Section 7.1: Angles and Their Measures

- **Def:** A ray is a portion of a line that starts at a point \( V \) on the line and extends indefinitely in one direction. The vertex of a ray is the starting point, \( V \), of the ray.

- **Def:** If two rays are drawn with a common vertex then the angle between the two rays is a measure of how much you have to rotate one ray to get to the other ray. When there is an angle between two rays, the ray you start at is called the initial side and the ray you end at, after rotating the initial side, is the terminal side.

- **Notation:** Angles are usually symbolized by Greek letters, such as \( \alpha, \beta, \gamma, \theta \).

- **Note:** The angle between two rays is not uniquely measured. You can rotate the initial side clockwise or counterclockwise to get to the terminal side. Also, you can rotate the initial side directly to the terminal side or you can rotate the initial side through one full rotation before rotating it to the terminal side or you can rotate the initial side through two full rotations before rotating it to the terminal side, etc.

- **Def:** An angle \( \theta \) is said to be in standard position if the vertex is at the origin and its initial side is along the positive side of the \( x \)-axis.

- If the terminal side of an angle \( \theta \) lies in one of the four quadrants, then we say that \( \theta \) lies in the quadrant in which the terminal side lies. If the terminal side of \( \theta \) lies on the \( x \)- or \( y \)-axis, then we say that \( \theta \) is a quadrantal angle. For example, in the first figure below, \( \theta \) is in quadrant III and in the second figure, \( \theta \) is a quadrantal angle.
• One of the two most common units used to measure angles is degrees. If you were to rotate the initial side of an angle through one full revolution (so the terminal side lies on top of the initial side), the angle would be $360^\circ$. Thus, each degree is $\frac{1}{360}$ of a revolution; i.e., $1^\circ = \frac{1}{360}$ revolutions.

• **Def**: A right angle is an angle measuring $90^\circ$. A straight angle is an angle measuring $180^\circ$.

• ex. Draw each angle:
  (a) $120^\circ$

(b) $270^\circ$

(c) $-45^\circ$

(d) $-405^\circ$

• **Def**: A central angle is a positive angle whose vertex is the center of a circle. The rays of a central angle intersect an arc on the circle.
• The other of the two most common units used to measure angles is radians. One radian is defined to be the measure of the central angle whose rays intersect the circle of radius \( r \) to form an arc of length \( r \).

\begin{center}
\begin{tikzpicture}
\draw (0,0) circle (1cm);
\draw[->] (0,0) -- (1,0) node [above] {\( r \)};
\draw[->] (0,0) -- (0,1) node [right] {\( 1 \) rad};
\draw[->] (0,0) -- (0.5,1) node [below] {\( r \)};
\end{tikzpicture}
\end{center}

• **Theorem:** The length, \( s \), of the arc on the circle of radius \( r \) which is intersected by a central angle of \( \theta \) radians is

\[ s = r \theta. \]

• ex. Find the missing angle:

(a) \( r = 10 \text{ m}, \theta = 3 \text{ rad} \)

(b) \( s = 4 \text{ ft}, \theta = \frac{3}{5} \text{ rad} \)

• Relationship between degrees and radians: For a full revolution around a circle of radius \( r \), \( s = 2\pi r \) (if we go one full revolution then the initial and terminal sides lie on top of each other, so the arc between them is the circle itself and the length of the circle is the circumference of the circle). So, \( 2\pi r = r \theta \) so \( \theta = 2\pi \). So, one revolution is \( 2\pi \) radians. But in terms of degrees, one revolution is \( 360^\circ \), so \( 2\pi \text{ rad} = 360^\circ \), so \( \pi \text{ rad} = 180^\circ \). Thus, we have the following relations:

(i) \( 1^\circ = \frac{\pi}{180} \text{ rad} \)
(ii) \( 1 \text{ rad} = \frac{180^\circ}{\pi} \)

- ex. Convert each angle in degrees to radians. Express your answer as a multiple of \( \pi \).

(a) \( 45^\circ \)

(b) \( 135^\circ \)

(c) \( -210^\circ \)

(d) \( -450^\circ \)

- Convert each angle in radians to degrees.

(a) \( \frac{\pi}{4} \)

(b) \( -\frac{7\pi}{4} \)

(c) \( -3\pi \)

- ex. Draw the angle \( \frac{9\pi}{4} \).

