Rational Catalan Combinatorics

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This talk will advertise a definition.

Here is it.

**Definition**

Let $x$ be a positive rational number written as $x = a/(b - a)$ for $0 < a < b$ coprime. Then we define the **Catalan number**

$$\text{Cat}(x) := \frac{1}{a + b} \binom{a + b}{a, b} = \frac{(a + b - 1)!}{a!b!}.$$ 

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*Please note the $a, b$-symmetry.*
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Special cases.

When $b = 1 \mod a$...

- **Eugène Charles Catalan (1814-1894)**
  
  $(a < b) = (n < n + 1)$ gives the **good old Catalan number**
  
  $$\text{Cat}(n) = \text{Cat}\left(\frac{n}{1}\right) = \frac{1}{2n + 1} \binom{2n + 1}{n}.$$ 

- **Nicolaus Fuss (1755-1826)**
  
  $(a < b) = (n < kn + 1)$ gives the **Fuss-Catalan number**
  
  $$\text{Cat}\left(\frac{n}{(kn + 1) - n}\right) = \frac{1}{(k + 1)n + 1} \binom{(k + 1)n + 1}{n}.$$
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Euclidean Algorithm & Symmetry.

**Definition**

Again let $x = a/(b - a)$ for $0 < a < b$ coprime. Then we define the derived Catalan number

$$\text{Cat}'(x) := \frac{1}{b} \binom{b}{a} = \begin{cases} \text{Cat}(1/(x - 1)) & \text{if } x > 1 \\ \text{Cat}(x/(1 - x)) & \text{if } x < 1 \end{cases}$$

This is a “categorification” of the Euclidean algorithm.

**Remark**

If we define $\text{Cat} : \mathbb{Q} \setminus [-1, 0] \rightarrow \mathbb{N}$ by $\text{Cat}(-x - 1) := \text{Cat}(x)$ then the formula is simpler:

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Problem

Describe a recurrence for the Cat function, perhaps in terms of the Calkin-Wilf sequence

\[
\begin{align*}
\frac{1}{1} &\leftrightarrow \frac{1}{2} \leftrightarrow \frac{2}{1} \leftrightarrow \frac{1}{3} \leftrightarrow \frac{3}{2} \leftrightarrow \frac{2}{3} \leftrightarrow \frac{3}{1} \leftrightarrow \frac{1}{4} \leftrightarrow \frac{4}{3} \leftrightarrow \cdots
\end{align*}
\]

which is defined by

\[
x \mapsto \frac{1}{\lfloor x \rfloor + 1 - \{x\}}.
\]

See Aigner and Ziegler: “Proofs from THE BOOK”, Chapter 17.
What?

Well, that was fun. But *perhaps untethered to reality*...
Motivation 1: Cores

Definition

- An integer partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots) \vdash n$ is called $p$-core if it has no cell with hook length $p$.
- Say $\lambda \vdash n$ is $(a, b)$-core if it has no cell with hook length $a$ or $b$.

Example

The partition $(5, 4, 2, 1, 1) \vdash 13$ is $(5, 8)$-core.
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\[
\begin{array}{ccccc}
9 & 6 & 4 & 3 & 1 \\
7 & 4 & 2 & 1 \\
4 & 1 \\
2 \\
1 \\
\end{array}
\]
A few facts.

**Theorem (Anderson, 2002)**

The number of \((a, b)\)-cores is finite if and only if \(a, b\) are coprime, in which case the number is

\[
\text{Cat} \left( \frac{a}{b-a} \right) = \frac{1}{a+b} \binom{a+b}{a, b}.
\]

**Theorem (Olsson-Stanton, 2005, Vandehey, 2008)**

For \(a, b\) coprime \(\exists\) unique largest \((a, b)\)-core of size \(\frac{(a^2-1)(b^2-1)}{24}\), which contains all others as subdiagrams.

**Problem**

Study Young’s lattice restricted to \((a, b)\)-cores.
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A few facts.

**Theorem (Ford-Mai-Sze, 2009)**

For $a, b$ coprime, the number of self-conjugate $(a, b)$-cores is 

$$\left(\left\lfloor \frac{a^2}{2} \right\rfloor + \left\lfloor \frac{b^2}{2} \right\rfloor \right).$$

**Note:** Beautiful bijective proof! (omitted)

**Observation/Problem**

$$\left(\left\lfloor \frac{a}{2} \right\rfloor + \left\lfloor \frac{b}{2} \right\rfloor \right) = \frac{1}{[a + b]_q} \left\lfloor \begin{array}{c} a + b \\ a, b \end{array} \right\rfloor_q \bigg|_{q = -1}$$

**Conjecture (Armstrong, 2011)**

The average size of an $(a, b)$-core and the average size of a self-conjugate $(a, b)$-core are both equal to $$\frac{(a+b+1)(a-1)(b-1)}{24}.$$
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Anderson’s beautiful proof of \( \frac{1}{a+b} \binom{a+b}{a,b} \).

