A B_n-analogue of the chessboard complex

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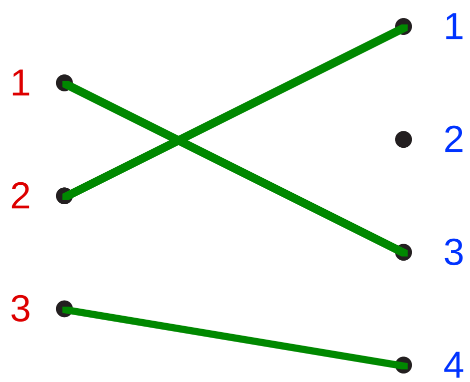
Chessboard Complex $M_{m,n}$

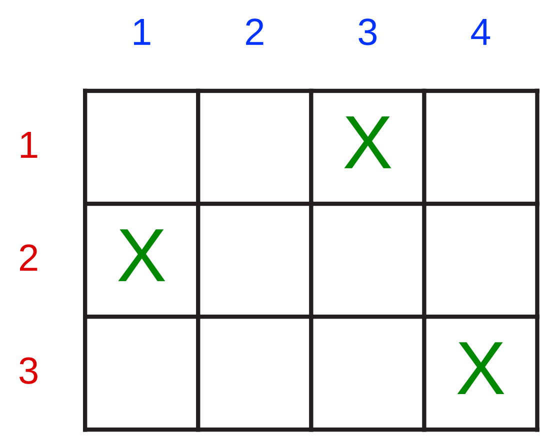
vertex set $:= [m] \times [n]$

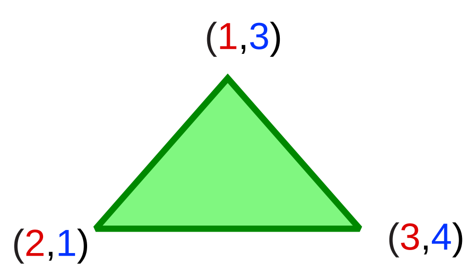
face $:= \{(i_1, j_1), \ldots, (i_k, j_k)\}$

$i_s \neq i_t, \;\; j_s \neq j_t \;\forall s, t$

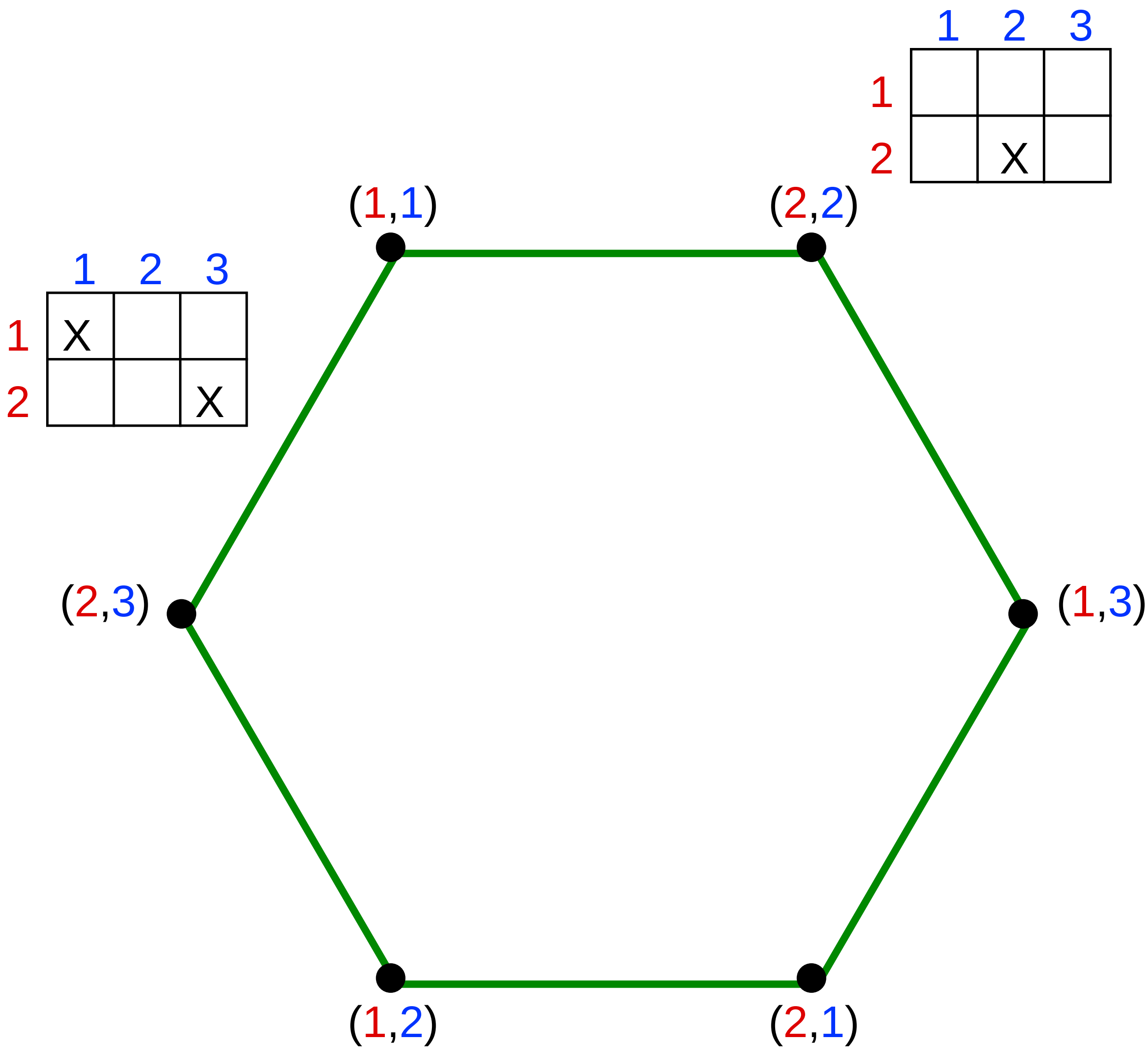
facet of $M_{3,4}$:







Chessboard Complex $M_{2,3}$



Topology of the Chessboard Complex

Garst(1979):

- $M_{m,n}$ is Cohen-Macaulay iff $n \geq 2m - 1$
- Computation of top homology

Björner, Lovász, Vrécica, Živaljević (1994): Let

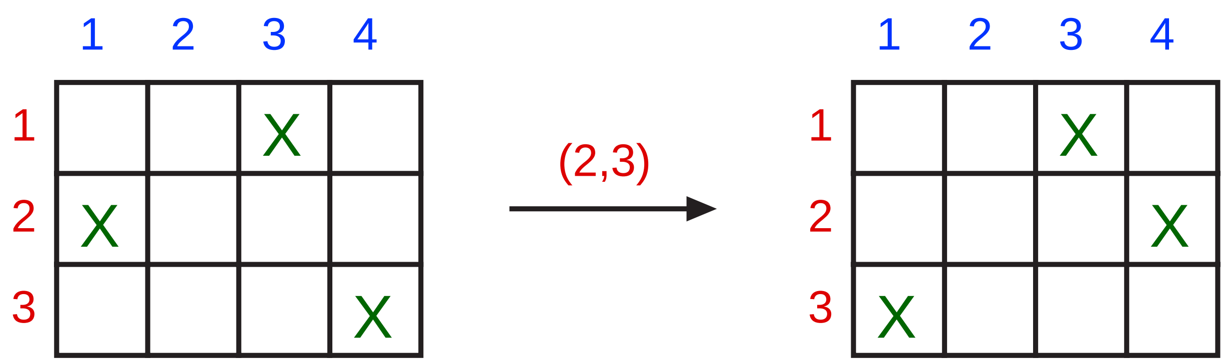
$$\nu(m,n) = \min\{m, \lfloor \frac{m+n+2}{3} \rfloor\} - 1$$