- ex. Find the missing quantity: \( r = 8 \text{ in}, \theta = 45^\circ \)
Section 7.2: Right Triangle Trigonometry

- **Def**: The trigonometric functions of acute angles are the six ratios which can be obtained from a right triangle. The six trigonometric functions are defined as follows:

1. sine of $\theta = \sin \theta = \frac{b}{c} = \frac{\text{opposite}}{\text{hypotenuse}}$
2. cosine of $\theta = \cos \theta = \frac{a}{c} = \frac{\text{adjacent}}{\text{hypotenuse}}$
3. tangent of $\theta = \tan \theta = \frac{b}{a} = \frac{\text{opposite}}{\text{adjacent}}$
4. cosecant of $\theta = \csc \theta = \frac{c}{b} = \frac{\text{hypotenuse}}{\text{opposite}}$
5. secant of $\theta = \sec \theta = \frac{c}{a} = \frac{\text{hypotenuse}}{\text{adjacent}}$
6. cotangent of $\theta = \cot \theta = \frac{a}{b} = \frac{\text{adjacent}}{\text{opposite}}$

- **ex.** Find the value of the six trigonometric functions of the angle $\theta$ in the figure.

- Among the six trigonometric functions, there are some relationships between some of them.
  - Reciprocal Identities:
    \[
    \begin{align*}
    \csc \theta &= \frac{1}{\sin \theta} & \sec \theta &= \frac{1}{\cos \theta} & \cot \theta &= \frac{1}{\tan \theta} \\
    \sin \theta &= \frac{1}{\csc \theta} & \cos \theta &= \frac{1}{\sec \theta} & \tan \theta &= \frac{1}{\cot \theta}
    \end{align*}
    \]
Quotient Identities:

\[ \tan \theta = \frac{\sin \theta}{\cos \theta} \quad \cot \theta = \frac{\cos \theta}{\sin \theta} \]

- ex. Use the definition or identities to find the exact value of each of the remaining five trigonometric functions of the acute angle \( \theta \).

(a) \( \cos \theta = \frac{\sqrt{2}}{4} \)

(b) \( \cot \theta = 3 \)

Pythagorean Identities:

\[ \sin^2 \theta + \cos^2 \theta = 1 \quad \tan^2 \theta + 1 = \sec^2 \theta \quad \cot^2 \theta + 1 = \csc^2 \theta \]

- Note: The second and third of the Pythagorean identities are obtained from the first identity by dividing each term by either \( \cos^2 \theta \) or by \( \sin^2 \theta \), respectively, and using the reciprocal or quotient identities to simplify.

- Collectively, the reciprocal identities, the quotient identities, and the Pythagorean identities are called the Fundamental identities.

Def: Two acute angles of a right triangle are called complementary if their sum is 90°. In the diagram below, the angles \( \alpha \) and \( \beta \) are complementary angles.
• **Note:** For the complementary angles \( \alpha \) and \( \beta \), the following relationships between the trigonometric functions exist:

\[
\sin \alpha = \frac{b}{c} = \cos \beta \quad \cos \alpha = \frac{a}{c} = \sin \beta \quad \tan \alpha = \frac{b}{a} = \cot \beta
\]

\[
csc \alpha = \frac{c}{b} = \sec \beta \quad \sec \alpha = \frac{c}{a} = \csc \beta \quad \cot \alpha = \frac{a}{b} = \tan \beta
\]

• **Def:** Trigonometric functions which are related by having the same value at complementary angles are called cofunctions. Thus, sine and cosine are cofunctions, cosecant and secant are cofunctions, and tangent and cotangent are cofunctions.

• **Complementary Angle Theorem:** Cofunctions of complementary angles are equal.

• The Complementary Angle Theorem just says in words what the relationships between the trigonometric functions of complementary angles above say in equations.

