Homology of posets of partitions, graphs, and trees

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Based on joint work with John Shareshian

What do these 3 simplicial complexes have in common?

- order complex of partition lattice Π_n
- complex of disconnected graphs on $[n]$
- complex of homeomorphically irreducible trees on $[n+1]$.

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- order complex of partition lattice Π_n
- complex of disconnected graphs on $[n]$
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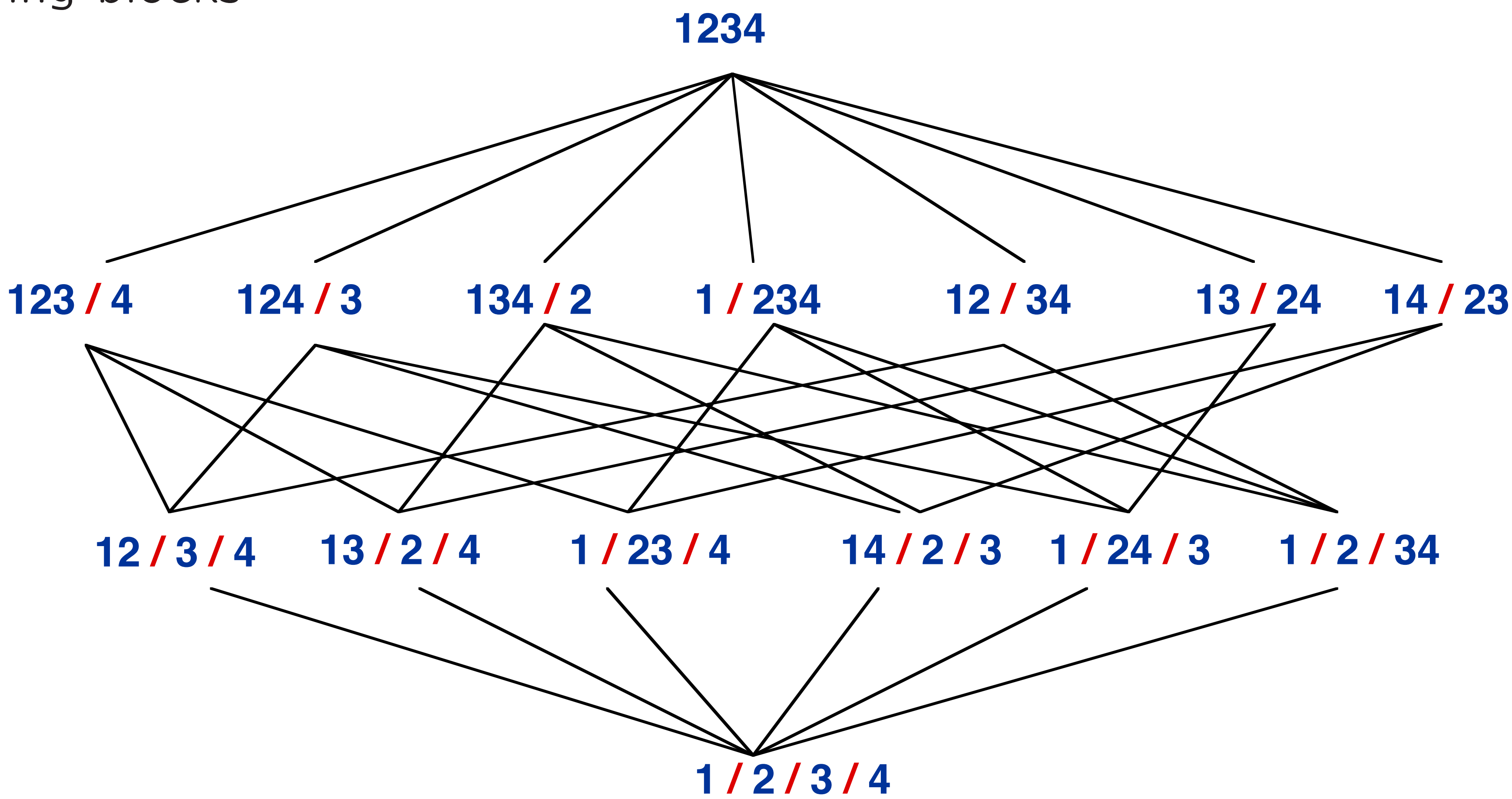
They all have the homotopy type of

$$\bigvee_{(n-1)!} \mathbb{S}^{n-3}$$

Homology of the partition Lattice

$\Pi_n :=$ poset of partitions of $[n] := \{1, 2, \ldots, n\}$ ordered by merging blocks



Homology of $\bar{\Pi}_n := \Pi_n \setminus \{0, 1\}$ vanishes below the top dimension.