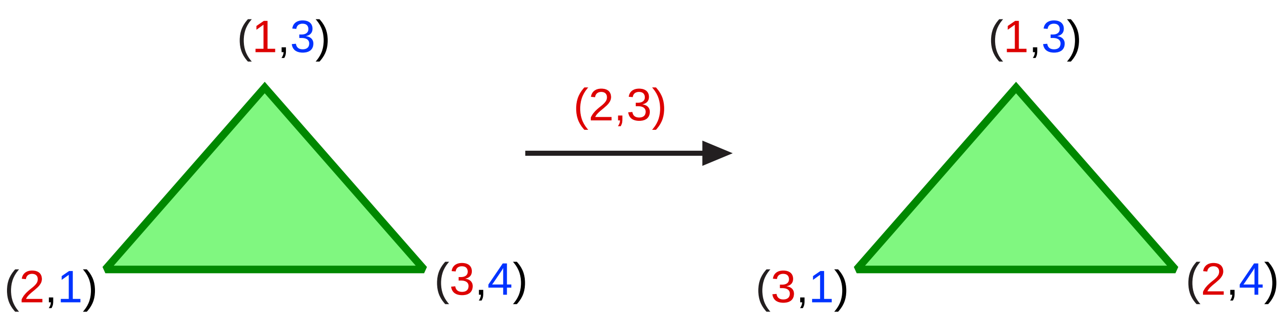
Then $M_{m,n}$ is $(\nu(m,n) - 1)$-connected

BLVZ Conjecture: $\pi_{\nu(m,n)}(M_{m,n}) \neq 0$

Action of the Symmetric Group

The symmetric groups $\mathfrak{S}_m$ and $\mathfrak{S}_n$ act on $M_{m,n}$ by permuting the rows and columns , resp., of the rook placement.



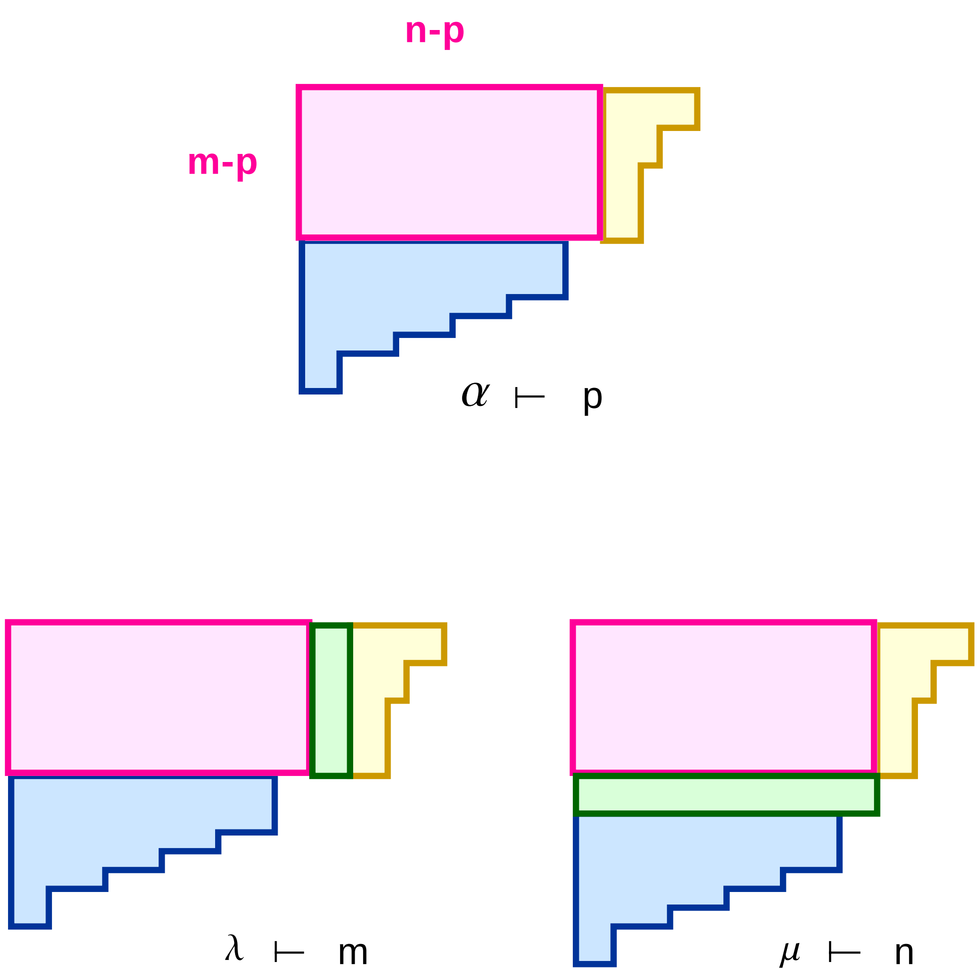


This induces a representation of $\mathfrak{S}_m \times \mathfrak{S}_n$ on $\tilde{H}_*(M_{m,n}; \mathbb{C})$.

