Vector Fields

$M = \text{manifold}$

$\mathbf{X} = \text{vector field on } M$

$\mathbf{X} \quad \text{pe} M \rightarrow \mathcal{T}_{p} M$

If in a coordinate chart $(\mathcal{U}, \mathbf{X}) = (\mathbb{R}, \mathbf{X}')$ on $\mathcal{U}$

$\mathbf{X} = \frac{\partial}{\partial x^i} X^i$

$X^i: \mathcal{U} \subset M \rightarrow \mathbb{R}$

Components of $X$ wrt $(\mathcal{U}, \mathbf{X})$

$X$ is smooth if its components wrt the coordinates are smooth.

$\mathfrak{X}(M) = \text{collection of smooth vector fields}$

$X, Y \in \mathfrak{X}(M)$

$X + Y \in \mathfrak{X}(M)$

$X \in \mathfrak{X}(M), f \in C^\infty(M)$

$fX \in \mathfrak{X}(M)$

Operations defined pointwise:

**Directional Derivative aspect of vector fields**

$\mathbf{V} \in \mathcal{T}_{p} M$

$\mathbf{V}: C^\infty(p) \rightarrow \mathbb{R}$

$\mathbf{V}(f) = \mathbf{V}^i \partial_i f$

$= \frac{d}{dt} f\circ \gamma(t) \bigg|_{t=0}$
\[ \forall x \in \mathcal{X}(M), f \in C^\infty(M) \]
\[ X(f) \in C^\infty(M) \]
\[ X(f(x)) = X_x(f) \]

3: \( U \subset \mathbb{R}^n \rightarrow M \)

on \( \mathcal{Z}(U) \), \( X = \sum_{i=1}^n x^i \frac{\partial}{\partial x^i} \)

\[ X(f) = \left( \sum_{i=1}^n x^i \frac{\partial}{\partial x^i} \right)f = \sum_{i=1}^n x^i \frac{\partial f}{\partial x^i} \]

Smooth on \( \mathcal{Z}(U) \)

\[ X : \mathcal{C}^\infty(M) \rightarrow C^\infty(M) \]
\[ f \rightarrow X(f) \]

This map satisfies:
1) \( X(af + bg) = aX(f) + bX(g) \) (linearity)
   \[ a, b \in \mathbb{R}, f, g \in C^\infty(M) \]
2) \( X(fg) = X(f)g + fX(g) \) (product rule)

Remark: Any mapping \( X : C^\infty(M) \rightarrow C^\infty(M) \) satisfying 1 & 2) is called a derivation.

Note: Shorthand \( Xf = X(f) \)

But be careful, \( Xf \neq fX \)

\[ X \text{ acting on } f \quad f \text{ times } X \]
Lie Bracket of Vector Fields

\[ X, Y \in C^\infty(M) \]

\[ X : C^\infty(M) \to C^\infty(M) \]

\[ Y : C^\infty(M) \to C^\infty(M) \]

\[ XY : C^\infty(M) \to C^\infty(M) \]

\[ XY(f) = X(Y(f)) \]

(by general this is not a vector field)

\[ M = \mathbb{R}^2 \quad (x, y) \]

\[ X = \frac{\partial}{\partial x}, \quad Y = \frac{\partial}{\partial y} \]

\[ XY(f) = X(Y(f)) = \frac{\partial}{\partial x} \left( \frac{\partial}{\partial y} f \right) = \frac{\partial^2 f}{\partial x \partial y} \neq 0 \frac{\partial}{\partial x} \frac{\partial}{\partial y} \]

\[ \Rightarrow \text{not a vector field} \]

\[ [X, Y] = XY - YX \]

\[ [X, Y] : C^\infty(M) \to C^\infty(M) \]

\[ [X, Y](f) = (XY - YX)(f) = XY(f) - YX(f) = X(Y(f)) - Y(X(f)) \]

\[ M = \mathbb{R}^2, \quad (x, y) \]

\[ X = y \frac{\partial}{\partial y}, \quad Y = x \frac{\partial}{\partial y} \]

Compute \([X, Y]\)

\[ [X, Y](f) = \left[ y \frac{\partial}{\partial y}, x \frac{\partial}{\partial y} \right] = \left( y \frac{\partial}{\partial y} \right) \left( x \frac{\partial}{\partial y} \right) f - \left( x \frac{\partial}{\partial y} \right) \left( y \frac{\partial}{\partial y} \right) f \]
\[ \begin{align*}
\text{(1)} & \quad y \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = y \frac{\partial^{2} f}{\partial y^{2}} \\
\text{(2)} & \quad x \frac{\partial}{\partial y} \left( y \frac{\partial f}{\partial y} \right) = x \left( \frac{\partial f}{\partial y} + y \frac{\partial^{2} f}{\partial y^{2}} \right) = x \frac{\partial f}{\partial y} + xy \frac{\partial^{2} f}{\partial y^{2}} \\
[ x, y ](f) & = (1 - \Theta) = -x \frac{\partial f}{\partial y} = (-x \frac{\partial f}{\partial y})(f) \\
[ x, y ] & = -x \frac{\partial}{\partial y} \\
[ x, y ] & = -y
\end{align*} \]

