Torsion in the matching complex and the chessboard complex

John Shareshian
&
Michelle Wachs

www.math.miami.edu/∼wachs

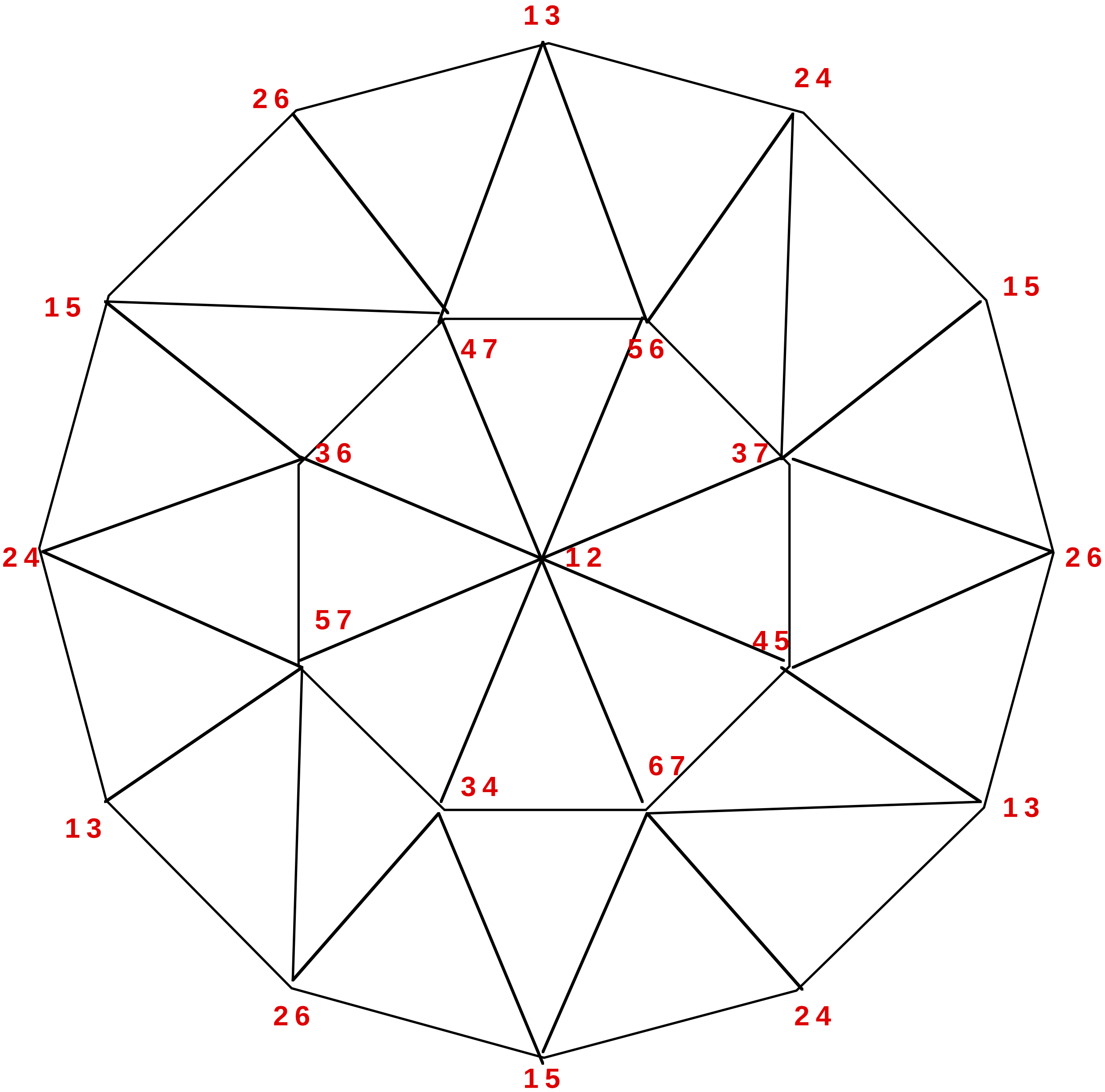
The matching complex M_n is the simplicial complex of partial matchings on