Friedman & Hanlon(1998): As $(\mathfrak{S}_m \times \mathfrak{S}_n)$-modules,

$$\tilde{H}_{p-1}(M_{m,n};\mathbb{C}) \cong \bigoplus_{(\lambda,\mu)\in\mathcal{R}(m,n,p)} S^{\lambda'} \otimes S^{\mu}$$

where $\mathcal{R}(m,n,p)$ is the set of all pairs of partitions $(\lambda \vdash m,\ \mu \vdash n)$ which can be obtained from a partition $\alpha \vdash p$ in the following way



Shareshian and Wachs (2000):

- Proof of BLVZ conjecture

- If $n \geq 11$ then $\tilde{H}_{\nu(n,n)}(M_{n,n})$ is a nontrivial 3-group of exponent at most 9

- For $n \geq 5$ and $n \equiv 2 \bmod 3$,

$$\tilde{H}_{\nu(n,n)}(M_{n,n}) = \mathbb{Z}_3$$

Tits Coset Complexes (Garst 1979)

Let G be a group and $G_1, \ldots, G_m$ a family of subgroups. Form a simplicial complex

$$\Delta(G; G_1, \ldots, G_m)$$

vertex set := $\{gG_i \mid g \in G,\ i \in [m]\}$
face := $\{gG_{i_1}, \ldots, gG_{i_k}\}$ where $g \in G$.

G acts on $\Delta(G; G_1, \ldots, G_m)$ by left multiplication: $h \cdot (gG_i) = hgG_i$

Examples: Coxeter complexes, Tits buildings, chessboard complexes

Let $G = \mathfrak{S}_n$ and $G_i = \{\sigma \in \mathfrak{S}_n \mid \sigma(i) = i\}$ for $i = 1, \ldots, m$. Then

$$gG_i = \{\sigma \in \mathfrak{S}_n \mid \sigma(i) = g(i)\} \mapsto (i, g(i)) \in [m] \times [n]$$

$$\{gG_{i_1}, \ldots, gG_{i_k}\} \mapsto \{(i_1, g(i_1)), \ldots, (i_k, g(i_k))\}$$

$$\Delta(G; G_1, \ldots, Gm) \xrightarrow{\cong} M_{m,n}$$

Hyperoctahedral group B_n

Group of barred permutations

$$B_2 = \{12, 1\bar{2}, \bar{1}2, \bar{1}\bar{2}, 21, 2\bar{1}, \bar{2}1, \bar{2}\bar{1}\}$$

Let $G = B_n$ and $G_i = \{\sigma \in B_n \mid \sigma(i) = i\}$ for $i = 1, \ldots, m$.

Then

$$gG_i = \{\sigma \in B_n \mid \sigma(i) = g(i)\} \mapsto (i, g(i)) \in [m]\times[n]\times\{\pm 1\}$$

$$\{gG_{i_1}, \ldots, gG_{i_k}\} \mapsto \{(i_1, g(i_1)), \ldots, (i_k, g(i_k))\}$$

