An covariant derivative operator $\nabla$ which is symmetric:

\[ [X,Y] = \nabla_X Y - \nabla_Y X \]

obeys metric product rule (it is compatible):

\[ X(y,z) = \langle \xi, Y \rangle + \langle Y, \xi \rangle \]

In coord chart $(U, \xi)$ $X^i$

$X, Y \in \mathfrak{X}(M)$ on $\xi(U)$

\[ \nabla_X Y = \left[ X(Y^k) + \Gamma_{ij}^{k} X^i Y^j \right] \partial_k \]

\[ \Gamma_{ij}^{k} = \frac{1}{2} g^{kl} \left[ \partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij} \right] \]

$\nabla_X Y \big|_p T_P M$ depends only on $X_p$

$X, Y \in \mathfrak{X}(M)$

\[ \nabla_N Y \in T_P M, \quad N = \sum N^i \frac{\partial}{\partial x^i} \big|_p \]

\[ \nabla_N Y = \left[ N(Y^k) + \Gamma_{ij}^{k} N^i Y^j (p) \right] \partial_k \big|_p \]

ex. $T^3$, standard metric, standard connection $D$

ex. $M^2 \subset T^3$ surfaces in $T^3$

\[ \langle \cdot, \cdot \rangle \text{ induced metric} \]

Induced connection in $M^2$: $X, Y \in \mathfrak{X}(M^2)$

\[ \nabla_X Y = \text{tan}(D_X Y) \]
OR
\[ \nabla_x y = \tan(\mathcal{D}_x \tilde{y}) \]
where \( \tilde{x}, \tilde{y} \in \mathcal{X}(\mathbb{R}^n) \) are extensions of \( x, y \in \mathcal{X}(M) \) to \( \mathbb{R}^n \)
\[ \tilde{x}|_m = x, \quad \tilde{y}|_m = y \]
\( \nabla_x y|_p \in T_p M \)
\[ \nabla_x y|_p = \tan(\mathcal{D}_x \tilde{y}|_p) \]

Proposition: Given \( M^2 \subset \mathbb{R}^3 \) the induced connection \( \nabla \) on \( M^2 \) is symmetric & compatible w/ the induced metric
1. \[ [x, y] = \nabla_x y - \nabla_y x \]
2. \[ x \langle y, z \rangle = \ldots \]
This is the Levi-Civita connection

\(*\) Exercise: Prove this
Fact needed to prove this: let \( \tilde{x}, \tilde{y} \in \mathcal{X}(\mathbb{R}^n) \) be extensions of \( x, y \in \mathcal{X}(\mathbb{R}^n) \). Then, \( [\tilde{x}, \tilde{y}]|_m = [x, y] \)
(This fact isn't hard to show but we will just trust it is true)

This ends Chapter 3
Chapter 4: Parallelism & Geodesics

\( \sigma' \) = velocity vector field

**Def:** Let \( \sigma : (a,b) \to M \), \( t \to \sigma(t) \) be a smooth curve in \( M \). A vector field \( X \) along \( \sigma \) is a function which assigns to each \( t \in (a,b) \) a vector \( X(t) \in T_{\sigma(t)}M \)

\[ X = X(t) \]
\[ t \to \sigma(t) \]

**Coordinate Expression**

\[ \sigma^{-1}(\mathbb{R}^n) \]

\[ \sigma^{-1}(\mathbb{R}^n) \to \mathbb{R} \]

\[ (t_0, t) \to \mathbb{R} \]

\( t_0 \in (a,b), \sigma(t_0) \in M \)

\( (U, \phi), X^i \) be a coordinate chart about \( \sigma(t_0) \)

\( \sigma^{-1}(\mathbb{R}^n) \)

Let \( X = X(t) \) v.f. along \( \sigma \)

For each \( t \in \sigma^{-1}(\mathbb{R}^n) \)

\[ X(t) \in T_{\sigma(t)}M \]
\[ X(t) = \sum_{i=1}^{n} X^i(t) \frac{\partial}{\partial x^i} |_{\sigma(t)} \]

\[ X^i = X^i(t) \] components of \( X \) wrt \((\mathbb{R}, U)\)

**Def (Smoothness Criteria):** Let \( X = X(t) \) be a vf along a curve \( \sigma: (a, b) \to M \). \( X = X(t) \) is smooth provided for each \( t \in (a, b) \) there is a coord chart \((U, \varphi)\) about \( \sigma(t) \in M \) s.t.

the components \( X^i = X^i(t) \) of \( X \) wrt \((\mathbb{R}, U)\) are smooth.