• Another way of stating the Complementary Angle Theorem is given by the following relationships (each relation is stated for \( \theta \) given in degrees or in radians):

\[
\sin \theta = \cos (90^\circ - \theta) \quad \sin \theta = \cos \left( \frac{\pi}{2} - \theta \right)
\]

\[
\cos \theta = \sin (90^\circ - \theta) \quad \cos \theta = \sin \left( \frac{\pi}{2} - \theta \right)
\]

\[
tan \theta = \cot (90^\circ - \theta) \quad tan \theta = \cot \left( \frac{\pi}{2} - \theta \right)
\]

\[
csc \theta = \sec (90^\circ - \theta) \quad csc \theta = \sec \left( \frac{\pi}{2} - \theta \right)
\]

\[
sec \theta = \csc (90^\circ - \theta) \quad sec \theta = \csc \left( \frac{\pi}{2} - \theta \right)
\]

\[
cot \theta = \tan (90^\circ - \theta) \quad cot \theta = \tan \left( \frac{\pi}{2} - \theta \right)
\]

• Use Fundamental Identities and/or the Complementary Angle Theorem to find the exact value of each expression.

(a) \( csc^2 40^\circ - \cot^2 40^\circ \)
(b) \[ \frac{\sin 38^\circ}{\cos 52^\circ} \]

(c) \[ \sin 40^\circ \cdot \csc 50^\circ \cdot \cot 40^\circ \]

- ex. Given \( \cos 60^\circ = \frac{\sqrt{3}}{2} \), use trigonometric identities to find the exact value of
  (a) \( \sin 30^\circ \)

(b) \( \sin^2 60^\circ \)

(c) \( \csc \frac{\pi}{6} \)

(d) \( \sec \frac{\pi}{3} \)
Section 7.3: Computing the Values of Trigonometric Functions of Acute Angles

- The value of the six trigonometric functions for certain acute angles from geometry are easy to figure out. For $\sin \theta$ and $\cos \theta$ we have the following values:

<table>
<thead>
<tr>
<th>$\theta$ (Degrees)</th>
<th>$\theta$ (Radians)</th>
<th>$\sin \theta$</th>
<th>$\cos \theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>30°</td>
<td>$\frac{\pi}{6}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
</tr>
<tr>
<td>45°</td>
<td>$\frac{\pi}{4}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
</tr>
<tr>
<td>60°</td>
<td>$\frac{\pi}{3}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{1}{2}$</td>
</tr>
</tbody>
</table>

The values of the remaining trigonometric functions at these angles can be determined by using the Fundamental identities.

<table>
<thead>
<tr>
<th>$\theta$ (Degrees)</th>
<th>$\theta$ (Radians)</th>
<th>$\tan \theta$</th>
<th>$\csc \theta$</th>
<th>$\sec \theta$</th>
<th>$\cot \theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>30°</td>
<td>$\frac{\pi}{6}$</td>
<td>$\frac{\sqrt{3}}{3}$</td>
<td>2</td>
<td>$\frac{2\sqrt{3}}{3}$</td>
<td>$\sqrt{3}$</td>
</tr>
<tr>
<td>45°</td>
<td>$\frac{\pi}{4}$</td>
<td>1</td>
<td>$\sqrt{2}$</td>
<td>$\sqrt{2}$</td>
<td>1</td>
</tr>
<tr>
<td>60°</td>
<td>$\frac{\pi}{3}$</td>
<td>$\sqrt{3}$</td>
<td>$\frac{2\sqrt{3}}{3}$</td>
<td>2</td>
<td>$\frac{\sqrt{3}}{3}$</td>
</tr>
</tbody>
</table>

- ex. Find the exact value of each expression. Do not use a calculator.
  
  (a) $2 \sin 30^\circ + 4 \cos 45^\circ$

  (b) $3 + \cot \frac{\pi}{4}$

  (c) $\csc^2 60^\circ - \sec^2 \frac{\pi}{4}$
Section 7.4: Trigonometric Functions of General Angles

- **Def:** Let \( \theta \) be an angle in standard position and let \((x, y)\) be any point on the terminal side of \( \theta \), except \((0, 0)\). Then the six trigonometric functions can be defined for any angle \( \theta \) (not just acute angles) as follows:

\[
\begin{align*}
\sin \theta &= \frac{y}{r} \\
\cos \theta &= \frac{x}{r} \\
\tan \theta &= \frac{y}{x} \\
csc \theta &= \frac{r}{y} \\
\sec \theta &= \frac{r}{x} \\
cot \theta &= \frac{x}{y}
\end{align*}
\]

where \( r = \sqrt{x^2 + y^2} \) and none of the denominators is zero. If a denominator does equal zero, then the trigonometric function of that angle \( \theta \) is undefined.

- ex. A point on the terminal side of an angle \( \theta \) is given. Find the value of the six trigonometric functions of \( \theta \).