$$[n] := \{1, 2, \ldots, n\}$$

vertices := 2 element subsets of $[n]$

faces := sets of pairwise disjoint 2 element subsets of $[n]$

$M_4 =$



Piece of M_7

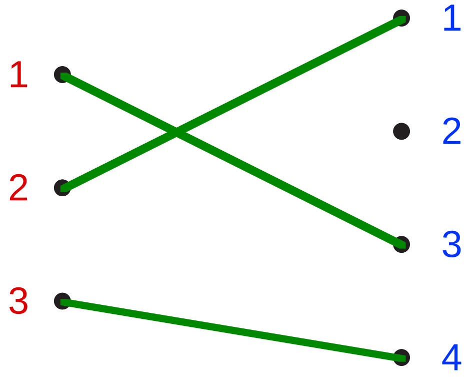
Chessboard Complex $M_{m,n}$

$$\text{vertex set} := [m] \times [n]$$

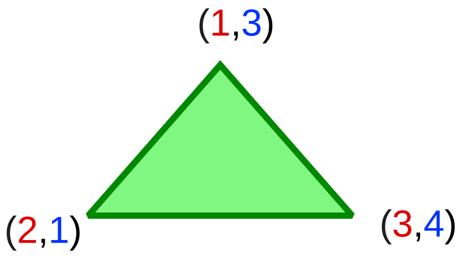
$$\text{face} := \{(i_1, j_1), \ldots, (i_k, j_k)\}$$

$$i_s \neq i_t, \;\; j_s \neq j_t \; \forall s, t$$

facet of $M_{3,4}$:



	1	2	3	4
1			X	
2	X			
3				X



Theorem (Björner, Lovász, Vrécica, Živaljević 1994) Let

$$\nu_n = \lfloor \frac{n+1}{3} \rfloor - 1$$

$$\nu_{m,n} = \min\{m, n, \lfloor \frac{m+n+1}{3} \rfloor\} - 1.$$

Then

- M_n is $(\nu_n - 1)$-connected
- $M_{m,n}$ is $(\nu_{m,n,-1})$-connected

Conjecture (Björner, Lovász, Vrécica, Živaljević 1994)

- $\tilde{H}_{\nu_n}(M_n; \mathbb{Z}) \neq 0$
- $\tilde{H}_{\nu_{m,n}}(M_{m,n}; \mathbb{Z}) \neq 0$

Bouc (1990):

If $n \equiv 1 \bmod 3$, $n \geq 7$ then $\tilde{H}_{\nu_n}(M_n) \cong \mathbb{Z}_3$

If $n \equiv 0 \bmod 3$, $n \geq 12$ then $\tilde{H}_{\nu_n}(M_n)$ is a nontrivial finite 3-group with exponent ≤ 9

Proof idea: Bouc shows for $n = 0, 1 \bmod 3$, $\tilde{H}_{\nu_n}(M_n)$ is generated by cycles of the form

$$\alpha * \beta$$

where

$$\alpha \in \tilde{H}_0(M_{\{a,b,c\}}), \quad \beta \in \tilde{H}_{\nu_{n-3}}(M_{[n]\setminus\{a,b,c\}}).$$

Since

$$3(\alpha * \beta) = \alpha * 3\beta$$

induction can be used.

Base step of induction for 1 mod 3:

$$\tilde{H}_{\nu_7}(M_7) = \mathbb{Z}_3$$

Shareshian & MW (1999): $\tilde{H}_{\nu_n}(M_n)$ is nonvanishing for all n. Moreover, for all $n \geq 12$ (except possibly $n = 14$)

$$\tilde{H}_{\nu_n}(M_n) \cong \mathbb{Z}_3^{r_n}$$

where $r_n \geq 1$.

Our improvement:

Base step for $n \equiv 0$ mod 3: Computer calculation using homology software of Heckenbach, Welker, Dumas, Saunders:

$$\tilde{H}_{\nu_{12}}(M_{12}) = \mathbb{Z}_3^{56}$$

$n \equiv 2$ mod 3: We show that $\tilde{H}_{\nu_n}(M_n)$ is generated by cycles of the form

$$\alpha * \beta,$$

$$\alpha \in \tilde{H}_0(M_{\{a,b,c,d,e\}}), \ \ \beta \in \tilde{H}_{\nu_{n-5}}(M_{[n]\setminus\{a,b,c,d,e\}})$$

Since $3(\alpha * \beta) = \alpha * 3\beta$ and $n - 5 \equiv 0$ mod 3, we can apply the 0 mod 3 case when $n - 5 \geq 12$

