Rational Parking Functions

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CONVENTION
Given \( x \in \mathbb{Q} \setminus [-1, 0] \), there exist unique coprime \((a, b) \in \mathbb{N}^2\) such that
\[
x = \frac{a}{b - a}.
\]
We will always identify \( x \leftrightarrow (a, b) \).

Definition
For each \( x \in \mathbb{Q} \setminus [-1, 0] \) we define the Catalan number:
\[
\text{Cat}(x) = \text{Cat}(a, b) := \frac{1}{a + b} \binom{a + b}{a, b} = \frac{(a + b - 1)!}{a!b!}.
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Special cases

When $b = 1 \text{ mod } a \ldots$

- **Eugène Charles Catalan (1814-1894)**
  
  $(a, b) = (n, n + 1)$ gives the **good old Catalan number**:

  \[
  \text{Cat}(n) = \text{Cat} \left( \frac{n}{(n + 1) - n} \right) = \frac{1}{2n + 1} \binom{2n + 1}{n}.
  \]

- **Nicolaus Fuss (1755-1826)**
  
  $(a, b) = (n, kn + 1)$ gives the **Fuss-Catalan number**:

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Symmetry

Definition

By definition we have $\text{Cat}(a, b) = \text{Cat}(b, a)$, which translates to

$$\text{Cat}(x) = \text{Cat}(-x - 1)$$

(i.e. symmetry about $x = -1/2$), which implies that

$$\text{Cat} \left( \frac{1}{x - 1} \right) = \text{Cat} \left( \frac{x}{1 - x} \right).$$

We call this the derived Catalan number:

$$\text{Cat}'(x) := \text{Cat} \left( \frac{1}{x - 1} \right) = \text{Cat} \left( \frac{x}{1 - x} \right).$$
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Euclidean Algorithm

Observation

The process $\text{Cat}(x) \mapsto \text{Cat}'(x) \mapsto \text{Cat}''(x) \mapsto \cdots$ is a categorification of the Euclidean algorithm.

Example: $x = 5/3$ and $(a, b) = (5, 8)$

Subtract the smaller from the larger:

- $\text{Cat}(5, 8) = 99,$
- $\text{Cat}'(5, 8) = \text{Cat}(3, 5) = 7,$
- $\text{Cat}''(5, 8) = \text{Cat}'(3, 5) = \text{Cat}(2, 3) = 2,$
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How to put it in Sloane’s OEIS

Suggestion

The Calkin-Wilf sequence is defined by \( q_1 = 1 \) and

\[
q_n := \frac{1}{2[q_{n-1}] - q_{n-1} + 1}.
\]

Theorem: \((q_1, q_2, \ldots) = \mathbb{Q}_{>0}\).
Proof: See “Proofs from THE BOOK”, Chapter 17.

Study the function \( n \mapsto \text{Cat}(q_n) \).

<table>
<thead>
<tr>
<th>( q )</th>
<th>( \frac{1}{1} )</th>
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Well, that was fun.
Consider the “Dyck paths” in an $a \times b$ rectangle.

Example $(a, b) = (5, 8)$
Again let $x = a/(b - a)$ with $a, b$ positive and coprime.

Example $(a, b) = (5, 8)$
Let $D(x) = D(a, b)$ denote the set of Dyck paths.

Example $(a, b) = (5, 8)$
The number of Dyck paths is the Catalan number:

\[ |D(x)| = \text{Cat}(x) = \frac{1}{a+b} \binom{a+b}{a, b}. \]

Claimed by Grossman (1950), “Fun with lattice points, part 22”.


Proof: Break \( \binom{a+b}{a, b} \) lattice paths into cyclic orbits of size \( a + b \).
Each orbit contains a unique Dyck path.
Theorem (Grossman 1950, Bizley 1954)

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Theorem (Armstrong 2010, Loehr 2010)

- The number of Dyck paths with $k$ vertical runs equals
  \[ \text{Nar}(x; k) := \frac{1}{a} \binom{a}{k} \binom{b - 1}{k - 1}. \]
  Call these the Narayana numbers.

- And the number with $r_j$ vertical runs of length $j$ equals
  \[ \text{Krew}(x; r) := \frac{1}{b} \binom{b}{r_0, r_1, \ldots, r_a} = \frac{(b - 1)!}{r_0!r_1! \ldots r_a!}. \]
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Call these the **Kreweras numbers**.
Motivation: Core Partitions

Definition
Let $\lambda \vdash n$ be an integer partition of “size” $n$.

- Say $\lambda$ is a $p$-core if it has no cell with hook length $p$.
- Say $\lambda$ is an $(a, b)$-core if it has no cell with hook length $a$ or $b$.

