Problem 1.4-16

Note that we look at two scenarios for b)

You are a member of a class of $N = 18$ students. A bowl contains $N = 18$ chips: 1 blue and 17 red. Each student is to take one chip from the bowl without replacement. The student who draws the blue chip is guaranteed an A for the course.

a) If you have a choice of drawing first, fifth, or last, which position would you choose?

b) Suppose the bowl contains $b = 2$ blue and $N - b = 16$ red chips. What position would you now choose?

(I) if any student who draws a blue chip wins (there are $b = 2$ winners)

(II) if the winner is the first who draws a blue chip

Solution. The probability we want to know is

$$f(x) = P(A), \quad A = \{\text{the } x\text{-th draw is blue}\}, \quad x = 1, 2, \ldots$$

$$B_k = \{\text{there are } k \text{ blue chips among the first } x - 1 \text{ draws}\}$$

$$f(x) = P(A) = \sum_{k=0}^{b-1} P(A \cap B_k) = \sum_{k=0}^{b-1} P(A|B_k) P(B_k)$$

$$P(A|B_k) = \frac{b-k}{N-(x-1)} = \frac{b-k}{N-x+1}$$

There are $b-k$ blue remaining to choose from $N-(x-1)$ chips after $x-1$ draws;

$$P(B_k) = \frac{\binom{b}{k} \binom{N-b}{x-1-k}}{\binom{N}{x-1}}$$

we have to pick $k$ blue out of $b$ blue, then pick $(x-1)-k$ red out of $N-b$ red to achieve exactly $k$ red in the first $x-1$ draws. These numbers are multiplied because each choice of red chips, together with each choice of blue chips gives us a valid combination. We divide this product by the grand total (the denominator): the
number of choices of \( x - 1 \) out of \( N \) if we would not care about color.

We apply this formula to

a) \( b = 1, \ k = 0 \) (there are no other cases)

\[
f(x) = \frac{1}{N - x + 1} \binom{N-1}{x-1} = \frac{1}{N}
\]

all are equal, quite expected.

b) \( (1) b = 2, \ k = 0, 1 \)

\[
f(x) = \sum_{k=0}^{1} \frac{b - k}{N - x + 1} \binom{b}{k} \binom{N-b}{x-1-k} \binom{N}{x-1} = \frac{2}{N} = \frac{b}{N}
\]

are equal again!

b) \( (2) b = 2, \ k = 0 \)

\[
f(x) = \frac{b - k}{N - x + 1} \binom{b}{k} \binom{N-b}{x-1-k} \binom{N}{x-1} = \frac{2}{N} = \frac{b}{N}
\]

This decreases with \( x = 1, 2, \ldots N - 1 \).

You want to be the first!

Remarks.

- Comment on \( x = 1 \) (why is it familiar?) and \( x = N - 1 \) (why is this the last value?).
- Since \( f(x) \) are probabilities, it follows that

\[
1 + 2 + 3 + \ldots (N - 2) + (N - 1) = \frac{N(N-1)}{2},
\]

something we could have calculated independently by adding the same numbers in reversed order (sum of an arithmetic progression).

- Now try to solve this problem by combinatorial methods, without probability.