\(X, Y \in \mathcal{X}(\mathbb{R}^n)\)

\[ Y = \sum_{i=1}^{n} Y^i \frac{\partial}{\partial x^i} \]

\[ D_x Y = \sum_{i=1}^{n} X^i (Y^i) \frac{\partial}{\partial x^i} \in \mathcal{X}(\mathbb{R}^n) \]

Directional derivative

\[ D_x Y \big|_p = \sum_{i=1}^{n} X^i (Y^i) \frac{\partial}{\partial x^i} \big|_p \in T_p \mathbb{R}^n \]

**D:** \(\mathcal{X}(\mathbb{R}^n) \times \mathcal{X}(\mathbb{R}^n) \rightarrow \mathcal{X}(\mathbb{R}^n)\)

\[(X, Y) \rightarrow D_x Y\]

1) This map is linear in \(Y\) over \(\mathbb{R}\)

2) (product rule)

\[ D_x (fY) = X(f)Y + fD_x Y \quad \forall f \in C^\infty(\mathbb{R}^n) \]

3) \((X, Y) \rightarrow D_x Y\) is linear in \(X\) over \(C^\infty(\mathbb{R}^n)\)

\[ D_{x+y} z = D_x z + D_y z \quad \forall f \in C^\infty(\mathbb{R}^n) \]

\[ D_{fX} z = f D_x z \]

**Def:** A linear connection \(\nabla\) on a manifold \(M\) is a map

\[ \nabla : \mathcal{X}(M) \times \mathcal{X}(M) \rightarrow \mathcal{X}(M) \]

\[(X, Y) = \nabla_X Y\]

Called covariant derivative

(differentiation of \(v\) that is independent of coordinates)

1) \((X,Y) \rightarrow \nabla_X Y\) is linear in \(X\) over \(C^\infty(M)\)
\[ \nabla_{(fX+gY)}^2 z = f \nabla_x z + g \nabla_y z \]

2) \((X,Y) \mapsto \nabla_x Y\) is linear in \(Y\) wrt \(\mathbb{R}\)
\[ \nabla_x (aY + bZ) = a \nabla_x Y + b \nabla_x Z \quad \forall \ a, b \in \mathbb{R} \]

3) (product rule) \((X,Y) \mapsto \nabla_x Y\) obeys
\[ \nabla_x (fY) = X(f)Y + f \nabla_x Y \quad \forall \ f \in C^\infty(M) \]

\(\nabla_x Y\) = covariant derivative of \(Y\) wrt \(X\)
\(\nabla\) = covariant derivative operator
(covariant meaning coordinate-free)

ex. \(\mathbb{R}^n\), \(\nabla = D = \) standard connection on \(\mathbb{R}^n\)

\(\nabla\) on \(M \subset \mathbb{R}^3\)
\(\nabla = \) induced connection

\(\text{surface in } \mathbb{R}^3\)
\(\text{tangent space}\)

\(\text{tan} T_p \mathbb{R}^3 \rightarrow T_p M\)
\(\text{tan } N = \) component of \(N\) tangent to \(M\)
\(\text{tan } N = N - \langle N, n \rangle n \)
Check $\langle \tan \nabla, n \rangle = 0$ (from the formula above).

Side note: recall $\nabla \in T_p \mathbb{R}^3$ can be written as

$\nabla = \nabla' \mathbf{x}_1 + \nabla' \mathbf{x}_2 + \nabla' \mathbf{x}_3$

$\tan \nabla = \nabla' \mathbf{x}_1 + \nabla' \mathbf{x}_2$

$\tan$ is linear over $\mathbb{R}$

$\tan (a \nabla + b \nabla) = a \tan \nabla + b \tan \nabla$

$X = \nabla f$ along $M$ (not necessarily tangent to $M$)

