Problem 1. Logical Analysis.
(a) Let $Q$ and $R$ be logical statements. Use a truth table to prove that $\neg(Q \lor R)$ is logically equivalent to $\neg Q \land \neg R$. [This is called de Morgan’s law.]
(b) Let $P$, $Q$, and $R$ be logical statements. Use a truth table to prove that $(Q \lor R) \Rightarrow P$ is logically equivalent to $(Q \Rightarrow P) \land (R \Rightarrow P)$.
(c) Apply the principles from (a) and (b) to prove that for all integers $m$ and $n$ we have “$mn$ is even” $\iff$ “$m$ is even or $n$ is even”.
[Hint: Let $P =$“$mn$ is even”, $Q =$“$m$ is even”, and $R =$“$n$ is even”. Use part (a) for the $\Rightarrow$ direction and use part (b) for the $\iff$ direction.]

Problem 2. Absolute Value. Given an integer $a$ we define its absolute value as follows: 
\[
|a| := \begin{cases} 
  a & \text{if } a > 0 \\
  0 & \text{if } a = 0 \\
  -a & \text{if } a < 0 
\end{cases}
\]
Prove that for all integers $a$ and $b$ we have $|ab| = |a||b|$. [Hint: Your proof will break into at least five separate cases. You may assume without proof the properties $(-a)(-b) = ab$ and $(-a)b = a(-b) = -(ab)$; we’ll prove them later.]

Problem 3. Divisibility. Given integers $m$ and $n$ we will write “$m|n$” to mean that “there exists an integer $k$ such that $n = mk$” and when this is the case we will say that “$m$ divides $n$” or “$n$ is divisible by $m$”. Now let $a$, $b$, and $c$ be integers. Prove the following properties.
(a) If $a|b$ and $b|c$ then $a|c$.
(b) If $a|b$ and $a|c$ then $a|(bx + cy)$ for all integers $x$ and $y$.
(c) If $a|b$ and $b|a$ then $a = \pm b$. [Hint: Use the fact that $uv = 0$ implies $u = 0$ or $v = 0$.]
(d) If $a|b$ and $b$ is nonzero then $|a| \leq |b|$. [Hint: Use the result of Problem 2.]

Problem 4. The Square Root of 5. Prove that $\sqrt{5}$ is not a ratio of integers, in two steps.
(a) First prove the following lemma: Let $n$ be an integer. If $n^2$ is divisible by 5, then so is $n$. [Hint: Use the contrapositive and note that there are four separate ways for an integer to be not divisible by 5. Sorry it’s a bit tedious; we will find a better way to do this later.]
(b) Use the method of contradiction to prove that $\sqrt{5}$ is not a ratio of integers. Explicitly quote your lemma in the proof. [Hint: Your proof should begin as follows: “Assume for contradiction that $\sqrt{5}$ is a ratio of integers. In this case, . . . ”]