Top homology is a subspace of the chain space $C_{n-3}(\bar{\Pi}_n)$.

Top cohomology is a quotient of the chain space $C_{n-3}(\bar{\Pi}_n)$.

Suppose we have

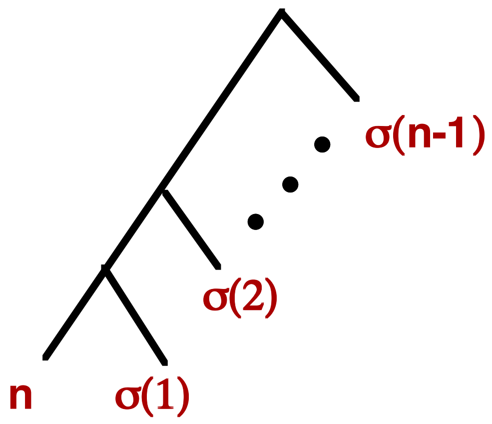
- A collection of cycles $\{\rho_i : i = 1, \ldots, (n-1)!\}$

- A collection of maximal chains, $\{c_i : i = 1, \ldots, (n-1)!\}$

If coef c_i in $\rho_j = \delta_{i,j}$ then

$\{\rho_i : i = 1, \ldots, (n-1)!\}$ is a basis for homology
$\{c_i : i = 1, \ldots, (n-1)!\}$, is a dual basis for cohomology

Basis for top cohomology - **comb basis**



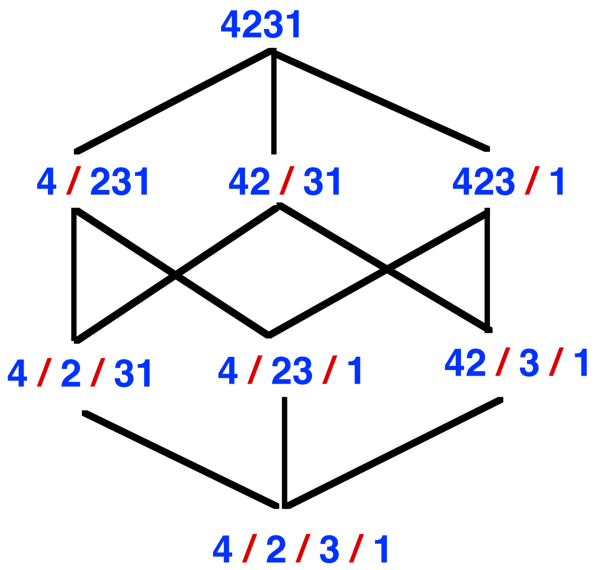
$\sigma \in \mathfrak{S}_{n-1}$

$$\{n, \sigma(1)\} \lessdot \{n, \sigma(1), \sigma(2)\} \lessdot \cdots \lessdot \{n, \sigma(1), \sigma(2), \ldots, \sigma(n-2)\}$$

unique nonsingleton blocks

Basis for top homology - **splitting basis**

For each $\sigma \in \mathfrak{S}_n$, let Π_σ be the subposet of Π_n consisting of partitions obtained by splitting σ.