Example
The partition $(5, 4, 2, 1, 1) \vdash 13$ is a $(5, 8)$-core.
Motivation: Core Partitions

Theorem (Anderson 2002)

The number of \((a, b)\)-cores (of any size) is finite if and only if \((a, b)\) are coprime, in which case they are counted by the Catalan number

\[
\text{Cat}(a, b) = \frac{1}{a+b} \binom{a+b}{a, b}.
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Theorem (Olsson-Stanton 2005, Vandehey 2008)

For \((a, b)\) coprime \(\exists\) unique largest \((a, b)\)-core of size \(\frac{(a^2-1)(b^2-1)}{24}\), which contains all others as subdiagrams.

Suggestion

Study Young's lattice restricted to \((a, b)\)-cores.
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Example: The poset of (3, 4)-cores.
Motivation: Core Partitions

Theorem (Ford-Mai-Sze 2009)

For $a, b$ coprime, the number of self-conjugate $(a, b)$-cores is \( \left\lfloor \frac{a}{2} \right\rfloor + \left\lfloor \frac{b}{2} \right\rfloor \).  

Note: Beautiful bijective proof! (omitted)

Observation/Problem

\[
\left( \left\lfloor \frac{a}{2} \right\rfloor + \left\lfloor \frac{b}{2} \right\rfloor \right) = \frac{1}{[a+b]_q} \left[ a+b \right]_{[a,b]_q} \bigg|_{q=-1}
\]

Conjecture (Armstrong 2011)

The average size of an $(a, b)$-core and the average size of a self-conjugate $(a, b)$-core are both equal to \( \frac{(a+b+1)(a-1)(b-1)}{24} \).
Motivation: Core Partitions

Theorem (Ford-Mai-Sze 2009)

For $a, b$ coprime, the number of self-conjugate $(a, b)$-cores is $\left(\left\lfloor \frac{a}{2} \right\rfloor + \left\lfloor \frac{b}{2} \right\rfloor, \left\lfloor \frac{a}{2} \right\rfloor, \left\lfloor \frac{b}{2} \right\rfloor \right)$. Note: Beautiful bijective proof! (omitted)

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$$\left(\left\lfloor \frac{a}{2} \right\rfloor + \left\lfloor \frac{b}{2} \right\rfloor, \left\lfloor \frac{a}{2} \right\rfloor, \left\lfloor \frac{b}{2} \right\rfloor \right) = \frac{1}{[a+b]_q \left\lbrack a, b \right\rbrack_q} \bigg|_{q=-1}$$

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**Proof.**

Bijection: \((a, b)\)-cores ↔ Dyck paths in \(a \times b\) rectangle

---

**Example (The \((5, 8)\)-core from earlier.)**

\[
\begin{array}{cccccc}
9 & 6 & 4 & 3 & 1 \\
7 & 4 & 2 & 1 \\
4 & 1 \\
2 & 1 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
40 & 35 & 30 & 25 & 20 & 15 & 10 & 5 & 0 \\
32 & & & & & & & & \\
24 & & & & & & & & \\
16 & & & & & & & & \\
8 & & & & & & & & \\
0 & & & & & & & & \\
\end{array}
\]
Proof.

Bijection: \((a, b)\)-cores \(\leftrightarrow\) Dyck paths in \(a \times b\) rectangle

Example (Label the rectangle cells by “height”.)
Anderson’s Beautiful Proof

Proof.

Bijection: \((a, b)\)-cores ↔ Dyck paths in \(a \times b\) rectangle

Example (Label the first column hook lengths.)

\[
\begin{array}{cccc}
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4 & 1 & & \\
2 & & & \\
1 & & & \\
\end{array}
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Example (Voila!)
Proof.

Bijection: \((a, b)\)-cores \(\leftrightarrow\) Dyck paths in \(a \times b\) rectangle

Example (Observe: Conjugation is a bit strange.)
Rational Parking Functions

Definition

- Label the up-steps by \( \{1, 2, \ldots, a\} \), increasing up columns.

- Call this a parking function.

- Let \( PF(x) = PF(a, b) \) denote the set of parking functions.

- Classical form \((z_1, z_2, \ldots, z_a)\) has label \( z_i \) in column \( i \).