(a) \((6, -8)\)

(b) \(\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)\)
The value of the six trigonometric functions at the four quadrantal angles are:

<table>
<thead>
<tr>
<th>θ (Degrees)</th>
<th>θ (Radians)</th>
<th>sin θ</th>
<th>cos θ</th>
<th>tan θ</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>90°</td>
<td>π/2</td>
<td>1</td>
<td>0</td>
<td>not defined</td>
</tr>
<tr>
<td>180°</td>
<td>π</td>
<td>0</td>
<td>−1</td>
<td>0</td>
</tr>
<tr>
<td>270°</td>
<td>3π/2</td>
<td>−1</td>
<td>0</td>
<td>not defined</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>θ (Degrees)</th>
<th>θ (Radians)</th>
<th>csc θ</th>
<th>sec θ</th>
<th>cot θ</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>0</td>
<td>not defined</td>
<td>1</td>
<td>not defined</td>
</tr>
<tr>
<td>90°</td>
<td>π/2</td>
<td>1</td>
<td>not defined</td>
<td>0</td>
</tr>
<tr>
<td>180°</td>
<td>π</td>
<td>not defined</td>
<td>−1</td>
<td>not defined</td>
</tr>
<tr>
<td>270°</td>
<td>3π/2</td>
<td>−1</td>
<td>not defined</td>
<td>0</td>
</tr>
</tbody>
</table>

- **Def:** Two angles in standard position are said to be coterminal if they have the same terminal side.

- **Note:** For an angle θ measured in degrees, the angles θ + 360°n, where n is any integer, are coterminal to θ. For an angle measured in radians, the angles θ + 2πn, where n is any integer, are coterminal to θ. Thus, we have the following relationships for the values of the six trigonometric functions of various angles:
\[ \theta \text{ (Degrees)} \quad \theta \text{ (Radians)} \]

\[
\begin{align*}
\sin (\theta + 360^\circ n) &= \sin \theta & \sin (\theta + 2\pi n) &= \sin \theta \\
\cos (\theta + 360^\circ n) &= \cos \theta & \cos (\theta + 2\pi n) &= \cos \theta \\
\tan (\theta + 360^\circ n) &= \tan \theta & \tan (\theta + 2\pi n) &= \tan \theta \\
\csc (\theta + 360^\circ n) &= \csc \theta & \csc (\theta + 2\pi n) &= \csc \theta \\
\sec (\theta + 360^\circ n) &= \sec \theta & \sec (\theta + 2\pi n) &= \sec \theta \\
\cot (\theta + 360^\circ n) &= \cot \theta & \cot (\theta + 2\pi n) &= \cot \theta 
\end{align*}
\]

- ex. Use a coterminal angle to find the exact value of each expression.

(a) \( \cos 480^\circ \)

(b) \( \sin (-585^\circ) \)

(c) \( \cot 1215^\circ \)

- The sign of the six trigonometric functions depends on which quadrant the angle is in.
  - Quadrant I: All six trigonometric functions are positive.
  - Quadrant II: sin and csc are positive. The rest are negative.
  - Quadrant III: tan and cot are positive. The rest are negative.
  - Quadrant IV: cos and sec are positive. The rest are negative.

A helpful way to remember this is through the phrase: All Students Take Calculus. The first letter of each word tells you which of sin, cos, and/or tan is positive in each quadrant. In the first quadrant, All the trigonometric functions are positive. In the second quadrant, Sine (and hence its reciprocal cosecant) are positive. In the third quadrant, Tangent (and hence its reciprocal cotangent) are positive. And in the fourth quadrant, Cosine (and hence its reciprocal secant) are positive.

- ex. If \( \sin \theta > 0 \) and \( \cot \theta < 0 \), name the quadrant in which the angle \( \theta \) lies.
• **Def:** Let $\theta$ denote an angle that lies in one of the four quadrants. The acute angle formed by the terminal side of $\theta$ and the $x$-axis is called the reference angle for $\theta$.

• **ex.** Find the reference angle for each of the following angles:

  (a) $135^\circ$

  (b) $-600^\circ$

  (c) $\frac{7\pi}{4}$

  (d) $-\frac{5\pi}{6}$

• **Reference Angles Theorem:** The value of each of the six trigonometric functions is the same at any angle as it is at its reference angle, with the possible exception of a sign difference.

• **ex.** Use the reference angle to find the exact value of each expression.