$$\Delta(G; G_1, \ldots, G_m) \xrightarrow{\cong} \overline{M}_{m,n}$$

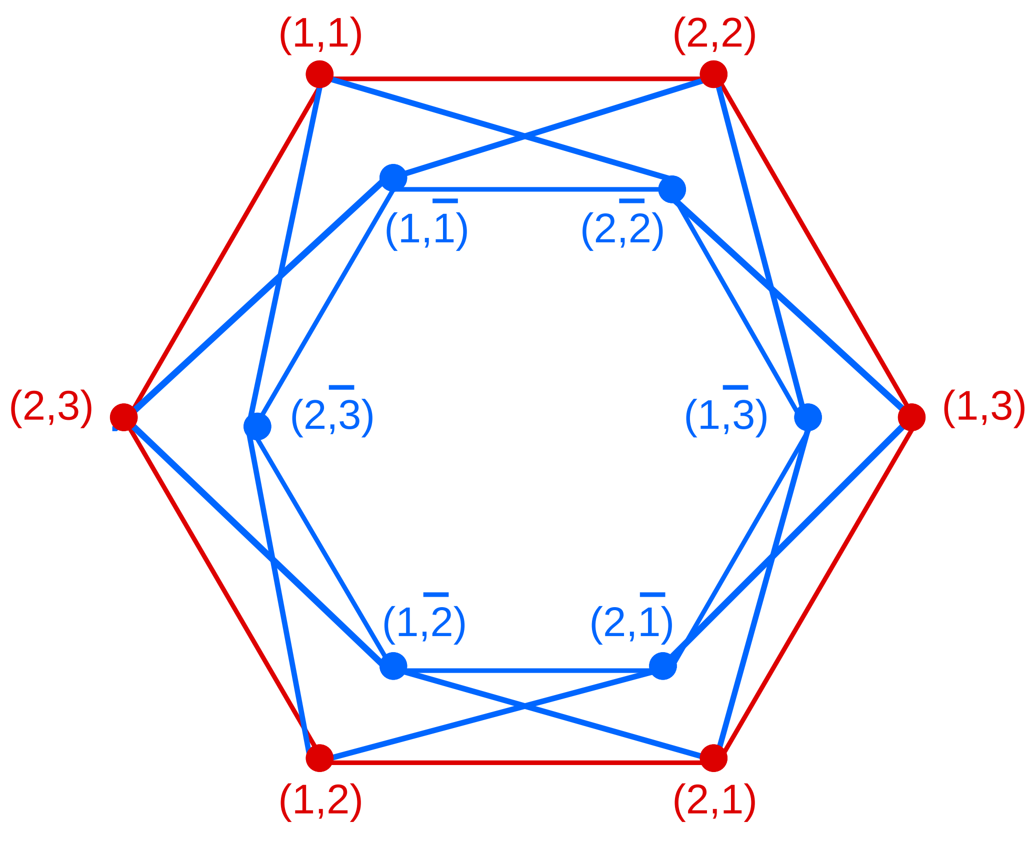
Example: $g = 3\bar{1}52\bar{4} \in B_5$

$$\{gG_2, gG_3\} \mapsto \{(2, \bar{1}), (3, 5)\}$$

$\overline{M}_{m,n}$ has vertex set $[m] \times [n] \times \{-1, 1\}$
Faces have form

$$\{(v_1, \epsilon_1), (v_2, \epsilon_2), \ldots, (v_k, \epsilon_k)\},$$

where $\{v_1, v_2, \ldots, v_k\}$ is a k-face of $M_{m,n}$ and $\epsilon_i \in \{-1, 1\}\ \forall i$.



$\overline{M}_{2,3}$

r-Inflation

Let Δ be a simplicial complex on vertex set V. The r-inflation Δ^r is the simplicial complex with vertex set $V \times [r]$ and faces of the form $\{(v_1, c_1), \ldots, (v_k, c_k)\}$ where $\{v_1, \ldots, v_k\}$ is a k-face of Δ.

$\overline{M}_{m,n}$ is the 2-inflation of $M_{m,n}$

Björner (1994): If Δ is connected

$$\Delta^2 \simeq \bigvee_{F \in \Delta} \mathrm{Susp}^{|F|}(\mathrm{lk}_\Delta F)$$