**Remark:** As usual, this definition of smoothness does not depend on the particular choice of coord chart.

**Notation:** \( X(\sigma) \) = collection of smooth \( \text{vf} \) in \( M \) along \( \sigma \).

**Ex. 1.** \( \sigma: (a, b) \to M, \ t \to \sigma(t) \)

**Velocity vf along \( \sigma \):**

\[ t \to \sigma'(t) \]

velocity vectors act on things

\[ \sigma'(t) : C^\infty(\sigma(t)) \to \mathbb{R} \]

\[ \sigma'(t)(f) = \frac{df}{dt} \bigg|_{\sigma(t)} \]

In coord chart, \( X^i \)

\[ \sigma(t) = (X^1(t), X^2(t), \ldots, X^n(t)) \]

\[ \hat{\sigma} : x^i = x^i(t) \]

\[ \vdots \]

\[ x^n = x^n(t) \]

\[ \sigma'(t) = \sum_{i=1}^{n} \frac{dx^i(t)}{dt} \frac{\partial}{\partial x^i} |_{\sigma(t)} \]
ex. 2. $X \in T(M)$, $\sigma : (a,b) \to M$

$X|_{\sigma} = X_{\sigma(t)}$

$X_\sigma(t) \in T_{\sigma(t)} M$

$X_\sigma(t) = X_{\sigma(t)}$

In coord chart $(U, \tau)$, $X^i$

on $\tau(U)$, $X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}$, $X \in C^\infty(\tau(U))$

$V$ along $\sigma$

$t \to X_\sigma(t) = X_{\sigma(t)}$

$X_\sigma(t) = \sum_{i=1}^n X^i(\sigma(t)) \frac{\partial}{\partial x^i}|_{\sigma(t)}$

$X_\sigma(t) = \sum_{i=1}^n X^i \circ \sigma(t) \frac{\partial}{\partial x^i}|_{\sigma(t)}$

along $\sigma_1$, $X_\sigma = \sum_{i=1}^n X^i \circ \sigma \frac{\partial}{\partial x^i}|_{\sigma(t)}$

Covariant Differentiation of $V$ along a curve

Consider $\mathbb{R}^n$ w/ standard metric $\langle \cdot, \cdot \rangle$ & standard connection $\nabla$.

Given: $\sigma : (a,b) \to \mathbb{R}^n$

$X = X(t)$ smooth $V$ along $\sigma$

$x^1, x^2, \ldots, x^n$ Cartesian coords provide us w/ a canonical way of representing $X = X(t)$

\[ X(t) = \sum_{i=1}^n X^i(t) \frac{\partial}{\partial x^i}|_{\sigma(t)} \]
\[ X^i = X^i(t), \quad X^i \in C^\infty((a,b)) \quad \forall x \]

\[
\frac{dX}{dt}(t) = \sum_{i=1}^{n} \frac{\partial X^i}{\partial t}(t) \frac{2}{\partial x^i} \bigg|_{t(\sigma)}
\]

\[
\frac{dX}{dt} = \sum_{i=1}^{n} \frac{dX}{dt} \frac{2}{\partial x^i} \bigg|_{t(\sigma)}
\]

This differentiation satisfies certain properties:

1) Linearity over reals
\[
\frac{d}{dt}(\alpha X + \beta Y) = \alpha \frac{dX}{dt} + \beta \frac{dY}{dt}
\]
for \( \alpha, \beta \in \mathbb{R} \)

2) Product rule
\[
\frac{d}{dt} f X = \frac{df}{dt} X + f \frac{dX}{dt}
\]
\( f = f(t) \)

3) There is a connection between \( \frac{d}{dt} \) and \( D \)
\[
X \in \mathcal{X}(\mathbb{R}^n), \quad \sigma: (a,b) \rightarrow \mathbb{R}^n \quad t \rightarrow X_\sigma(t) \quad (X_\sigma(t) = X_{\sigma(t)})
\]
\[
\frac{d}{dt} X_\sigma(t) = D_{\sigma'(t)} X
\]

**Proof of 3**: \( X \in \mathcal{X}(\mathbb{R}^n) \)
\[
X = \sum_{i=1}^{n} X^i \partial_i, \quad X^i \in C^\infty(\mathbb{R}^n)
\]
\[
X_\sigma = \sum_{i=1}^{n} X^i \circ \sigma \partial_i |_{\sigma}
\]
\[
\frac{d}{dt} X_\sigma(t) = \sum_{i=1}^{n} \frac{d}{dt} X^i \circ \sigma(t) \partial_i \bigg|_{\sigma(t)}
\]

\[
\bigg| = D_{\sigma'(t)} X^i
\]

\[
\frac{d}{dt} X_\sigma(t) = \sum_{i=1}^{n} \sigma'(t)(X^i) \partial_i \bigg|_{\sigma(t)} = D_{\sigma'(t)} X
\]