  (a) $\sin 135^\circ$

  (b) $\cos -600^\circ$
(c) \( \tan \frac{7\pi}{4} \)

(d) \( \sec \left( -\frac{5\pi}{6} \right) \)

- ex. Find the exact value of each of the remaining trigonometric functions of \( \theta \).

(a) \( \sin \theta = -\frac{10}{24}, \) \( \theta \) is quadrant IV

(b) \( \cos \theta = \frac{3}{5}, \frac{3\pi}{2} < \theta < 2\pi \)

(c) \( \cot \theta = -\frac{16}{9}, \sin \theta > 0 \)
Section 7.5: Unit Circle Approach; Properties of the Trigonometric Functions

• **Def:** The unit circle is the circle, centered at the origin, of radius 1.

• Let 
  \[ P = (x, y) \]
  be a point on the unit circle. Then the radius along the positive \( x \)-axis and the radius touching \( P \) form an angle \( \theta \). The coordinates of the point \( P \) can be given in terms of trigonometric functions. In particular, \( \sin \theta = y \) and \( \cos \theta = x \), so the point \( P = (x, y) \) can be given by \( P = (\cos \theta, \sin \theta) \).

• For the point \( P \) on the unit circle, we can use \( x = \cos \theta \) and \( y = \sin \theta \) to define the remaining trigonometric functions. Doing so we get \( \tan \theta = \frac{y}{x} \), \( \csc \theta = \frac{1}{y} \), \( \sec \theta = \frac{1}{x} \), \( \cot \theta = \frac{x}{y} \).

• Since the trigonometric functions can be defined by using the unit circle, they are sometimes called circular functions.

• ex. Find the values of the six trigonometric functions if \( P = \left( \frac{1}{2}, -\frac{\sqrt{3}}{2} \right) \) is a point on the unit circle.

• If we have a circle of radius \( r \) (which is given by the equation \( x^2 + y^2 = r^2 \)) and a point \( P = (x, y) \) on the circle, then the coordinates are given by \( (r \cos \theta, r \sin \theta) \). Using these, we can define the remaining trigonometric functions on the circle of radius \( r \) by \( \tan \theta = \frac{y}{x} \), \( \csc \theta = \frac{r}{y} \), \( \sec \theta = \frac{r}{x} \), \( \cot \theta = \frac{x}{y} \).

• The domains and ranges of the trigonometric functions are as follows:
<table>
<thead>
<tr>
<th>Function</th>
<th>Domain</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(\theta) = \sin \theta$</td>
<td>$(-\infty, \infty)$</td>
<td>$[-1, 1]$</td>
</tr>
<tr>
<td>$f(\theta) = \cos \theta$</td>
<td>$(-\infty, \infty)$</td>
<td>$[-1, 1]$</td>
</tr>
<tr>
<td>$f(\theta) = \tan \theta$</td>
<td>$\mathbb{R \setminus \left{ \frac{\pi}{2} + n\pi \right}}$, where $n$ is an integer</td>
<td>$(-\infty, \infty)$</td>
</tr>
<tr>
<td>$f(\theta) = \csc \theta$</td>
<td>$\mathbb{R \setminus {\pi n}}$, where $n$ is an integer</td>
<td>$(-\infty, -1] \cup [1, \infty)$</td>
</tr>
<tr>
<td>$f(\theta) = \sec \theta$</td>
<td>$\mathbb{R \setminus \left{ \frac{\pi}{2} + n\pi \right}}$, where $n$ is an integer</td>
<td>$(-\infty, -1] \cup [1, \infty)$</td>
</tr>
<tr>
<td>$f(\theta) = \cot \theta$</td>
<td>$\mathbb{R \setminus {\pi n}}$, where $n$ is an integer</td>
<td>$(-\infty, \infty)$</td>
</tr>
</tbody>
</table>

- **Def:** Suppose $f$ is a function and $\theta$ is in the domain of $f$. We call $f$ a periodic function if there is a positive number $p$ such that $\theta + p$ is also in the domain of $f$ and $f(\theta + p) = f(\theta)$. If $f$ is periodic, there may be more than one value of $p$ with this property. The smallest $p$ is called the period of $f$.

- All six trigonometric functions are periodic functions. $\sin \theta$, $\cos \theta$, $\csc \theta$, $\sec \theta$ all have period $2\pi$ and $\tan \theta$, $\cot \theta$ have period $\pi$.