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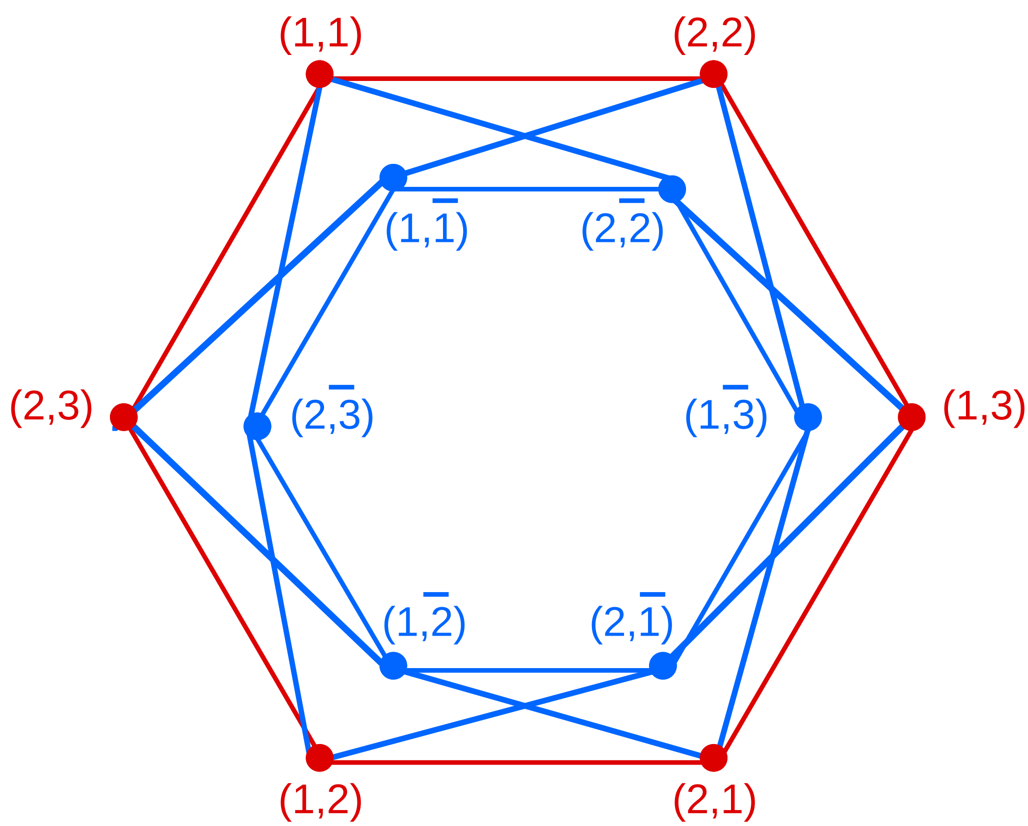

$\overline{M}_{2,3}$

Since $\mathrm{lk}_{M_{m,n}} F = M_{n-|F|,m-|F|}$ we have:

Theorem:

- $\overline{M}_{m,n}$ is Cohen-Macaulay iff $n \geq 2m - 1$ Garst(1979)

- $\overline{M}_{m,n}$ is $(\nu(m,n) - 1)$-connected.

- $\tilde{H}_{\nu(m,n)}(\overline{M}_{m,n}) \neq 0$

- If $n \geq 11$ then $\tilde{H}_{\nu(n,n)}(\overline{M}_{n,n})$ is a nontrivial 3-group of exponent at most 9.

- For $n \geq 5$ and $n \equiv 2 \bmod 3$,

$$\tilde{H}_{\nu(n,n)}(\overline{M}_{n,n}) = \mathbb{Z}_3$$

B_n-Analogue of Friedman-Hanlon

Equivariant version of Björner's result: Let group G act simplicially on Δ and Δ^2. Suppose the actions commute with the natural map $\Delta^2 \to \Delta$. Then

$$\tilde{H}_p(\Delta^2) \cong \bigoplus_{F \in \Delta/G} \left(\tilde{H}_{|F|-1}(\hat{F}^2) \otimes \tilde{H}_{p-|F|}(\mathrm{lk}_\Delta F) \right) \Big\uparrow_{\mathrm{Stab}_G F}^{G}$$

where $\hat{F}$ is the simplicial complex whose only facet is F.

Theorem: As $(B_m \times B_n)$-modules,

$$\tilde{H}_{p-1}(\overline{M}_{m,n}; \mathbb{C}) \cong \bigoplus_{\beta,\lambda,\mu} S^{(\lambda',\beta')} \otimes S^{(\mu,\beta)}$$

summed over all partitions β, λ, μ such that $|\beta| \leq p$, $(\lambda, \mu) \in \mathcal{R}(m - |\beta|, n - |\beta|, p - |\beta|)$.

$S^{(\mu,\beta)}$ denotes the irreducible B_n-module indexed by (μ, β)

$$S^{(\mu,\beta)} = \left(S^{\mu} \otimes (S^{\beta} \otimes \text{par})\right)\Big\uparrow_{B_{|\mu|}\times B_{|\beta|}}^{B_n}$$

- B_j acts as $\mathfrak{S}_j$ on S^{λ}
- par is the 1-dimensional parity representation of B_j: Character is $(-1)^{\#\text{bars}}$

Proof Sketch (for B_n action):

Let $F_j = \{(1,1),(2,2),\ldots,(j,j)\}$.

