Last day of the semester. 😞

Recall:

\[ \text{Isom}(\mathbb{AV}, \mathbb{Q}) = \mathbb{V} \times O(\mathbb{V}, \mathbb{Q}) \]  
and \( \forall \varphi \in \text{Isom}(\mathbb{AV}, \mathbb{Q}) \) \( \exists \) reflections

\[ \varphi = R_1 \circ R_2 \circ \cdots \circ R_k \]

where \( k \leq \dim(\mathbb{V}) + 1 \)

Today: Finite groups of isometries.

Fixed Point Theorem:
If \( G \leq \text{Isom}(\mathbb{AV}, \mathbb{Q}) \) is finite,
then \( G \) fixes a point, i.e., \( G \) is isomorphic to a subgroup of \( O(\mathbb{V}, \mathbb{Q}) \).

(This is easy, but the proof is nice.)

Define the full affine group

\[ \text{Aff}(\mathbb{V}) := \mathbb{V} \times \text{GL}(\mathbb{V}) \]
Lemma: Affine maps preserve affine combinations, i.e. \( \forall \varphi \in \text{Aff}(\mathbf{V}) \), vectors \( \mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbf{V} \) and scalars \( c_1, \ldots, c_n \in \mathbb{F} \) with \( c_1 + \cdots + c_n = 1 \), have

\[
\varphi(c_1 \mathbf{x}_1 + \cdots + c_n \mathbf{x}_n) = c_1 \varphi(\mathbf{x}_1) + \cdots + c_n \varphi(\mathbf{x}_n).
\]

Proof: Let \( \varphi = t_\alpha \circ A \in \mathbf{V} \times \text{GL}(\mathbf{V}) \).

Then \( t_\alpha \circ A(c_1 \mathbf{x}_1 + \cdots + c_n \mathbf{x}_n) \)

\[
= t_\alpha(c_1 A \mathbf{x}_1 + \cdots + c_n A \mathbf{x}_n)
\]

\[
= (c_1 A \mathbf{x}_1 + \cdots + c_n A \mathbf{x}_n) + \alpha
\]

\[
= (c_1 A \mathbf{x}_1 + \cdots + c_n A \mathbf{x}_n) + (c_1 \alpha + \cdots + c_n \alpha)
\]

\[
= c_1 (A \mathbf{x}_1 + \alpha) + \cdots + c_n (A \mathbf{x}_n + \alpha)
\]

\[
= c_1 [t_\alpha \circ A(\mathbf{x}_1)] + \cdots + c_n [t_\alpha \circ A(\mathbf{x}_n)]
\]

\[
= c_1 \varphi(\mathbf{x}_1) + \cdots + c_n \varphi(\mathbf{x}_n)
\]

Could have defined affine maps this way.
Proof of Fixed Point Theorem:

Let $G \leq \text{Isom}(AV, \mathbb{Q})$ be finite and let $x \in AV$ be any point. Consider the $G$-orbit

$$G(x) := \{g(x) \in AV : g \in G\}$$

Suppose $G(x) = \{x_1, x_2, \ldots, x_n\}$ (in particular, $n ||G||$) and consider the centroid

$$x' = \frac{1}{n}(x_1 + \cdots + x_n).$$

(Let's assume $|G|$ and char $IF$ are coprime)

Now let $\varphi \in G$, since $\text{Isom}(AV, \mathbb{Q}) \leq \text{Aff}(V)$, the Lemma says

$$\varphi(x') = \frac{1}{n} \varphi(x_1) + \cdots + \frac{1}{n} \varphi(x_n).$$

$$= \frac{1}{n} (\varphi(x_1) + \cdots + \varphi(x_n)).$$

But note $\sum \varphi(x_1), \ldots, \varphi(x_n) = \sum x_1, \ldots, x_n$

$$\implies \varphi(x') = x'.$$
Idea: If \(|G(x)| = |G|\) then \(G(x)\) is a geometric realization of \(G\), with possibly nice structure (polytope, etc.)

Finally, back to topological fields.

\(F = \mathbb{R}, \mathbb{C}, \mathbb{H}\).

**DEF**: We say \(G\) is a topological group if

1. \(G\) is a group
2. \(G\) is a topological space
3. The maps

   \[\mu : G \times G \to G, \quad \mu(g, h) = gh\]

   \[\text{inv} : G \to G, \quad \text{inv}(g) = g^{-1}\]

are continuous.
(Def: $G$ is a group object in the category of topological spaces.)

Say subgroup $H \leq G$ is discrete if the relative topology is discrete, i.e., each $h \in H$ is isolated in $G$. ($\exists$ open $G$-nbhd $h \in U$ with $U \cap H = \{ h \}$)

Example: $(\mathbb{R}^n, +)$ is topological in the usual way.

A discrete subgroup of $(\mathbb{R}^n, +)$ is called a "lattice".

Picture $\mathbb{R}^2$. 

\[ \begin{array}{c}
\text{\ldots} \\
\downarrow \\
\text{\ldots} \\
\text{\ldots} \\
\text{\ldots} \\
\end{array} \]
Nontrivial (1) Theorem:
Every lattice $\Lambda \subseteq \mathbb{R}^n$ has a basis, i.e. $\exists$ $\mathbb{R}$-linearly independent $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}^n$ $(k \leq n)$ with
$\mathbb{Z}^k \cong \Lambda = \{ \sum c \cdot \alpha_i : c \in \mathbb{Z}, \sum \leq \mathbb{R}^n \}$

This $k$ is called the "rank" of $\Lambda$.
(We reserve the word "dimension" for vector spaces over fields.)

Observe: If $\Lambda \subseteq \mathbb{R}^n$ is full-rank then $\mathbb{R}^n / \Lambda$ is a "torus".

Now consider

$\mathbb{R}^n = \text{Euclidean space}$
$\text{Isom}(\mathbb{R}^n) = \text{Euclidean geometry}$
(in the sense of Klein).
If $\Gamma \leq \text{Isom}(\mathbb{R}^n)$ is discrete, one can show that

$$\Gamma = \Lambda \times G,$$

where $G$ is a finite subgroup of $O(n)$, stabilizing a lattice $\Lambda \leq \mathbb{R}^n$.

i.e. $\forall g \in G, x \in \Lambda$ we have $g(x) \in \Lambda$

[Key step: $O(n)$ is compact, hence $G \leq O(n)$ discrete $\implies G$ finite]

Sometimes discrete $\Gamma \leq \text{Isom}(\mathbb{R}^n)$ are called crystallographic groups.

Naive Goal:

Classify crystallographic groups

Case $n = 2$

There are 17, called the "wallpaper groups" (M. Artin pg. 174)

and 7 "frieze groups"
Case $n = 3$. (1892 Fyodorov–Schönflies)

There are 230. (Ugh.)

Hilbert's 18th problem (1900):

Are there $< \infty$ groups for each $n$?

Theorem (Bieberbach 1910): Yes.

However ...

Classify discrete $\Gamma \leq \text{Isom}(\mathbb{R}^n)$

Classify lattices $\Lambda \leq \mathbb{R}^n$

Classify finite $G \leq O(n)$.

(Too hard.) (Impossible.)

Oh Well.
Remark: There is one very special case. The finite subgroups of $SO(3)$ are

1. cyclic
2. dihedral
3. rotations of tetrahedron
4. rotations of cube/octahedron
5. rotations of icosidodecahedron

I.O.U. What this means (McKay Correspondence)

To go further, we need some natural restriction.

Idea: Study discrete $\Gamma \subseteq Isom(\mathbb{R}^n)$ generated by reflections

BINGO!

The END?