- ex. Use the fact that the trigonometric functions are periodic to find the exact value of each expression.
  
  (a) $\sin 1125^\circ$

(b) $\sec \frac{8\pi}{3}$

(c) $\cot \frac{13\pi}{4}$

- All six trigonometric functions have symmetry. $\sin \theta$, $\csc \theta$, $\tan \theta$, $\cot \theta$ are all odd functions and $\cos \theta$, $\sec \theta$ are even functions. Thus,
  
  $\sin (-\theta) = -\sin \theta$  
  $\cos (-\theta) = \cos \theta$  
  $\tan (-\theta) = -\tan \theta$
  
  $\csc (-\theta) = -\csc \theta$  
  $\sec (-\theta) = \sec \theta$  
  $\cot (-\theta) = -\cot \theta$
• ex. Use the even-odd properties to find the exact value of each expression.
  
  (a) \( \cot(-90^\circ) \)

  (b) \( \sec(-45^\circ) \)

  (c) \( \cos\left(-\frac{\pi}{3}\right) \)
Section 7.6: Graphs of the Sine and Cosine Functions

- The sine function, \( f(x) = \sin x \), has the following properties:
  1. The domain is the set of all real numbers.
  2. The range is \([-1, 1]\).
  3. It is an odd function so it is symmetric about the origin.
  4. It is periodic with period \(2\pi\).
  5. The \(x\)-intercepts occur at all multiples of \(\pi\); i.e., at \(x = \ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots\)
      The \(y\)-intercept is 0.
  6. The maximum value is 1 and it occurs at \(2\pi\) increments of \(\frac{\pi}{2}\); i.e., at
      \(x = \ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots\)
      The minimum value is \(-1\) and it occurs at \(2\pi\)
      increments of \(\frac{3\pi}{2}\); i.e., at \(x = \ldots, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots\).
  7. The graph looks like

- The cosine function, \( f(x) = \cos x \), has the following properties:
  1. The domain is the set of all real numbers.
  2. The range is \([-1, 1]\).
  3. It is an even function so it is symmetric about the \(y\)-axis.
  4. It is periodic with period \(2\pi\).
  5. The \(x\)-intercepts occur at \(\pi\) increments of \(\frac{\pi}{2}\); i.e., at
      \(x = \ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots\)
      The \(y\)-intercept is 1.
  6. The maximum value is 1 and it occurs at all multiples of \(2\pi\); i.e., at
      \(x = \ldots, -2\pi, 0, 2\pi, 4\pi, \ldots\)
      The minimum value is \(-1\) and it occurs at \(2\pi\)
      increments of \(\pi\); i.e., at \(x = \ldots, -\pi, \pi, 3\pi, \ldots\).
7. The graph looks like

- **Note**: From the graphs of \( \sin x \) and \( \cos x \), we can see that \( \sin x = \cos \left(x - \frac{\pi}{2}\right)\).
- **Def**: The amplitude of the sine or cosine graph is the maximum distance of the graph from the zeros (before any vertical shifts have been applied to the graph).
- **Recall**: The period of a function is the distance it takes for the functions to get back to where it started or to repeat itself.
- **Theorem**: If \( k > 0 \) then the functions \( y = A \sin (kx) \) and \( y = A \cos (kx) \) have amplitude \( |A| \) and period, denoted by \( T \), of \( T = \frac{2\pi}{k} \).
• **Def:** One period of the graph of $y = \sin (kx)$ or $y = \cos (kx)$ is called a cycle.

• **ex.** Determine the amplitude and period of each function then graph the function. Show at least two cycles.

(a) $y = -2 \sin (4x)$

(b) $y = \cos \left( \frac{1}{2} x \right) - 1$
(c) \( y = \frac{1}{2} \cos (-\pi x) \)

- ex. Find an equation for the graph.
Section 7.6: Example Answers

• ex. Determine the amplitude and period of each function then graph the function. Show at least two cycles.