Chessboard complex - homology groups

$\tilde{H}_{\nu_{m,n}}(M_{m,n})$

$m \backslash n$	2	3	4	5	6	7
1	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^3$	$\mathbb{Z}^4$	$\mathbb{Z}^5$	$\mathbb{Z}^6$
2	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^5$	$\mathbb{Z}^{11}$	$\mathbb{Z}^{19}$	$\mathbb{Z}^{29}$
3		$\mathbb{Z}^4$	$\mathbb{Z}^2$	$\mathbb{Z}^{14}$	$\mathbb{Z}^{47}$	$\mathbb{Z}^{104}$
4			$\mathbb{Z}^{15}$	$\mathbb{Z}^{20}$	$\mathbb{Z}^5$	$\mathbb{Z}^{225}$
5				$\mathbb{Z}_3$	$\mathbb{Z}^{152}$	$\mathbb{Z}^{98}$
6					$\mathbb{Z}^{25} \oplus \mathbb{Z}_3^{10}$	$\mathbb{Z}_3$
7						$\mathbb{Z}^{588} \oplus \mathbb{Z}_3^{66}$

Torsion in the Chessboard Complex

Shareshian & MW (2000): $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is non-vanishing for all m and n. Moreover

- $n+m \equiv 1 \bmod 3$
  If $5 \leq m \leq n \leq 2m-5$ then
  $$\tilde{H}_{\nu_{m,n}}(M_{m,n}) \equiv \mathbb{Z}_3$$
- $n+m \equiv 0 \bmod 3$
  If $9 \leq m \leq n \leq 2m-9$ then $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is a 3-group with exponent at most 9.
- $n+m \equiv 2 \bmod 3$
  If $13 \leq m \leq n \leq 2m-13$ then $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is a 3-group with exponent at most 9.

Note: If $2m-1 \leq n$ then $\nu_{m,n} = m-1$. So $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is top homology which is free.

The proof is more difficult than for the matching complex.

Base step limitation: Computer provides only

$$\tilde{H}_{\nu_{5,5}}(M_{5,5}) \cong \mathbb{Z}_3.$$

$\tilde{H}_{\nu_{6,6}}(M_{6,6})$ and $\tilde{H}_{\nu_{7,7}}(M_{7,7})$ have free parts.

$\tilde{H}_{\nu_{9,9}}(M_{9,9})$ is too big for the computer.

If $\tilde{H}_{\nu_{9,9}}(M_{9,9})$ turns out to have exponent 3 then we can bring the exponent down to 3 in all cases.

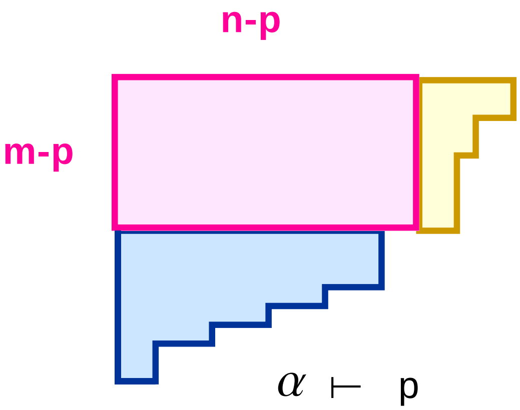
Induction step difficulty: We can show that generating cycles have the form $\alpha * \beta$ where
$\alpha \in \tilde{H}_0(M_{2,1})$, $\beta \in \tilde{H}_{\nu_{m-2,n-1}}(M_{m-2,n-1})$ or
$\alpha \in \tilde{H}_0(M_{1,2})$, $\beta \in \tilde{H}_{\nu_{m-1,n-2}}(M_{m-1,n-2})$.

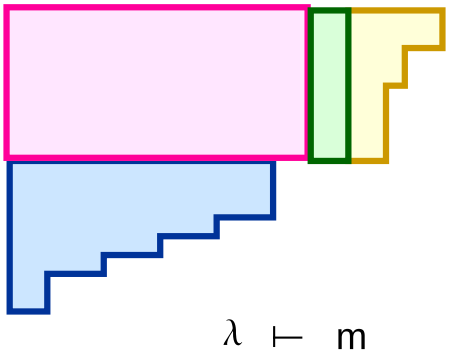
We need to decompose cycles in the top homology.

