space is effectively the subset \( B \). We are now confronted with the problem of defining a probability set function with \( B \) as the “new” sample space. That is, for a given event

| Table 1.3-1: Tulip bulb colors
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Early (E)</td>
<td>Late (L)</td>
</tr>
<tr>
<td>Red (R)</td>
<td>5</td>
<td>8</td>
</tr>
<tr>
<td>Yellow (Y)</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>Totals</td>
<td>8</td>
<td>12</td>
</tr>
</tbody>
</table>