**Problem 1. Limits and Colimits.** Let $C$ be any category and let $I$ be a small category. A diagram of shape $I$ in $C$ is just a covariant functor $D : I \to C$. The limit of the diagram (if it exists) is a structure $\lim \leftarrow D = (L, \{\lambda_I\}_{I \in I})$ where

- $L \in C$ is an object and $\lambda_I : L \to D(I)$ are morphisms such that for all objects $I, J \in I$ and morphisms $\alpha \in \text{Hom}_C(I, J)$ we have $D(\alpha) \circ \lambda_I = \lambda_J$.
- If $M \in C$ is another object with morphisms $\mu_I : M \to D(I)$ satisfying the first property, then there exists a unique morphism $M \to L$ such that:

$$M \xrightarrow{\mu_I} L \xrightarrow{\lambda_I} D(I) \xrightarrow{D(\alpha)} D(J)$$

In other words, the limit $\lim \leftarrow D$ is a final object in a certain category (of “cones”). We define the colimit $\lim \rightarrow D$ by reversing all arrows in the definition.

(a) Given objects $A, B \in C$, express the categorical product $A \times B$ as a limit.

(b) Suppose there exists a zero object $0 \in C$. Given a morphism $\varphi : X \to Y$ in $C$, express the categorical kernel of $\varphi$ as a limit.

(c) (Optional) Let $R$ be a ring and think of $I = (\mathbb{N}, \leq)$ as a category with a unique arrow $i \to j$ for each $i \leq j$ in $\mathbb{N}$. Now let $D : I \to R\text{-Mod}$ be a diagram in which each morphism $D(\alpha)$ is injective. Prove that the colimit $\lim \rightarrow D$ exists.

**Problem 2. Length Equals Dimension For Vector Spaces.** Consider a field $K$ and a vector space $V \in K\text{-Vec}$. Recall that a composition series of length $n$ is a chain of subspaces

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = V$$

in which each quotient $V_{i+1}/V_i$ is simple (i.e. has no nontrivial subspaces). Prove that $V$ has a composition series of length $n$ if and only if it has a basis of size $n$. Hence the length of a vector space is the same as its dimension.

**Problem 3. Localization of a Module.** Let $M$ be a module over a commutative ring $R$ and let $S \subseteq R$ be a submonoid. Then we define the set of “fractions”

$$S^{-1}M := \left\{ \left[\frac{m}{s}\right] : m \in M, s \in S \right\}$$

with the equivalence relation

$$\left[\frac{m}{s}\right] = \left[\frac{n}{t}\right] \iff \exists u \in S, u(sn - tm) = 0.$$ 

(a) Prove that this is indeed an equivalence relation.

(b) Prove that the operations

$$\left[\frac{m}{s}\right] + \left[\frac{n}{t}\right] = \left[\frac{tm + sn}{st}\right] \quad \text{and} \quad r \left[\frac{m}{s}\right] = \left[\frac{rm}{s}\right]$$

are well-defined.
(c) Prove that the operations from part (b) make $S^{-1}M$ into an $R$-module and that the map $M \to S^{-1}M$ defined by $m \mapsto [m/1]$ is an $R$-module homomorphism.

**Problem 4. Rank Exists Over a Domain.** Let $R$ be an integral domain and let $S = R \setminus \{0\}$. Then we can identify the field of fractions $K = \text{Frac}(R)$ with the localization $S^{-1}R$. Let $M$ be any $R$-module and let $A \subseteq M$ be any subset.

(a) Show that we can regard $S^{-1}M$ as a $K$-module.
(b) If $A$ is $R$-linearly independent in $M$ prove that the image of $A$ under $M \to S^{-1}M$ (let’s call it $S^{-1}A$) is $K$-linearly independent in $S^{-1}M$.
(c) If $A \subseteq M$ is a maximal $R$-linearly independent subset prove that $S^{-1}A \subseteq S^{-1}M$ is a maximal $K$-linearly independent subset. Conclude that any two such sets have the same cardinality. [Hint: Let $A \subseteq M$ be $R$-linearly independent. Then we know from part (a) that $S^{-1}A \subseteq S^{-1}M$ is $K$-linearly independent. Suppose there exists $n \in S^{-1}M \setminus S^{-1}A$ such that $S^{-1}A \cup \{n\}$ is $K$-linearly independent. Then we can write $n = [m/s]$ for some $m \in M \setminus A$ and $s \in S$. Show that the set $A \cup \{m\} \subseteq M$ is $R$-linearly independent.]

**Problem 5. The Category $R$-Alg.** Let $R$ be a ring. We define an $R$-algebra as a pair $(\iota, S)$ where $S$ is a ring and $\iota : R \to S$ is a ring homomorphism such that $\text{im} \iota \subseteq Z(S)$. A morphism of $R$-algebras $\varphi : (\iota_1, S_1) \to (\iota_2, S_2)$ is defined as a ring homomorphism $\varphi : S_1 \to S_2$ such that

\[
\begin{array}{ccc}
S_1 & \xrightarrow{\varphi} & S_2 \\
\downarrow \iota_1 & & \downarrow \iota_2 \\
R & & R
\end{array}
\]

(a) Explain why $Z$-Alg = Rng.
(b) Prove that an $R$-algebra $(\iota, S)$ is also an $R$-module in a natural way (i.e. by forgetting the monoid structure on $S$).
(c) Conversely, given any set $A$ there exists an $R$-algebra $R(A)$ (called the free $R$-algebra generated by $A$) with the following universal property: For all set functions $f : A \to S$ there exists a unique $R$-algebra homomorphism $\varphi : R(A) \to S$ such that

\[
\begin{array}{ccc}
A & \xrightarrow{f} & R(A) \\
\downarrow & & \downarrow \varphi \\
R & & S
\end{array}
\]

If $|A| = n$ then we can identify $R(A)$ with the $R$-algebra $R(x_1, \ldots, x_n)$ of polynomials in the $n$ noncommuting indeterminates $x_1, \ldots, x_n$. Use this idea to find the initial object in the category $R$-Alg.

(d) Given a subset $A \subseteq S$ of an $R$-algebra, let $i_A : R(A) \to S$ be the unique $R$-algebra morphism defined in part (c). If $i_A$ is injective we say that the set $A$ is $R$-algebraically independent in $S$ and if $i_A$ is surjective we say $A$ is an $R$-algebraic generating set for $S$. If $i_A$ is bijective we say that $A$ is an $R$-algebra basis for $S$. Prove that an algebraic basis is necessarily a maximal algebraically independent set and a minimal algebraic generating set. [To make notation easier you can assume that the basis is finite.]