$\tan X$ is a $\nabla f$ tangent to $M$

$\tan X |_p = \tan X_p$

Consider the following map: $(M^3 \subset \mathbb{R}^3)$

$\nabla: \mathcal{F}(M) \times \mathcal{F}(M) \to \mathcal{F}(M)$

$X, Y \in \mathcal{F}(M)$

$\nabla_X Y = \tan (D_X Y)$

$D$ = standard connection on $\mathbb{R}^3$

$p \in M$, $D_X Y |_p = D_{X_p} Y$

$X_p = \partial^1 (0)$
Proposition: \( M^2 \subset \mathbb{R}^3 \) smooth surface in \( \mathbb{R}^3 \). The map
\[
\nabla: \mathcal{X}(M^2) \times \mathcal{X}(M^2) \rightarrow \mathcal{X}(M^2)
\]
\[(X,Y) \mapsto \nabla_X Y\]
defined by
\[
\nabla_X Y \equiv \text{lem} (D_x Y)
\]
is a linear connection on \( M \), called the induced connection.

Proof: \textbf{Exercise} (show it satisfies the 3 axioms of linear connections)

Proposition: Let \( \nabla \) be a linear connection (De Carmo calls it an affine connection) on a manifold \( M \). For \( X, Y \in \mathcal{X}(M) \)
\( \nabla_X Y \in \mathcal{X}(M) \) depends only on values of \( X \) & \( Y \) locally, i.e.
\( \nabla_X Y \big|_p \) depends only on values of \( X \) & \( Y \) near \( p \), i.e.
if \( X = \tilde{X} \) near \( p \) & \( Y = \tilde{Y} \) near then
\[
\nabla_X Y \big|_p = \nabla_{\tilde{X}} \tilde{Y} \big|_p
\]

We can break this proof into two steps:
1) \( \nabla_X Y \big|_p \) depends only on values of \( Y \) near \( p \)
2) \( \nabla_X Y \big|_p \) depends only on values of \( X \) near \( p \).

Proof of 1): Let \( Y, \tilde{Y} \in \mathcal{X}(M) \), \( Y = \tilde{Y} \) on a nbhd \( W \) about \( p \).
$\nabla_x \tilde{Y} \big|_p = \nabla_x \tilde{Y} \big|_p$

Set $z = y - \tilde{y} \in \mathfrak{T}(M)$

$z \equiv 0$ on $W$

Moreover, $\nabla_x z = \nabla_x (y - \tilde{y}) = \nabla_x y - \nabla_x \tilde{y}$

$\Rightarrow$ $\nabla_x z \big|_p = \nabla_x y \big|_p - \nabla_x \tilde{y} \big|_p$

It suffices to show $\nabla_x z \big|_p = 0$ where $z \equiv 0$ on $W$

The proof of $\nabla_x z \big|_p = 0$ uses a "bump function" argument.

**Bump Function Lemma**: Let $W$ be an open set in $M$, containing $p \in M$. Then there exists a smooth real-valued function on $M$ ($f \in C^\infty(M)$) s.t.

1) $0 \leq f \leq 1$
2) $f = 1$ on a small nbhd of $p$
3) $f = 0$ outside $W$

![Graph of a bump function](image)

By introducing coords to prove this lemma, it suffices to consider $M = \mathbb{R}^n$

(See Do Carmo P.30, remark 5.7 for how to construct bump functions)

Back to proof:
Let $f \in C^\infty(M)$ be such a bump function based at $p$, $f(p) = 1$, $f = 0$ on $M \setminus W$

$f \in C^\infty(M), \quad z \in \mathscr{X}(M)$

$\nabla_x f z = \nabla_x 0 = 0$

*Exercise: Show $\nabla_x 0 = 0$ (use the axioms)

On the other hand, $\nabla_x f z = X(f) z + f X(z)$

at $p$, $\nabla_x f z \big|_p = X_p(f) z_p + f(p) \nabla_x z \big|_p$

$= 0$

$\Rightarrow 0 = \nabla_x z \big|_p$

$\Rightarrow 0 = \nabla_x z \big|_p$

Thus, we've shown that $\nabla x Y \big|_p$ depends only on values near $p$ (we proved 1)

*Exercise: Prove 2) (it is similar to this proof)

Because of this local property, then it makes sense to compute cov. der. for $v$ only defined locally