Step 1

- Bijection: \((a, b)\)-cores \(\leftrightarrow\) Dyck paths in \(a \times b\) rectangle

Example (The \((5, 8)\)-core from earlier.)
Anderson’s beautiful proof of \( \frac{1}{a+b} \binom{a+b}{a,b} \).

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9 & 6 & 4 & 3 \\
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4 & 1 & & \\
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\[
\begin{array}{cccccccc}
40 & 35 & 30 & 25 & 20 & 15 & 10 & 5 & 0 \\
32 & & & & & & & & \\
24 & & & & & & & & \\
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Example (NB: Conjugation is weird, but...)

```
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```
\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
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```
Step 2

- **Theorem (Bizley, 1954):** \# Dyck paths is \( \frac{1}{a+b} \binom{a+b}{a,b} \).

Proof idea.

- The \( \binom{a+b}{a,b} \) lattice paths break into cyclic orbits of size \( a + b \).
- Each orbit contains a unique Dyck path.
- Coprimality of \( a, b \) is necessary.
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Motivation 2: Parking Functions

(with Haglund, Haiman, Loehr, Warrington et al.)

Definition

Again let \( x = a/(b - a) \) with \( 0 < a < b \) coprime.

- An \( x \)-parking function is a “decorated” Dyck path in the \( a \times b \) rectangle. (Decorate the vertical runs with the labels \( \{1, 2, \ldots, a\} \).

- Classical form: \((z_1, z_2, \ldots, z_a)\) where label \( i \) occurs in column \( z_i \).

- Symmetric group \( S_a \) acts on classical forms by permutation. Let \( PF(x) \) denote the corresponding \( S_a \)-module.
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Examples for \( x = \frac{5}{(8 - 5)}. \) (\( \text{Cat}(x) = 99. \))

- Here’s the 5/3-parking function with classical form \((3, 1, 4, 4, 1)\).

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A few facts.

**Theorems**

- # x-parking functions is $b^{a-1}$.

- # x-Dyck paths with $r_i$ vertical runs of length $i$ is \( \frac{1}{b} \binom{b}{r_0, r_1, \ldots, r_a} \):\[
PF(x) = \sum_{r \vdash a} \frac{1}{b} \binom{b}{r_0, r_1, \ldots, r_a} h_r,
\]
  where the sum is over $r = 0^{r_0}1^{r_1} \cdots a^{r_a} \vdash a$ with $\sum_i r_i = b$.

- # x-parking functions fixed by $\sigma \in S_a$ is $b^\#\text{cycles}(\sigma)-1$: \[
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**Theorems**

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Define a quasisymmetric function with coefficients in $\mathbb{N}[q, t]$ by

$$\text{PF}_{q,t}(x) := \sum_{P} q^{\text{qstat}(P)} t^{\text{tstat}(P)} F_{\text{iDes}(P)}.$$

- Sum over $x$-parking functions $P$.

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Idea
qstat is easy.

**Definition**

- Let qstat := area := \# boxes between the path and diagonal.
- Note: Maximum value of area is \((a - 1)(b - 1)/2\). (Frobenius) — see Beck and Robins, Chapter 1

**Example**

- This 5/3-parking function has area = 6.
qstat is easy.

Definition

- Let \( qstat := area := \# \) boxes between the path and diagonal.
- Note: Maximum value of area is \( (a - 1)(b - 1)/2 \). (Frobenius)
  — see Beck and Robins, Chapter 1

Example

- This 5/3-parking function has area = 6.
iDes is reasonable.

**Definition**

- Read labels by increasing “height” to get permutation \( \sigma \in S_a \).
- \( i\text{Des} := \) the descent set of \( \sigma^{-1} \).

**Example**

- This is a secret message.

  \[ i\text{Des} = \{1,4\} \]
iDes is reasonable.

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- $\text{iDes} = \{1, 4\}$. 
iDes is reasonable.

**Definition**
- Read labels by increasing “height” to get permutation $\sigma \in \mathfrak{S}_a$.
- $\text{iDes} :=$ the descent set of $\sigma^{-1}$.

**Example**
- Remember the “height”?

![Graph showing the descent set of a permutation]

$\text{iDes} = \{1, 4\}$. 
iDes is reasonable.

**Definition**

- Read labels by increasing “height” to get permutation $\sigma \in \mathfrak{S}_a$.
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**Example**

- Look at the heights of the vertical step boxes.