$$\tilde{H}_p(\overline{M}_{m,n}) \cong \bigoplus_j \binom{m}{j} \left(\tilde{H}_{j-1}(\hat{F}_j^2) \otimes \tilde{H}_{p-j}(\text{lk} F_j)\right)\Big\uparrow_{\mathbb{Z}_2^j\times B_{n-j}}^{B_n}$$

$$\left(\tilde{H}_{j-1}(\hat{F}_j^2) \otimes \tilde{H}_{p-j}(\mathsf{lk} F_j)\right)\Big\uparrow_{\mathbb{Z}_2^j \times B_{n-j}}^{B_n}$$

$$\begin{aligned}
&\cong \left(\tilde{H}_{j-1}(\hat{F}_j^2) \uparrow_{\mathbb{Z}_2^j}^{B_j} \otimes \tilde{H}_{p-j}(M_{m-j,n-j})\right)\Big\uparrow_{B_j \times B_{n-j}}^{B_n} \\
&\cong \Big((C_{j-1}(M_{j,j}) \otimes \mathsf{par}) \otimes \tilde{H}_{p-j}(M_{m-j,n-j})\Big)\Big\uparrow_{B_j \times B_{n-j}}^{B_n} \\
&\cong \Big(\big(\bigoplus_{\beta \vdash j} f^\beta S^\beta \otimes \mathsf{par}\big) \otimes \big(\bigoplus_{(\lambda,\mu)\in\mathcal{R}} f^\lambda S^\mu\big)\Big)\Big\uparrow_{B_j \times B_{n-j}}^{B_n} \\
&\cong \bigoplus_{\beta,\lambda,\mu} f^\lambda f^\beta \left((S^\beta \otimes \mathsf{par}) \otimes S^\mu\right)\Big\uparrow_{B_j \times B_{n-j}}^{B_n} \\
&\cong \bigoplus_{\beta,\lambda,\mu} f^\lambda f^\beta \, S^{(\mu,\beta)}
\end{aligned}$$

$$\tilde{H}_p(\overline{M}_{m,n}; \mathbb{C}) \cong_{B_n} \bigoplus_{\beta,\lambda,\mu} \binom{m}{|\beta|} f^\lambda f^\beta \, S^{(\mu,\beta)}$$

Wreath product generalization

$G = \mathfrak{S}_n \wr \mathbb{Z}_r$

View $w \in G$ as colored permutation

$$w = w_1 w_2 \dots w_n$$

where $w_i = (\sigma(i), c_i)$, $\sigma \in \mathfrak{S}_n$ and $c_i \in \mathbb{Z}_r$.

Let $G_i = \{g \in G \mid g_i = (i, 0)\}$.

$\Delta(G; G_1, \dots, G_m)$ is the r-inflation of $M_{m,n}$.

Björner, Wachs, Welker (2000): Let Δ be connected.

- $$\Delta^r \simeq \bigvee_{F \in \Delta} \mathrm{Susp}^{|F|}(\mathrm{lk}_\Delta F)^{\vee (r-1)^{|F|}}.$$
- Let group G act simplicially on Δ and Δ^r. Suppose the actions commute with the natural map $\Delta^r \to \Delta$. Then

$$\tilde{H}_p(\Delta^r) \cong \bigoplus_{F \in \Delta/G} \left(\tilde{H}_{|F|-1}(\hat{F}^r) \otimes \tilde{H}_{p-|F|}(\mathrm{lk}_\Delta F) \right) \Big\uparrow_{\mathrm{Stab}_G F}^{G}$$

The irreducible $(\mathfrak{S}_n \wr \mathbb{Z}_r)$-modules are indexed by r-tuples of partitions $(\mu, \beta_1, \ldots, \beta_{r-1})$ satisfying $|\mu| + |\beta_1| + \cdots + |\beta_{r-1}| = n$.

Theorem: As $((\mathfrak{S}_m \wr \mathbb{Z}_r) \times (\mathfrak{S}_n \wr \mathbb{Z}_r))$-modules,

$$\tilde{H}_{p-1}(M^r_{m,n}; \mathbb{C}) \cong \bigoplus S^{(\lambda', \beta'_1, \ldots, \beta'_{r-1})} \otimes S^{(\mu, \beta_1, \ldots, \beta_{r-1})}$$

summed over all partitions $\beta_1, \ldots, \beta_{r-1}, \lambda, \mu$ such that $\sum |\beta_i| \leq p$, $(\lambda, \mu) \in \mathcal{R}(m - \sum |\beta_i|, n - \sum |\beta_i|, p - \sum |\beta_i|)$.