(a) \( y = -2 \sin(4x) \)

(b) \( y = \cos\left(\frac{1}{2}x\right) - 1 \)
(c) $y = \frac{1}{2} \cos (-\pi x)$

- ex. Find an equation for the graph.

\[ y = -3 \sin \left( \frac{\pi}{3}x \right) + 1 \]
The tangent function, \( f(x) = \tan x \) has the following properties:

1. The domain is \( \mathbb{R} \setminus \left\{ \frac{\pi}{2} + n\pi \right\} \).
2. The range is \(( -\infty, \infty )\).
3. It is an odd function, so it is symmetric about the origin.
4. It is periodic with period \( \pi \).
5. The \( x \)-intercepts occur at all multiples of \( \pi \); i.e., at \( x = \ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots \). The \( y \)-intercept is 0.
6. The vertical asymptotes occur at \( \pi \) increments of \( \frac{\pi}{2} \); i.e., at \( x = \ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots \).
7. The graph of \( y = \tan x \) looks like

The cotangent function, \( f(x) = \cot x \) has the same properties as the tangent function, except that the \( x \)-intercepts occur at \( \pi \) increments of \( \frac{\pi}{2} \); i.e., at \( x = \ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots \) and the vertical asymptotes occur at all multiples of \( \pi \); i.e., at \( x = \ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots \). The graph of \( y = \cot x \) looks like
• The cosecant function, \( f(x) = \csc x \), has the following properties:

1. The domain is \( \mathbb{R} \setminus \{n\pi\} \).
2. The range is \((-\infty, -1] \cup [1, \infty)\).
3. It is an odd function, so it is symmetric about the origin.
4. It is periodic with period \( 2\pi \).
5. The is no \( x \)-intercept or \( y \)-intercept.
6. The vertical asymptotes occur at all multiples of \( \pi \); i.e., at \( x = \ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots \).
7. The graph of \( y = \csc x \) looks like
The secant function, \( f(x) = \sec x \) has the following properties:

1. The domain is \( \mathbb{R} \setminus \{ \frac{\pi}{2} + n\pi \} \).
2. The range is \( (-\infty, -1] \cup [1, \infty) \).
3. It is an even function, so it is symmetric about the \( y \)-axis.
4. It is periodic with period \( 2\pi \).
5. The is no \( x \)-intercept and the \( y \)-intercept hits at 1.
6. The vertical asymptotes occur at \( \pi \) increments of \( \frac{\pi}{2} \); i.e., at \( x = \ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots \).
7. The graph of \( y = \sec x \) looks like

\[ \begin{array}{c}
\text{Graph}
\end{array} \]

- ex. Graph each function. Be sure to show at least two cycles.
1. \( y = -2 \sec x \)

2. \( y = \tan (2x) \)
3. \( y = \frac{1}{2} \csc (\pi x) - 1 \)
Section 7.7: Example Answers

- ex. Graph each function. Be sure to show at least two cycles.

1. \( y = -2 \sec x \)

2. \( y = \tan (2x) \)
3. \( y = \frac{1}{2} \csc (\pi x) - 1 \)
Section 7.8: Phase Shifts

- In order to graph the functions \( y = A \sin (kx - \phi) \) or \( y = A \cos (kx - \phi) \), rewrite them as \( y = A \sin \left[ k \left( x - \frac{\phi}{k} \right) \right] \) or \( \cos \left[ k \left( x - \frac{\phi}{k} \right) \right] \), respectively. Then these functions have an amplitude of \( |A| \), a period of \( T = \frac{2\pi}{k} \) and a phase shift (horizontal shift) of \( \frac{\phi}{k} \). If \( \phi < 0 \) then the phase shift is to the left. If \( \phi > 0 \) then the phase shift is to the right.

- ex. Find the amplitude, period, and phase shift of each function. Then graph each function. Show at least two periods.

(a) \( y = 2 \sin (4x - \pi) \)
(b) \( y = -3 \cos \left( \frac{1}{2} x + \frac{\pi}{2} \right) \)

(c) \( y = -\tan (2\pi x + \pi) \)
(d) \( y = 2 \csc \left( \frac{\pi}{4} x - \frac{\pi}{2} \right) - 1 \)
Section 7.8: Example Answers

- ex. Find the amplitude, period, and phase shift of each function. Then graph each function. Show at least two periods.

(a) \( y = 2 \sin (4x - \pi) \)

(b) \( y = -3 \cos (\frac{1}{2}x + \frac{\pi}{2}) \)
(c) \( y = -\tan (2\pi x + \pi) \)

(d) \( y = 2 \csc \left( \frac{\pi}{4} x - \frac{\pi}{2} \right) - 1 \)