- Example: \((3, 1, 4, 4, 1)\)
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- Let \( \text{PF}(x) = \text{PF}(a, b) \) denote the set of parking functions.
- **Classical form** \((z_1, z_2, \ldots, z_a)\) has label \(z_i\) in column \(i\).
- Example: \((3, 1, 4, 4, 1)\)
Rational Parking Functions

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Theorems (with N. Loehr and N. Williams)

- The dimension of $PF(a, b)$ is $b^{a-1}$.
- The complete homogeneous expansion is
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  PF(a, b) = \sum_{r \vdash a} \frac{1}{b} \left( \binom{b}{r_0, r_1, \ldots, r_a} \right) h_r,
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  where the sum is over $r = 0^{r_0} 1^{r_1} \cdots a^{r_a} \vdash a$ with $\sum_i r_i = b$.
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Observation/Definition

The multiplicities of the **hook Schur functions** $s[k + 1, 1^{a-k-1}]$ in $\text{PF}(a, b)$ are given by the **Schröder numbers**

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**Special Cases:**

- **Trivial character:** $\text{Schrö}(a, b; a-1) = \text{Cat}(a, b)$.
- **Smallest $k$ that occurs is** $k = \max\{0, a-b\}$, in which case

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Problem

Given $a, b$ coprime we have an $\mathfrak{S}_a$-module $\text{PF}(a, b)$ of dimension $b^{a-1}$ and an $\mathfrak{S}_b$-module $\text{PF}(b, a)$ of dimension $a^{b-1}$.

▶ What is the relationship between $\text{PF}(a, b)$ and $\text{PF}(b, a)$?

▶ Note that hook multiplicities are the same:

$$\text{Schrö}(a, b; k) = \text{Schrö}(b, a; k + b - a).$$

▶ See Eugene Gorsky, Arc spaces and DAHA representations, 2011.
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How about $q$ and $t$?

We want a “Shuffle Conjecture”

Define a quasisymmetric function with coefficients in $\mathbb{N}[q, t]$ by

$$PF_{q,t}(a, b) := \sum_P q^{q\text{stat}(P)} t^{t\text{stat}(P)} F_{\text{iDes}(P)}.$$ 

- Sum over $(a, b)$-parking functions $P$.

- $F$ is a fundamental (Gessel) quasisymmetric function. — natural refinement of Schur functions

- We require $PF_{1,1}(a, b) = PF(a, b)$.

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- Must define \( q\text{stat}, t\text{stat}, \text{iDes} \) for \((a, b)\)-parking function \( P \).
qstat is easy

Definition

- Let $qstat := area := \# \text{ boxes between the path and diagonal}$.
- Note: Maximum value of area is $(a - 1)(b - 1)/2$. (Frobenius)
  — see Beck and Robins, Chapter 1

Example

- This $(5, 8)$-parking function has area $= 6$. 
iDes is reasonable

**Definition**

- Read labels by increasing “height” to get permutation $\sigma \in \mathfrak{S}_a$.
- $\text{iDes} :=$ the descent set of $\sigma^{-1}$.

**Example**

- Remember the “height”?

```
 40 35 30 25 20 15 10 5 0
32 27 22 17 12 7 2 -3 -8
24 19 14 9 4 -1 -6 -11 -16
16 11 6 1 -4 -9 -14 -19 -24
 8 3 -2 -7 -12 -17 -22 -27 -32
 0 -5 -10 -15 -20 -25 -30 -35 -40
```

- $\text{iDes} = \{1, 4\}$
iDes is reasonable

**Definition**

- Read labels by increasing "height" to get permutation $\sigma \in S_a$.
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**Example**

- Look at the heights of the vertical step boxes.

$\text{iDes} = \{1, 4\}$
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- Read labels by increasing “height” to get permutation $\sigma \in \mathcal{S}_a$.
- $i\text{Des} :=$ the descent set of $\sigma^{-1}$.

**Example**

Remember the labels we had before.

$$i\text{Des} = \{1, 4\}$$
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**Example**

- Read them by increasing height to get $\sigma = 2\bar{1}53\bar{4} \in \mathfrak{S}_5$.

$$
\begin{array}{cccccc}
40 & 35 & 30 & 25 & 20 & 15 & 10 & 5 & 0 \\
32 & 3 & 4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
24 & & & & & & & & \\
16 & & & & & & & & \\
8 & & & & & & & & \\
0 & & & & & & & & \\
\end{array}
$$

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tstat is hard (as usual)

Definition

▶ “Blow up” the \((a, b)\)-parking function.
▶ Compute “dinv” of the blowup.

Example

▶ Recall our favorite the \((5, 8)\)-parking function.
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Definition

- “Blow up” the \((a, b)\)-parking function.
- Compute “dinv” of the blowup.

Example

- Since \(2 \cdot 8 - 3 \cdot 5 = 1\) we “blow up” by 2 horiz. and 3 vert....

[Diagram of a grid with numbers 2, 1, 5, 3, 4]
tstat is hard (as usual)

Example

▶ To get this!
Example

To get this! Now compute $\text{dinv} = 7$. 
tstat is hard (as usual)

Example

▶ (There’s a scaling factor depending on the path, so $tstat = 3$.)
So our favorite \((5, 8)\)-parking function contributes \(q^6 t^3 F_{\{1,4\}}\).

