\( M^n = \text{smooth manifold} \)

\( \mathbb{R}^n \)

- vectors = directed line segment
  
  \[ \overrightarrow{PQ} \]

\[ \mathbf{v} = (v_1, \ldots, v_n) \]

We need to take advantage of the differentiability structure of manifolds to talk about tangents on a manifold because we don't have a notion of vectors on a manifold.

\( p \in \mathbb{R}^n \)

\( n \in T_p M \)

\[ f : \mathbb{R}^n \rightarrow \mathbb{R} \]

\( D_{\mathbf{v}} f = \text{directional derivative of } f \text{ in direction } \mathbf{v} \text{ at } p. \)

\( \mathbf{N} \subset D_{\mathbf{v}} f \)

Explore this correspondence.

Recall def of dir deriv

\( p \in \mathbb{R}^n \)

\( \mathbf{N} \in T_p \mathbb{R}^n \)

Let \( \sigma : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^n \) be s.t. \( \sigma(0) = p \), \( \sigma'(0) = \mathbf{v} \).
\( (e.g. \sigma(t) = p + tv) \)

\[
\frac{D}{dt} f \circ \sigma(t) \bigg|_{t=0}
\]

Claim: This definition does not depend on the particular choice of \( \sigma \).

\( \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \)

\( \sigma(t) \in \mathbb{R}^n \)

\( \sigma(t) = (x^1(t), x^2(t), \ldots, x^n(t)) \)

\( f = f(x^1, x^2, \ldots, x^n) \)

\( f \circ \sigma(t) = f(\sigma(t)) = f(x^1(t), x^2(t), \ldots, x^n(t)) \)

\[
\frac{d}{dt} f \circ \sigma(t) = \frac{d}{dt} f(x^1(t), \ldots, x^n(t)) = \frac{\partial f}{\partial x^1} \frac{dx^1}{dt} + \ldots + \frac{\partial f}{\partial x^n} \frac{dx^n}{dt}
\]

\( t = 0, \quad D \sigma = \frac{d}{dt} f \circ \sigma(t) \bigg|_{t=0} = \frac{dx^1}{dt}(0) \frac{\partial f}{\partial x^1}(p) + \ldots + \frac{dx^n}{dt}(0) \frac{\partial f}{\partial x^n}(p) \)

Theorem: \( \sigma'(0) = 0 \) \( \Rightarrow \) \( \frac{dx^i}{dt}(0) = \sigma_i \), \( i = 1, 2, \ldots, n \)

\[
D \sigma f = n^1 \frac{\partial f}{\partial x^1}(p) + \ldots + n^n \frac{\partial f}{\partial x^n}(p)
\]

\[
= \left( n^1 \frac{\partial f}{\partial x^1} \bigg|_p + n^2 \frac{\partial f}{\partial x^2} \bigg|_p + \ldots + n^n \frac{\partial f}{\partial x^n} \bigg|_p \right) (f)
\]
\[ N \epsilon \rightarrow D_N \]
\[ (N', \ldots, N^n) \leftrightarrow N' \frac{\partial^2}{\partial x^1} \bigg|_p + \ldots + N^n \frac{\partial^2}{\partial x^n} \bigg|_p \]

\[ e_1 = (1, 0, \ldots, 0) \]
\[ e_2 = (0, 1, \ldots, 0) \]
\[ e_n = (0, \ldots, 1) \]

\[ N^1 e_1 + N^2 e_2 + \ldots + N^n e_n \leftrightarrow N^1 \frac{\partial}{\partial x} \bigg|_p + N^2 \frac{\partial}{\partial x^2} \bigg|_p + \ldots + N^n \frac{\partial}{\partial x^n} \bigg|_p \]

\[ \nabla^+ \nabla \nabla^+ + c \nabla^+ \nabla \nabla^+ \leftrightarrow \alpha \frac{\partial^2}{\partial x} + b \frac{\partial}{\partial y} + c \frac{\partial^2}{\partial z} \]

"Modern" differential geometry

Tangent vectors to curves in \( M^n \).