```
40 35 30 25 20 15 10 5 0
32       12
24       4
16      1
 8      3
 0     5
```

- $i\text{Des} = \{1, 4\}$. 
iDes is reasonable.

Definition

- Read labels by increasing “height” to get permutation $\sigma \in S_a$.
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Example

- Remember the labels we had before.

![Diagram showing descent set]

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iDes is reasonable.

**Definition**

- Read labels by increasing “height” to get permutation $\sigma \in S_a$.
- $iDes :=$ the descent set of $\sigma^{-1}$.

**Example**

- Read them by increasing height to get $\sigma = 2\overline{1534} \in S_5$.
- $iDes = \{1, 4\}$. 
iDes is reasonable.

**Definition**

- Read labels by increasing “height” to get permutation $\sigma \in S_a$.
- $iDes :=$ the descent set of $\sigma^{-1}$.

**Example**

- Read them by increasing height to get $\sigma = 2\bar{1}53\bar{4} \in S_5$.

  ![Diagram](image)

  - $iDes = \{1, 4\}$.
tstat is bizarre (as usual).

**Definition**

- “Blow up” the x-parking function.
- Compute “dinv” of the blowup.

**Example**

- What?
tstat is bizarre (as usual).

**Definition**

- “Blow up” the $x$-parking function.
- Compute “dinv” of the blowup.

**Example**

- What?
tstat is bizarre (as usual).

Definition

- "Blow up" the $x$-parking function.
- Compute "dinv" of the blowup.

Example

- Remember our friend the 5/3-parking function.
tstat is bizarre (as usual).

Definition

- “Blow up” the $x$-parking function.
- Compute “dinv” of the blowup.

Example

- Since $2 \cdot 8 - 3 \cdot 5 = 1$ we “blow up” by 2 horiz. and 3 vert....
tstat is bizarre (as usual).

Example

- To get this!
tstat is bizarre (as usual).

Example

- To get this! Now compute “dinv”. (Computation omitted.)
Some things.

Things

- \( PF_{1,1}(x) = PF(x) \).
- \( PF_{q,t}(x) \) is symmetric and Schur-positive with coeffs \( \in \mathbb{N}[q, t] \).
  — via LLT polynomials
- Probably \( PF_{q,t}(x) = PF_{t,q}(x) \).
  — this will be impossible to prove (see Loehr’s Maxim)
- The coefficient of \( sgn \) is some \( Cat_{q,t}(x) \).
- Probably \( q^{(a-1)(b-1)/2} Cat_{q,\frac{1}{q}}(x) = \frac{1}{[a+b]_q} \left[ a+b \right]_q \).

Problems

- Does \( PF_{q,t}(x) \) occur “in nature”?
- How are \( PF_{q,t}(x) \) and \( PF_{q,t}(-x - 1) \) related?
Some things.

Things

- $\text{PF}_{1,1}(x) = \text{PF}(x)$.
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Things

- \( \text{PF}_{1,1}(x) = \text{PF}(x) \).
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  — *via LLT polynomials*
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  — *this will be impossible to prove (see Loehr’s Maxim)*
- The coefficient of \( \text{sgn} \) is some \( \text{Cat}_{q,t}(x) \).
- Probably \( q^{(a-1)(b-1)/2} \text{Cat}_{q,1/q}(x) = \frac{1}{[a+b]_q} [a,b]_q \).

Problems

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Problems

- Does $\text{PF}_{q,t}(x)$ occur “in nature”?
- How are $\text{PF}_{q,t}(x)$ and $\text{PF}_{q,t}(-x - 1)$ related?
Motivation 3: Lie Theory

(quoting from: Cellini-Papi, Haiman, Shi, Sommers et al.)
Consider Weyl group $\mathfrak{S}_a$ with $a, b$ coprime.
Consider Weyl group $\Gamma_a$ with $a, b$ coprime.

- These are the weight and root lattices $\Lambda < Q$ of $\Gamma_a$. 
Consider Weyl group $\Gamma_a$ with $a, b$ coprime.

Here is a **fundamental parallelepiped** for $\Lambda/b\Lambda$. 
Consider Weyl group $S_a$ with $a, b$ coprime.

- It contains $b^{a-1}$ elements (the “parking functions”).
Consider Weyl group $\mathfrak{S}_a$ with $a, b$ coprime.

- But they look better as a simplex...
Consider Weyl group $S_a$ with $a, b$ coprime.

...which is congruent to a nicer simplex.
Consider Weyl group $\mathfrak{S}_a$ with $a, b$ coprime.

- There are $\frac{1}{a+b} \binom{a+b}{a,b}$ elements of the root lattice inside.
Consider Weyl group $\mathfrak{S}_a$ with $a, b$ coprime.

- These are called $(a, b)$-cores (or $x$-Dyck paths).
“The same” works for all Weyl groups...

**Definition**

Consider a Weyl group $W$ with Coxeter number $h$ and let $p \in \mathbb{N}$ coprime to $h$. We define the **Catalan number**

$$\text{Cat}_q(W, p) := \prod_j \frac{[p + m_j]_q}{[1 + m_j]_q}$$

where $e^{2\pi i m_j / h}$ are the eigenvalues of a Coxeter element.
...but I’m out of time.