0. Compute the length of a chord of the unit circle subtended by an arc of length \( t \).

1. Given an arbitrary matrix 
\[
A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}
\]
we can define a function from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \) by \( x \mapsto Ax \),
in other words, 
\[
\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + cy \\ bx + dy \end{pmatrix}.
\]
Prove that this is a linear function.

Let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) be a linear function and consider the standard basis of \( \mathbb{R}^2 \) consisting of \( e_1 = (1, 0) \) and \( e_2 = (0, 1) \). If \( f(e_1) = (a, b) \) and \( f(e_2) = (c, d) \) then we define the matrix
\[
[f] = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.
\]
Given \( x \in \mathbb{R}^2 \) we will write \([x]\) for the corresponding column vector. Then we define the product of a matrix and a column by \([f][x] = [f(x)]\). [Why do we do this?]

2. Let \( f \) and \( g \) be linear functions from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \).
   (a) Prove that the composite \( f \circ g : \mathbb{R}^2 \to \mathbb{R}^2 \) is also linear.
   (b) We define the matrix product by \([f][g] := [f \circ g]\). If \([f] = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\) and \([g] = \begin{pmatrix} a' & c' \\ b' & d' \end{pmatrix}\), use the definition to compute the matrix product \([f][g]\).

3. Let \( R_t : \mathbb{R}^2 \to \mathbb{R}^2 \) be the (linear) function that rotates the plane counterclockwise by angle \( t \). Recall that we can express this in coordinates by
\[
[R_t] = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.
\]
   (a) Explain why \([R_t]^3 = [R_{3t}]\) without doing any work.
   (b) Use part (a) to express \( \cos(3t) \) as a polynomial in \( \cos(t) \). This is an example of a Chebyshev polynomial of the first kind.

4. Use the “angle sum formulas” to verify the following trigonometric identities.
   (a) \( 2 \sin \alpha \sin \beta = \cos(\alpha - \beta) - \cos(\alpha + \beta) \)
   (b) \( 2 \cos \alpha \cos \beta = \cos(\alpha - \beta) + \cos(\alpha + \beta) \)

5. Use the identities from Problem 4 to verify the following integrals.
   (a) \( \int_0^{2\pi} \sin(mt) \sin(nt) \, dt = \begin{cases} \pi & m = n \neq 0 \\ 0 & \text{otherwise} \end{cases} \)
   (b) \( \int_0^{2\pi} \cos(mt) \cos(nt) \, dt = \begin{cases} 2\pi & m = n = 0 \\ \pi & m = n \neq 0 \\ 0 & \text{otherwise} \end{cases} \)