Π_{4231}

Basis for top homology - **splitting basis**

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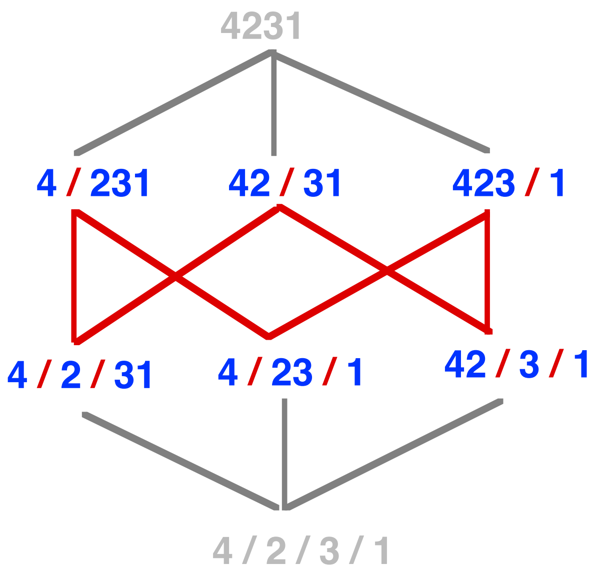
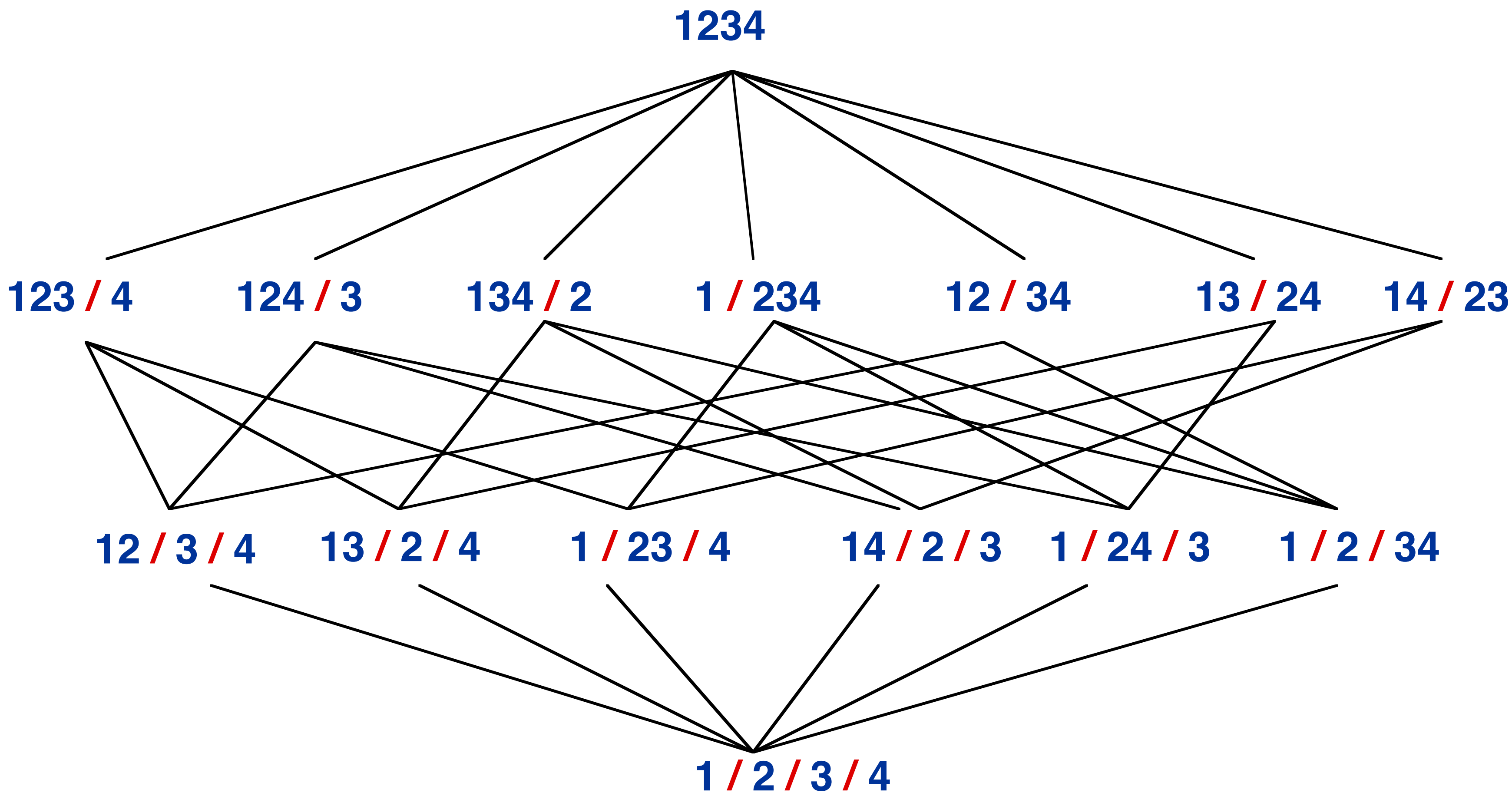


Π_{4231}

Π_σ is isomorphic to the lattice of subsets of $[n-1] \Rightarrow$ $\Delta(\bar{\Pi}_\sigma)$ is homeomorphic to an $(n-3)$-sphere.