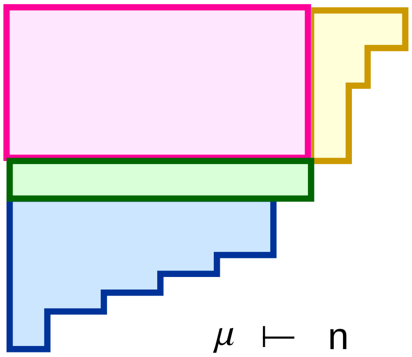
Friedman & Hanlon(1998): As $(\mathfrak{S}_m \times \mathfrak{S}_n)$-modules,

$$\tilde{H}_{p-1}(M_{m,n};\mathbb{C}) \cong \bigoplus_{(\lambda,\mu)\in\mathcal{R}(m,n,p)} S^{\lambda'} \otimes S^{\mu}$$

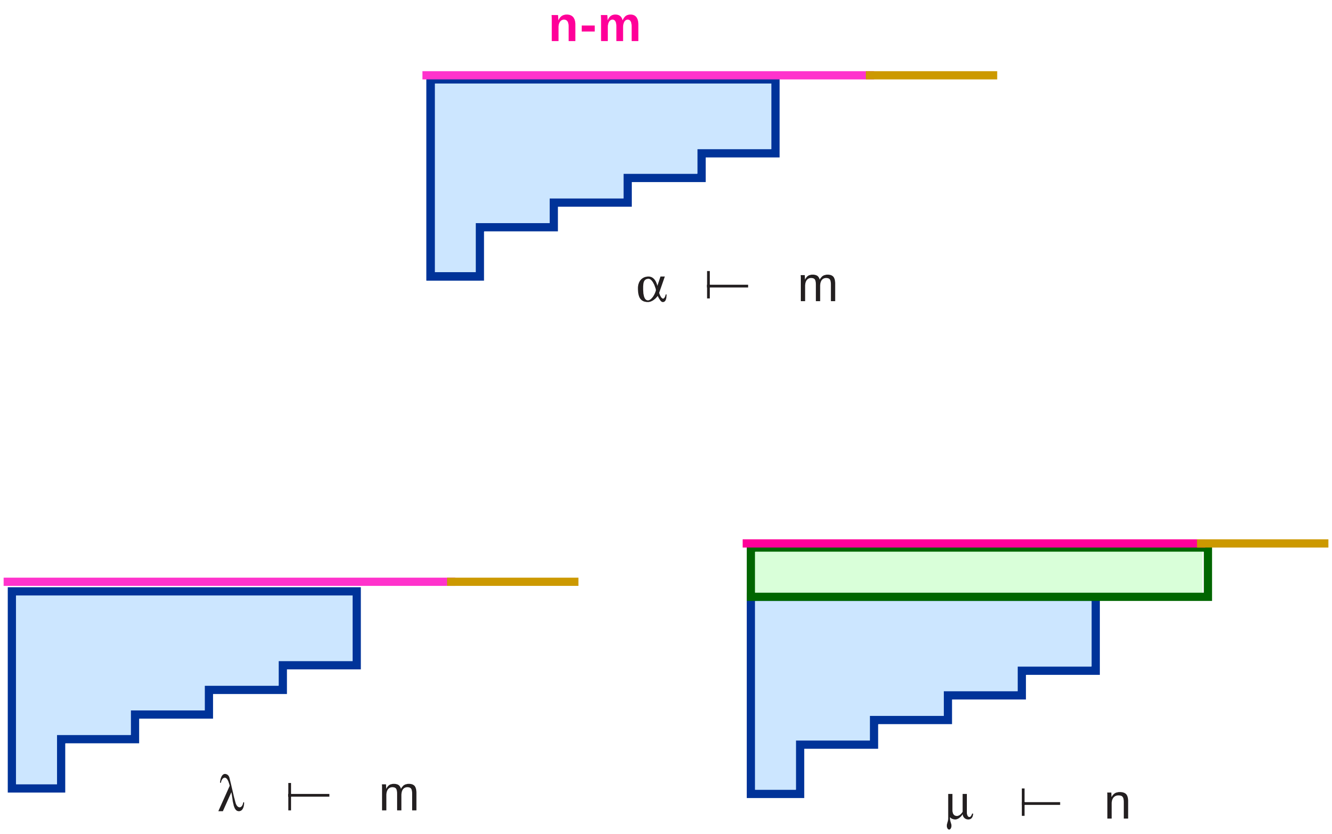
where $\mathcal{R}(m,n,p)$ is the set of all pairs of partitions $(\lambda \vdash m,\ \mu \vdash n)$ which can be obtained from a partition $\alpha \vdash p$ in the following way