$W \subset M$ open, $X, Y \in \mathscr{X}(M)$

$\nabla_x Y \in \mathscr{X}(W)$
Covariant Derivative in Coordinates

Let \( z: U \subset \mathbb{R}^n \rightarrow M \) be a coordinate chart on \( M \).

\[
\frac{\partial}{\partial x^i} \quad \text{on } g(U) = \left\{ \frac{\partial}{\partial x^i} \right\}
\]

Shorthand notation: \( \partial_i = \frac{\partial}{\partial x^i} \)

Coord \( U \)'s: \( \{ z_1, z_2, \ldots, z_n \} \) in \( g(U) \) \( U \subset \mathbb{R}^n \)

\[
\partial_i \in \mathfrak{X}(g(U))
\]

\[
\nabla_{\partial_i} \partial_j \in \mathfrak{X}(g(U))
\]

\[
\nabla_{\partial_i} \partial_j = \square \partial_i + \square \partial_2 + \ldots + \square \partial_n
\]

\[
= \Gamma_{ij}^{\alpha} \partial_\alpha + \Gamma_{ij}^{\alpha} \partial_\alpha + \ldots + \Gamma_{ij}^{\alpha} \partial_\alpha
\]

where \( \Gamma_{ij}^{\alpha} \) is our notation for the coefficients \( \square \)

\[
\nabla_{\partial_i} \partial_j = \frac{\partial}{\partial x^i} \Gamma_{ij}^{\alpha} \partial_\alpha
\]

\[
\Gamma_{ij}^{\alpha} \in C^\infty(g(U))
\]

\[
(\nabla_{\partial_i} \partial_j)_p = \frac{\partial}{\partial x^i} \Gamma_{ij}^{\alpha} (p) \partial_\alpha |_p
\]

\( \Gamma_{ij}^{\alpha} \) = connection coefficients w.r.t. \( \nabla \) & \( g: U \subset \mathbb{R}^n \rightarrow M \)

(once we introduce metrics, those connection coefficients will become our friends, the Christoffel symbols)

Using Einstein Summation Convention,

\[
\nabla_{\partial_i} \partial_j = \Gamma_{ij}^{h} \partial_h
\]
\( X, Y \in \mathfrak{X}(\mathbb{M}) \)
\[ \nabla_x Y \in \mathfrak{X}(\mathbb{M}) \]

Goal is to get a coordinate expression for this:

\[ X = X^i \delta_i, \quad Y = Y^j \delta_j \]

\[ \nabla_x Y = \nabla_{(X^i \delta_i)} (Y^j \delta_j) \]

\[ = X^i \nabla_{\delta_i} (Y^j \delta_j) \]

\[ = X^i \left[ \delta_i Y^j \delta_j + Y^j \nabla_{\delta_i} \delta_j \right] \]

Changing dummy index \( j \rightarrow h \)

\[ = X^i \left[ \delta_i Y^h \delta_h + Y^j \Gamma_{ij}^{\ h} \delta_h \right] \]

\[ = X^i \left[ \partial_i Y^h + \Gamma_{ij}^{\ h} Y^j \right] \partial_h \]

\( \gamma^h \quad \text{common notation in physics books but we won't use it here} \)

\[ \nabla_x Y = \left[ X^i \delta_i Y^h + \Gamma_{ij}^{\ h} X^i Y^j \right] \partial_h \]

\[ \nabla_x Y = \left[ X(Y^h) + \Gamma_{ij}^{\ h} X^i Y^j \right] \partial_h \]

\( (\nabla_x Y)^h = X(Y^h) + \Gamma_{ij}^{\ h} X^i Y^j = h^{th} \) component of \( \nabla_x Y \)

Note: In cartesian coords the \( \Gamma_{ij}^{\ h} \)'s = 0 & this is our usual expression
\[ \nabla_i \nabla_j = \Gamma^{k}_{ij} \nabla_k \]

\[ \Gamma^{k}_{ij} \text{ = Connection coefficients with } \nabla : U \subset \mathbb{R}^n \to M \]

- completely determine \( \nabla \) on \( (U, \Omega) \)