Let ρ_σ be the fundamental cycle of $\Delta(\bar{\Pi}_\sigma)$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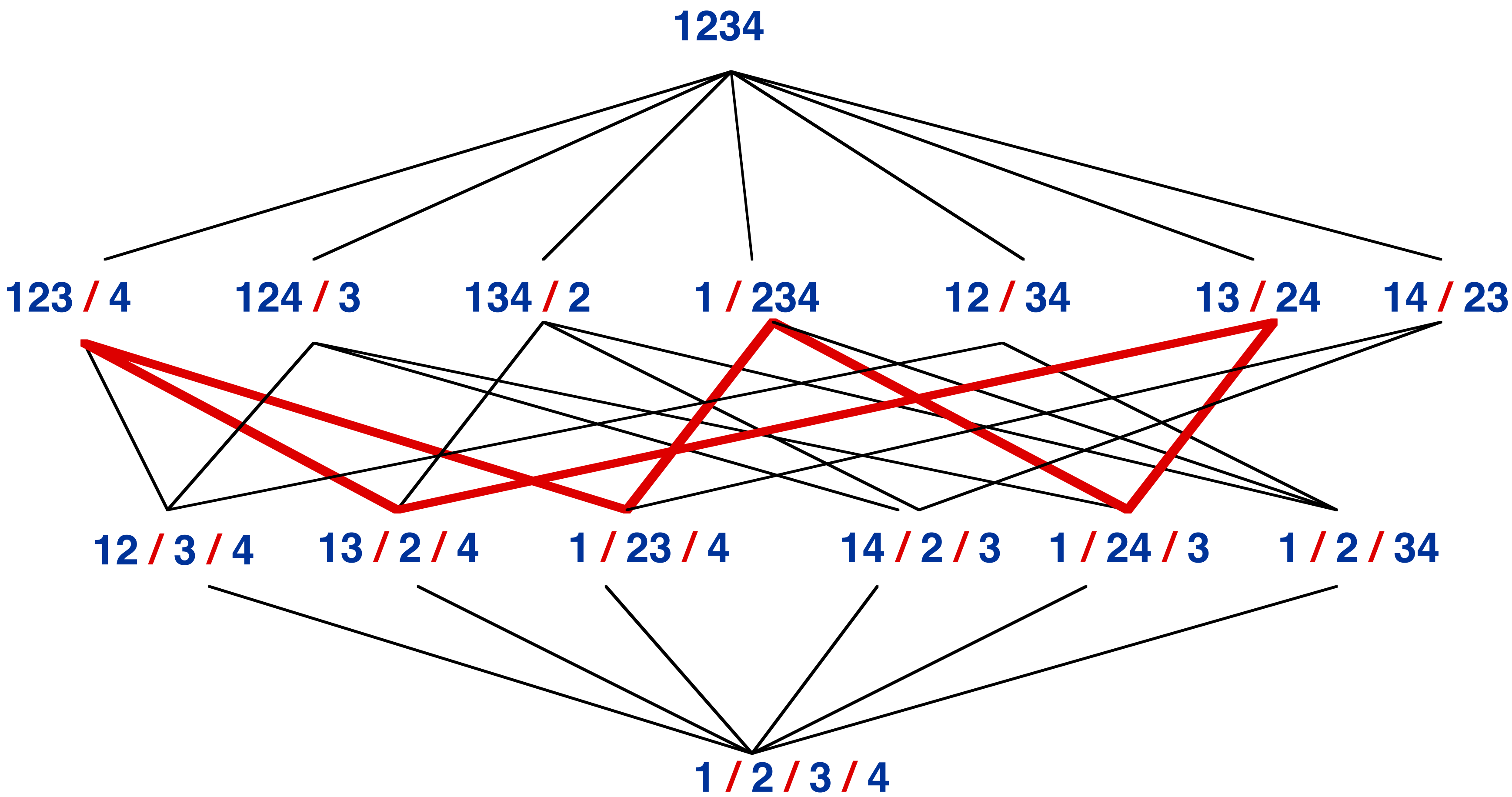
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