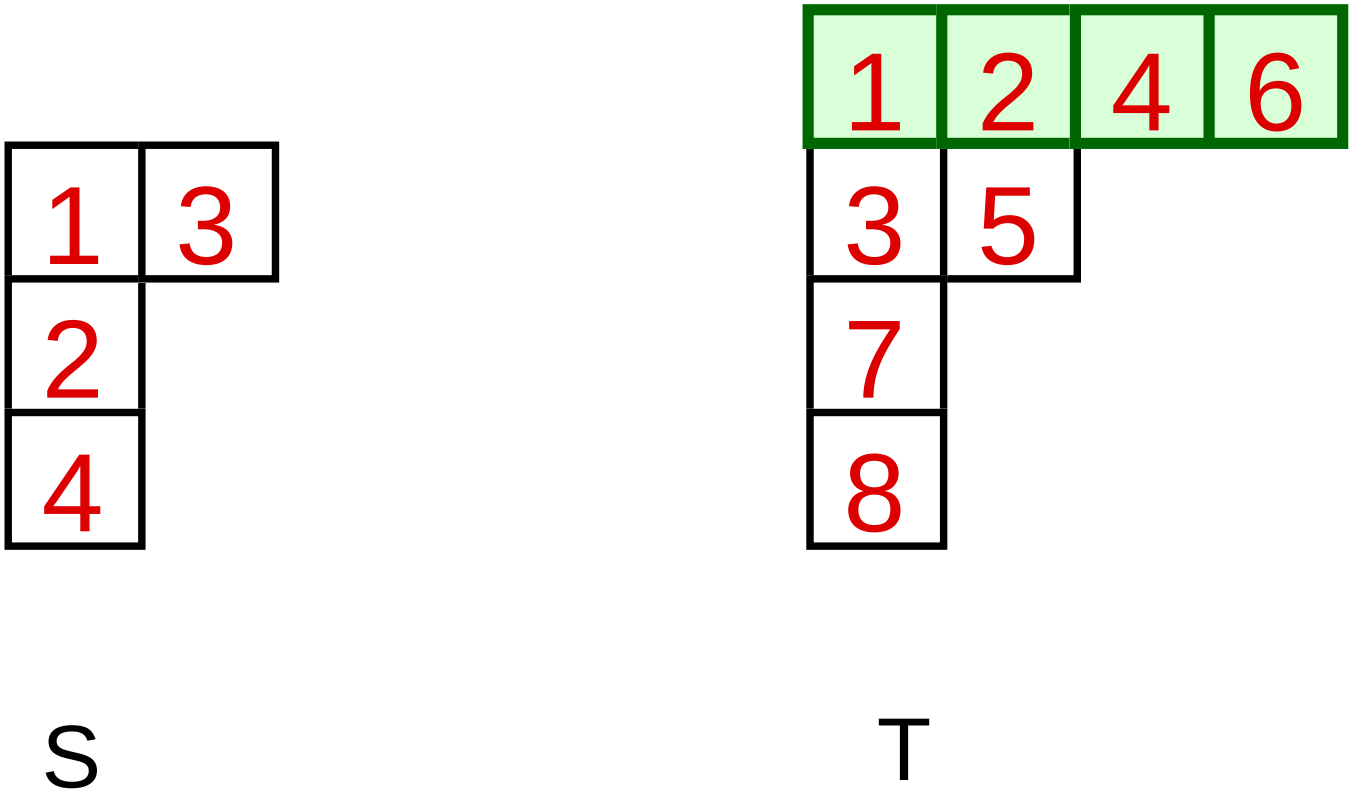
When $p = m \leq n$

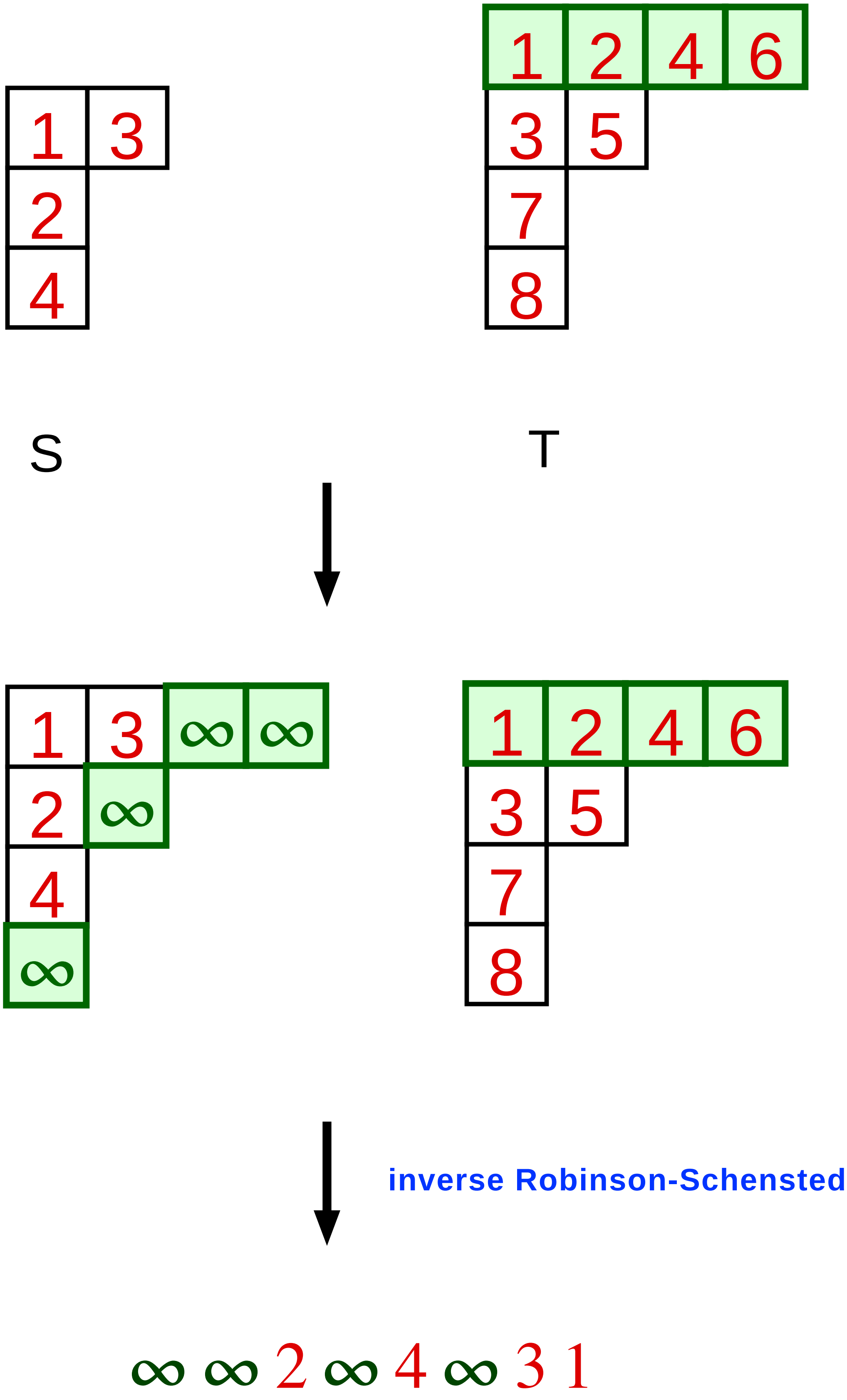


$\dim(S^\lambda) = \#$ standard tableaux T of shape λ

$\dim(\tilde{H}_{m-1}(M_{m,n};\mathbb{C})) = \#$ pairs of standard tableaux (S,T) of sizes m and n, whose shapes differ by a row.

From each such pair of standard young tableaux we construct a cycle and a cocycle of top dimension.