Comment: 2nd derivatives always cancel out

**Proposition:** Let \( X, Y \) be smooth vector fields on \( M \), \((U, \varphi)\) chart on \( M \), if

\[
X = \sum_{i} X^{i} \frac{\partial}{\partial x^{i}} \quad \text{and} \quad Y = \sum_{j} Y^{j} \frac{\partial}{\partial x^{j}}
\]

then

\[
[ X, Y ] = \sum_{i} \left[ \left( X^{i} \frac{\partial Y^{j}}{\partial x^{i}} - Y^{j} \frac{\partial X^{i}}{\partial x^{j}} \right) \right] \frac{\partial}{\partial x^{i}}
\]

**Proof:** Shorthand notation. \( \partial_{i} = \frac{\partial}{\partial x_{i}} \), \( \partial_{j} = \frac{\partial}{\partial x_{j}} \)

\[
X = \sum_{i} X^{i} \partial_{i} \quad \text{Einstein Summation Convention} \Rightarrow X = X^{i} \partial_{i}
\]

\[
Y = Y^{j} \partial_{j}
\]

\[
[ X, Y ](f) = XY(f) - YX(f) = \]

\[
XY(f) = \sum_{i} X^{i} \partial_{i} \left( Y^{j} \partial_{j} \right)(f) = \sum_{i} X^{i} \partial_{i} \left( Y^{j} \partial_{j} f \right) = \sum_{i} \left( X^{i} Y^{j} \partial_{j} + Y^{j} X^{i} \partial_{j} \right) \partial_{i} f
\]

\[
YX(f) = \sum_{j} Y^{j} \partial_{j} \left( X^{i} \partial_{i} f \right) = \sum_{j} Y^{j} \partial_{j} \left( X^{i} \partial_{i} f \right) = \sum_{j} \left( Y^{j} X^{i} \partial_{j} + X^{i} Y^{j} \partial_{j} \right) \partial_{i} f
\]

\[
[ X, Y ](f) = \sum_{i} \left( X^{i} Y^{j} \partial_{j} - Y^{j} X^{i} \partial_{j} \right) \partial_{i} f = \sum_{i} \left( X^{i} Y^{j} \partial_{j} - Y^{j} X^{i} \partial_{j} \right) \partial_{i} f
\]

\[
[X, Y] = \left( X^{i} Y^{j} - Y^{j} X^{i} \right) \partial_{j} = \left( X^{i} \frac{\partial Y^{j}}{\partial x^{i}} - Y^{j} \frac{\partial X^{i}}{\partial x^{j}} \right) \partial_{j}
\]
Remarks:

1. \[ [x, y] = \sum \frac{1}{3!} [x(y^i) - y(x^i)] \frac{\partial}{\partial x^i} \] (Component expression of Lie Bracket)

2. Given \((u, x^i)\)

\[ \left[ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right] = 0 \quad \forall i, j \]

\[ \left[ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right](f) = \frac{2}{\partial x^i} \left( \frac{\partial}{\partial x^j} \right)(f) - \frac{2}{\partial x^j} \left( \frac{\partial}{\partial x^i} \right)(f) = 0 \] because mixed partials are equal

i.e. Coordinate vector fields commute

Lie bracket measures extent to which \(\psi\)'s fail to commute

Further properties of the Lie bracket:

1. The Lie bracket is bilinear over the reals

i.e. \([a x + b y, z] = a [x, z] + b [y, z]\)

2. \([x, a y + b z] = a [x, y] + b [x, z]\)

Bilinear over the reals but not bilinear over \(C^\infty(M)\) in general

* Exercise: Construct an explicit example to show this

2. \(\forall f, g \in C^\infty(M) \& x, y \in \chi(M)\),

\[ [fx, gy] = fg [x, y] + f x(g) y - g y(f) x \]

\[ \text{ex. } g = 1 \Rightarrow x(g) = 0 \Rightarrow [fx, y] = f [x, y] - y(f) x \]

* Exercise: Prove property 2.

* Hint: Use the definition & no the coordinate expression
Exercise: a) Compute the Lie bracket of \( X = y \frac{\partial}{\partial y}, \ Y = x \frac{\partial}{\partial y} \) (vf on \( \mathbb{R}^2 \)) using property 2.

b) Compute the Lie bracket of \( X = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \),
\[ Y = z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z} \] (vf on \( \mathbb{R}^3 \))

3. The Lie bracket is skew symmetric, i.e. \([X, Y] = -[Y, X]\)

4. The Lie bracket obeys the Jacobi identity
\[
[[[X, Y], Z] + [[[Y, Z], X] + [[[Z, X], Y] = 0
\]

Exercise: Prove property 4

Several geometric interpretations of Lie bracket in terms of the flow of a v.f.

Flow line or integral curve of a v.f.: a curve where the velocity vector at each pt agrees w/ the vector field at that pt.

A flow line of a given vector field is a curve whose velocity vector at each pt agrees w/ the v.f. at that pt.

Given \( X \in \mathfrak{X}(M) \)
\[
\sigma: (t, \epsilon) \rightarrow M
\]
\[ t \rightarrow \sigma(t) \]
\[ \sigma(t) = X_{\sigma(t)} \]
Theorem: Given $X \in \mathcal{X}(M)$, for each $p \in M$ there is an integral curve $\sigma: (-\epsilon, \epsilon) \rightarrow M \in X$.

such that $\sigma(0) = p$.

Consider $X, Y \in \mathcal{X}(M)$.