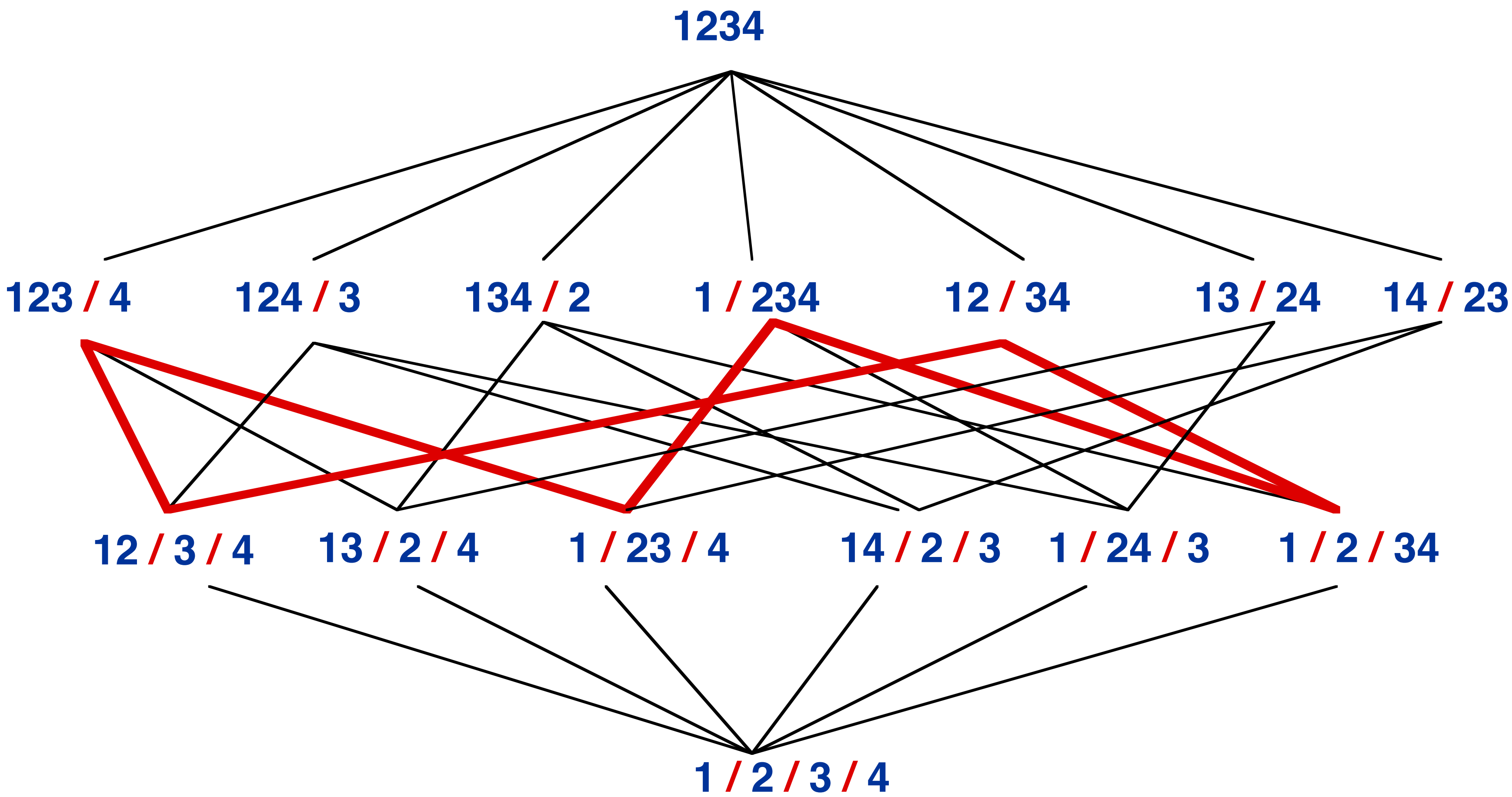
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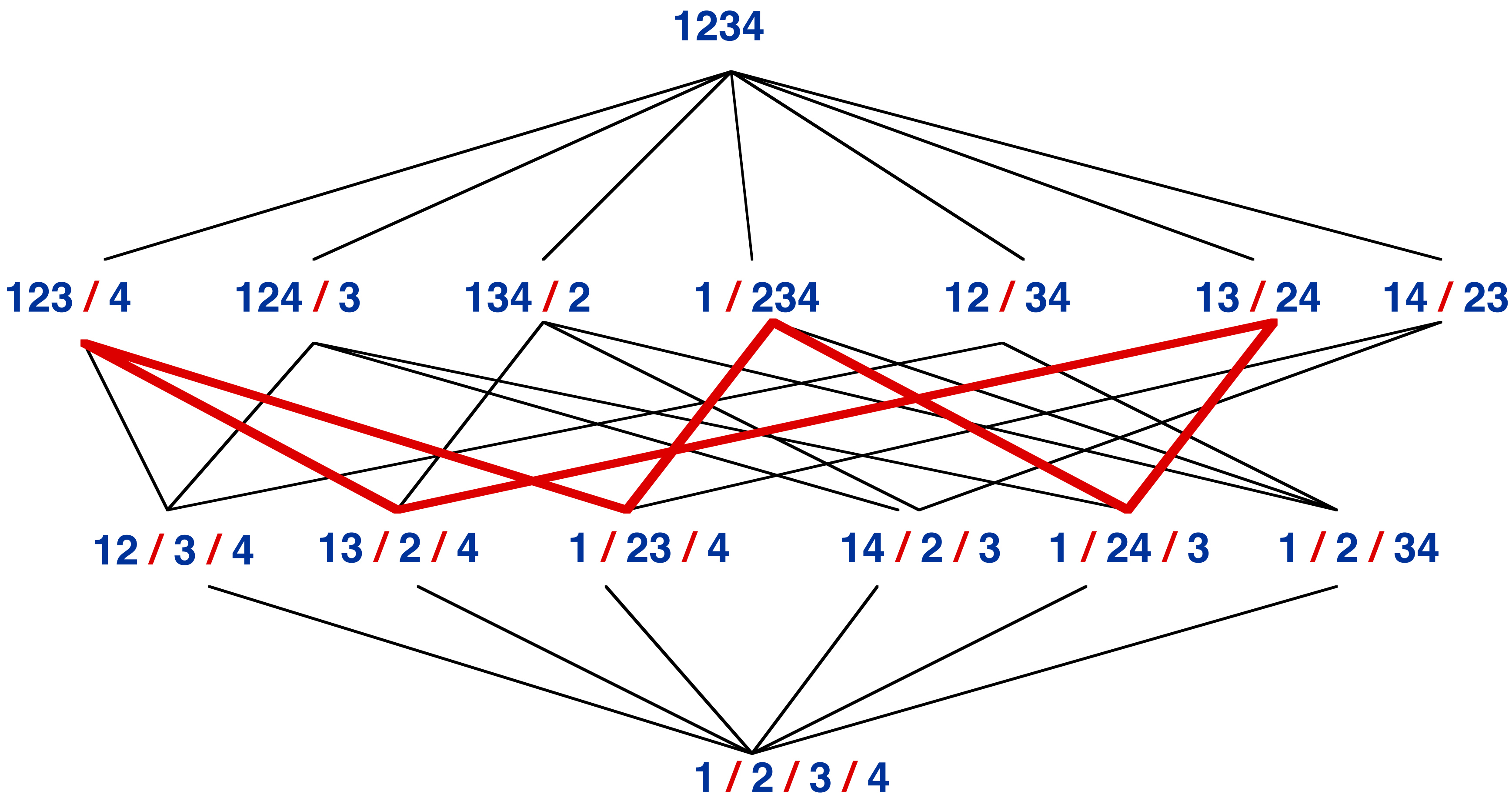
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Splitting basis for homology