Proof of Concept: The coefficient of \(s[2, 2, 1]\) in \(\text{PF}_{q,t}(5, 8)\) is:

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 3 & 4 & 3 & 2 & 1 \\
2 & 6 & 6 & 4 & 2 & 1 \\
2 & 7 & 7 & 4 & 2 & 1 \\
1 & 6 & 7 & 4 & 2 & 1 \\
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Facts

- \( \text{PF}_{1,1}(a, b) = \text{PF}(a, b) \).

- \( \text{PF}_{q,t}(a, b) \) is symmetric and Schur-positive with coeffs \( \in \mathbb{N}[q, t] \).
  — via LLT polynomials (HHLRU Lemma 6.4.1)

- Experimentally: \( \text{PF}_{q,t}(a, b) = \text{PF}_{t,q}(a, b) \).
  — this will be “impossible” to prove (see Loehr’s Maxim)

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\text{Schrö}_{q,\frac{1}{q}}(a, b; k) = \frac{1}{[b]_q} \left[ \begin{array}{c} a - 1 \\ k \end{array} \right]_q \left[ \begin{array}{c} b + k \\ a \end{array} \right]_q
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(centered)
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$$\text{Schrö}_{q,\frac{1}{q}}(a, b; k) = \frac{1}{[b]_q} \left[ b + k \right]_q \left[ a - 1 \right]_q \left[ a \right]_q$$

(centered)
A Few Facts

Facts

- \( \text{PF}_{1,1}(a, b) = \text{PF}(a, b). \)

- \( \text{PF}_{q,t}(a, b) \) is symmetric and Schur-positive with coeffs \( \in \mathbb{N}[q, t]. \)
  — via LLT polynomials (HHLRU Lemma 6.4.1)

- **Experimentally:** \( \text{PF}_{q,t}(a, b) = \text{PF}_{t,q}(a, b). \)
  — this will be “impossible” to prove (see Loehr’s Maxim)

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- **Experimentally:** Specialization \( t = 1/q \) gives

\[
\text{Schrö}_{q,\frac{1}{q}}(a, b; k) = \frac{1}{[b]_q} \left[ \begin{array}{c} a - 1 \\ k \end{array} \right]_q \left[ \begin{array}{c} b + k \\ a \end{array} \right]_q
\]

(centered)
Motivation: Lie Theory

The James-Kerber Bijection

- between $a$-cores and the root lattice of the Weyl group $\mathcal{G}_a$

\[
\begin{array}{ccccc}
9 & 6 & 4 & 3 & 1 \\
7 & 4 & 2 & 1 \\
4 & 1 \\
2 \\
1 \\
\end{array}
\quad \leftrightarrow \quad 
\begin{array}{cccccc}
5 & 6 & 7 & 8 & 9 \\
0 & 1 & 2 & 3 & 4 \\
-5 & -4 & -3 & -2 & -1 \\
\end{array}
\]

$(0, 1, -1, 1, -1)$

\[
\begin{array}{cccccc}
5 & 6 & 7 & 8 & 9 \\
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\end{array}
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9 & 6 & 4 & 3 & 1 \\
7 & 4 & 2 & 1 \\
4 & 1 \\
2 \\
1 \\
\end{array}
\]
These are the root and weight lattices $Q \subseteq \Lambda$ of $\mathcal{G}_a$. 
Here is a fundamental parallelepiped for $\Lambda/b\Lambda$. 

Here’s The Picture
It contains $b^{a-1}$ elements (these are the “parking functions”).
But they look better as a simplex...
...which is congruent to a nicer simplex.
There are $\text{Cat}(a, b) = \frac{1}{a+b} \binom{a+b}{a,b}$ elements of the root lattice inside.
These are the \((a, b)\)-Dyck paths (via Anderson, James-Kerber).
Other Weyl Groups?

Definition

Consider a Weyl group $W$ with Coxeter number $h$ and let $p \in \mathbb{N}$ be coprime to $h$. We define the **Catalan number**

$$
\text{Cat}_q(W, p) := \prod_j \frac{[p + m_j]_q}{[1 + m_j]_q}
$$

where $e^{2\pi i m_j / h}$ are the eigenvalues of a Coxeter element.

Observation

$$
\text{Cat}_q(S_a, b) = \frac{1}{[a + b]_q} \left[ a + b \right]_q
$$
Thank You

NIGHT STALKER
Mamiribius perhorridus

The night stalker's powerful front legs are developed from the wings of its ancestors. Its webbed feet, which were originally used for grasping and clutching, now serve as its shoulders and effectively form hands.

(Dixon 1981)