inverse Robinson-Schensted

∞ ∞ 2 ∞ 4 ∞ 3 1

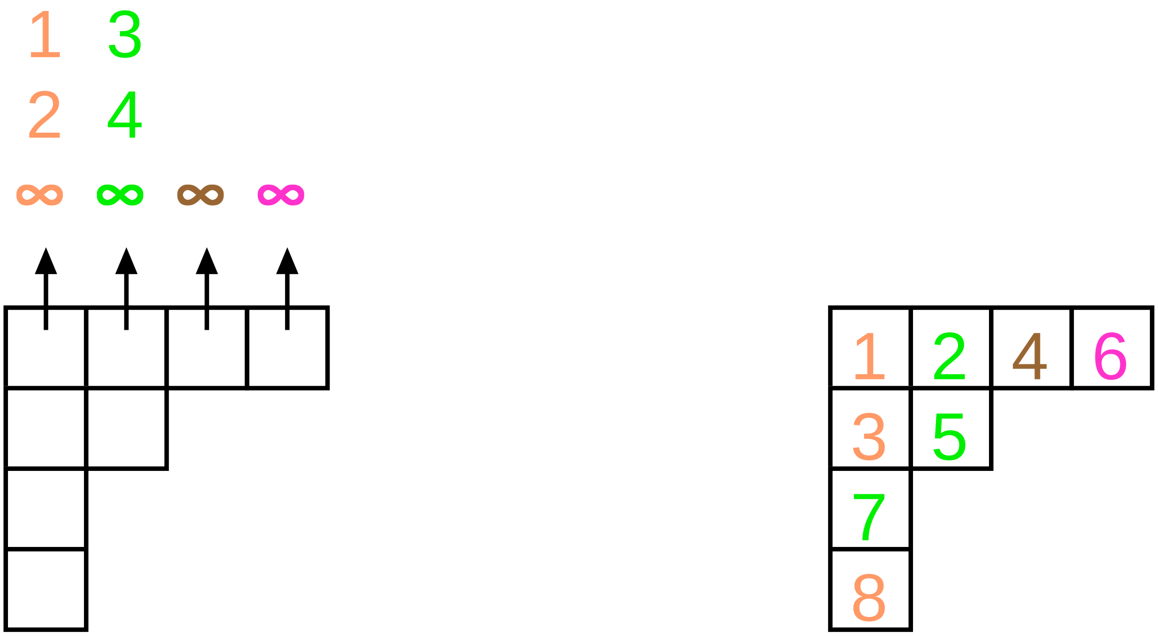
Cocycle:

$$\begin{array}{cccccccc} \infty & \infty & 2 & \infty & 4 & \infty & 3 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array}$$

$$\gamma(S,T) = (23, 45, 37, 18)$$

	1	2	3	4	5	6	7	8
1								X
2			X					
3							X	
4					X			

Cycle:



$M_{\{1,2\}\{1,3,8\}}$ and $M_{\{3,4\}\{2,5,7\}}$ are pseudomanifolds. So top homology is cyclic. Let

$$\begin{aligned}\tilde{H}_1(M_{\{1,2\}\{1,3,8\}}) &= \langle \alpha \rangle \\ \tilde{H}_1(M_{\{3,4\}\{2,5,7\}}) &= \langle \beta \rangle\end{aligned}$$

Define

$$\rho(S,T) = \alpha * \beta \in \tilde{H}_3(M_{4,8})$$

Shareshian & MW:

- $\{\rho(S,T) \mid (S,T)$ is a Friedman-Hanlon pair$\}$ is a basis for $\tilde{H}_{m-1}(M_{m,n})$
- $\{\gamma(S,T) \mid (S,T)$ is a Friedman-Hanlon pair$\}$ is a basis for the free part of $\tilde{H}^{m-1}(M_{m,n})$

Idea of Proof: We find an ordering of the pairs of standard tableaux

$$(S_1, T_1), \ldots, (S_k, T_k)$$

so that the matrix

$$(\langle \rho(S_i, T_i), \gamma(S_j, T_j) \rangle)_{i,j=1,\ldots,k}$$

is triangular with 1's on the diagonal.

What about $2m - n = 2, 3, 4, 6, 7, 10$?

Shareshian and MW (2001): Free when $2m - n = 2$

Conjecture:

- $3, 4$ - free

- $6, 7, 10$ and m, n sufficiently large - $\mathbb{Z}_3^r$

Generalizations and Variations

Hypergraph version: Björner, Lovász, Vrécica & Živaljević(1992), Björner & Eriksson(1999), Ksontini(2000), Shareshian(2000), Shareshian & MW(2001), Athanasiadis(2002)

Directed graph and multigraph versions: Björner & Welker(1998), MW(1999), Shareshian & MW (2002)

General bounded degree: Reiner & Roberts(1997), Karaguezian, Reiner & MW(1999), Dong(1999), MW(1999)

B_n-analogue and wreath product generalization: Garst(1979), MW(2001)

Coxeter-like complexes: Babson & Reiner(2001)