$$\{\rho_\sigma : \sigma \in \mathfrak{S}_n,\ \sigma(n) = n\}$$

dual to comb basis for cohomology

$$\{c_\sigma : \sigma \in \mathfrak{S}_n,\ \sigma(n) = n\}.$$

Proof: Let $\tau, \sigma \in \mathfrak{S}_n$ with $\sigma(n) = \tau(n) = n$.

Then c_τ is a chain of Π_σ iff $\tau = \sigma$.

Splitting basis for homology

$$\{\rho_\sigma : \sigma \in \mathfrak{S}_n,\ \sigma(n) = n\}$$

dual to comb basis for cohomology

$$\{c_\sigma : \sigma \in \mathfrak{S}_n,\ \sigma(n) = n\}.$$

Symmetric group $\mathfrak{S}_n$ acts on partition lattice Π_n

$$(2,3)\ [134/58/267] = [124/58/367]$$

Restriction of representation of $\mathfrak{S}_n$ to $\mathfrak{S}_{n-1}$ on $H_{n-3}(\Pi_n)$ is regular representation (Stanley 1982).

Stanley, Klyachko, Joyal (1982) As $\mathfrak{S}_n$ representations

$$H_{n-3}(\bar{\Pi}_n) \; \tilde{=}_{\mathfrak{S}_n} \; \mathsf{lie}_n \otimes \mathsf{sgn}_n$$

lie_n is multilinear component of free Lie algebra on n generators.

Combinatorial proofs:

Barcelo (1990): dual nbc basis $\rightarrow$ Lyndon basis

MW (1998): comb basis $\rightarrow$ comb basis

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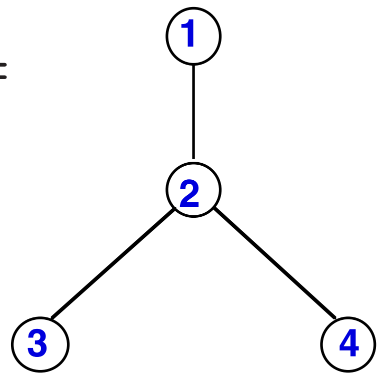
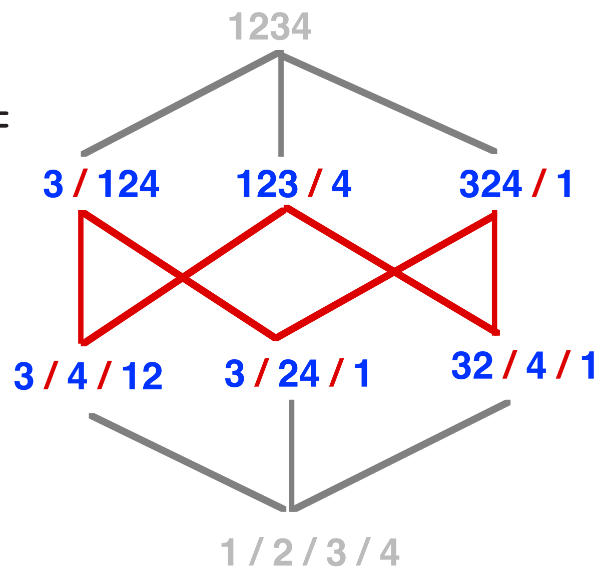
binary tree generating set $\longrightarrow$ binary tree generating set

nbc basis -Björner (1982): For each tree T on node set $[n]$, let Π_T be the subposet of Π_n consisting of partitions obtained by removing edges of T.

$T =$ 

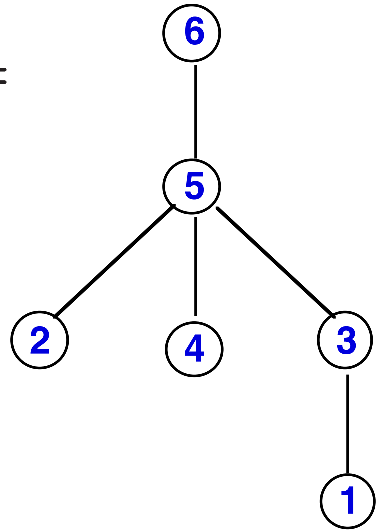
$\Pi_T =$ 

Π_T is isomorphic to the lattice of subsets of $[n-1]$

Let ρ_T be the fundamental cycle of $\Delta(\bar{\Pi}_T)$.