\( M^n \) = smooth manifolds
\( \sigma = \) smooth curve in \( M \)

\( \sigma : (a, b) \rightarrow M \), \( (a, b) \in \mathbb{R} \)

\( t \rightarrow \sigma(t) \)

\( z : U \subset \mathbb{R}^n \rightarrow M \)

chart in \( M \)

\( \sigma \) smooth means \( 3^{-1} \circ \sigma \circ \sigma = \hat{\sigma} \)

\( \hat{\sigma} = 3^{-1} \circ \sigma \)
\[ \hat{\sigma} : (a,b) \to \mathbb{R}^n \]
\[ \hat{\sigma}(t) = (x^1(t), \ldots, x^n(t)) \]  
(this is analogous to \[ \tau(t) = \pi(u^1(t), u^2(t)) \] from last semester)
\[ \sigma = \pi \circ \hat{\sigma} \]
\[ \sigma(t) = \pi(\hat{\sigma}(t)) \]
\[ \sigma(t) = \pi(x^1(t), \ldots, x^n(t)) \]  
coordinate representation for \( \sigma \)

\( M = \text{mani} \)

\( C^\infty(M) = \text{set of smooth real-valued functions on } M \)
\[ f : M \to \mathbb{R} \text{ smooth} \]

\( C^\infty(p) = \text{real-valued functions defined \& smooth in a nbhd of } p. \)

\[ \begin{array}{c}
\text{Def: Let } \sigma : (a,b) \to M \text{ be a smooth curve in } M. \text{ The tangent vector to } \sigma \text{ at } t=t_0 \text{ (} t_0 \in (a,b) \text{)} \text{, denoted } \sigma'(t_0) (\text{or } \frac{d}{dt} \sigma(t_0)) \text{ is the map:} \\
\sigma'(t_0) : C^\infty(p) \to \mathbb{R}, \text{ (} p = \sigma(t_0) \text{)} \\
\sigma'(t_0)(f) = \frac{d}{dt} f \circ \sigma(t) \bigg|_{t=t_0} \\
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
\sigma'(t_0) \\
\sigma(t_0) = p \\
\end{array}
\end{array} \]

\( N(f) = D_{\sigma(t)} f \)
Tangent vectors to coordinate curves

\[ \mathbb{R}^n \xrightarrow{z} M \]

chart about \( p \)

\[ \hat{p} = z^{-1}(p) = (p', \ldots, p^n) \]

\[ z(p', \ldots, p^n) = p \]

\( i^{th} \) coordinate curve \( \sigma^i \) thru \( p \), \( i = 1, \ldots, n \)

\[ x^i \rightarrow z(p', \ldots, x^i, \ldots, p^n) \]

let \( x^i \) vary; that is a curve

\[ (\sigma^i)'(p^i) \text{ is tangent vector to } \sigma^i \text{ at } x^i = p^i \]

what is this vector?

\[ (\sigma^i)'(p^i)(f) = \frac{d}{dx^i} f \circ \sigma^i \bigg|_{x^i = p^i} \]

\[ \sigma^i(x^i) = z(p^i, \ldots, x^i, \ldots, p^n) \]

\[ f \circ \sigma^i(x^i) = f(\sigma^i(x^i)) = f(z(p^i, \ldots, x^i, \ldots, p^n)) = \hat{z}(p', \ldots, x^i, \ldots, p^n) \]

\[ (\sigma^i)'(p) = \frac{d}{dx^i} f \circ \sigma^i \bigg|_{x^i = p^i} = \frac{2f}{\partial x^i}(p) = \frac{\partial f}{\partial x^i}(p) \]

\[ \frac{2f}{\partial x^i}(p) = \frac{\partial f}{\partial x^i}(p) = \frac{\partial f}{\partial x^i}(p)(f) \]
Notation:
1. \( \frac{\partial f}{\partial x^i}(p) = \frac{\partial f}{\partial x^i}(\hat{p}) \quad \hat{p} = \gamma^{-1}(p) = (p', ..., p^n) \)

2. \( \frac{\partial}{\partial x^i} \big|_p : C^\infty(p) \to \mathbb{R} \)

\[
\frac{\partial}{\partial x^i} \big|_p (f) = \frac{\partial f}{\partial x^i} \big|_p
\]

Coord vectors at \( p \):
\[ \left\{ \frac{\partial}{\partial x^1} \big|_p, \frac{\partial}{\partial x^2} \big|_p, ..., \frac{\partial}{\partial x^n} \big|_p \right\} \]

(analogous to \( \frac{\partial x^i}{\partial u^1}, \frac{\partial x^i}{\partial u^2} \))