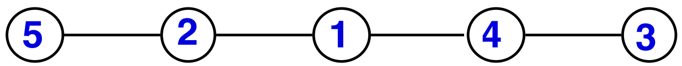
- nbc basis for homology

$$\{\rho_T : T \text{ increasing rooted tree on node set } [n]\}$$

$T =$ 

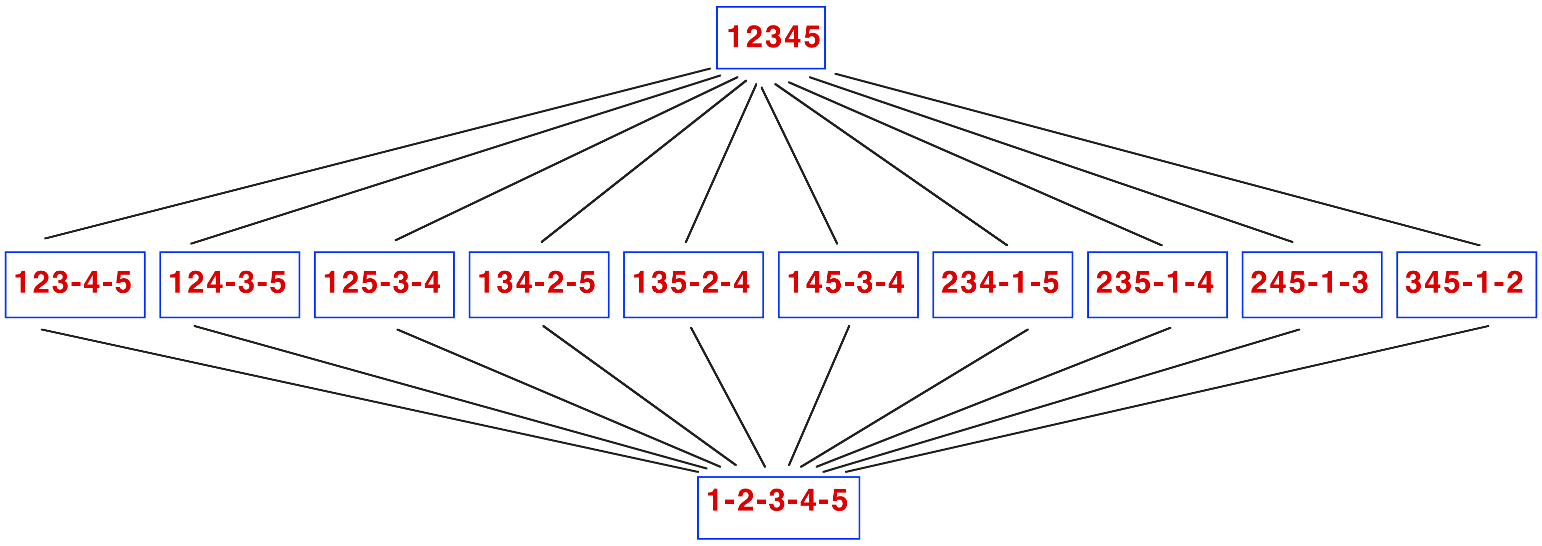
- splitting basis for homology

$$\{\rho_T : T \text{ linear rooted tree on node set } [n] \text{ with root } n\}$$

$T =$ 

The 1 mod k partition poset

$\Pi_n^{1 \bmod k}$ = subposet of Π_n consisting of partitions with block sizes $\equiv 1 \bmod k$



$\Delta(\Pi_5^{1 \bmod 2})$ has homotopy type of a wedge of 9 0-spheres.

Björner (1983): $\Pi_{kn+1}^{1 \bmod k}$ is shellable

Stanley (1983): $|\mu(\Pi_{2n+1}^{1 \bmod 2})| = (2n-1)!!^2$

$$(2n-1)!! := 1 \cdot 3 \cdot 5 \cdots (2n-1)$$

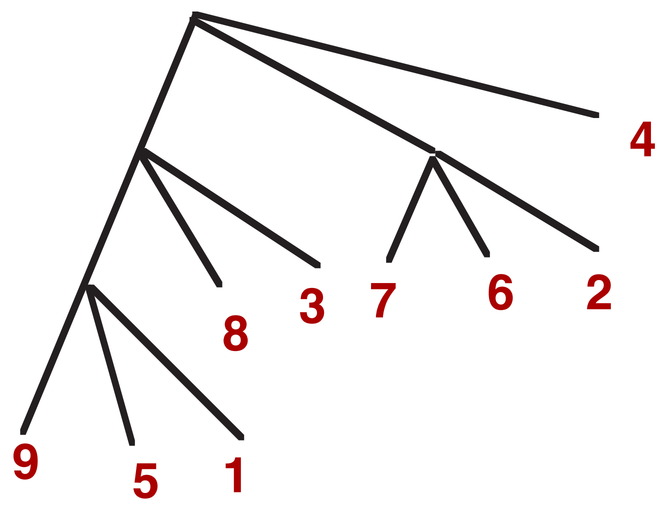
Calderbank-Hanlon-Robinson (1986): computed representation of symmetric group on homology of $\Pi_{kn+1}^{1 \bmod k}$

Hanlon-MW (1995): Lie k-algebra

$$H_{n-2}(\Pi_{kn+1}^{1 \bmod k}) \cong_{\mathfrak{S}_{kn+1}} \mathrm{lie}_n^k.$$

$\mathrm{lie}_n^k :=$ multilinear part of free Lie k-algebra

proof relies on brush basis

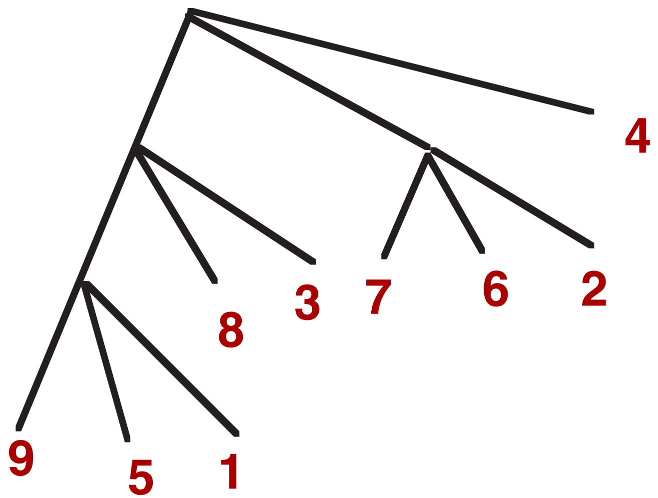


For each node x, let $m(x) =$ largest leaf in the tree rooted at x.

A k-brush is a $(k+1)$-ary tree on leaf set $[kn+1]$ such that for each node y, the child of y with the smallest m value is a leaf.

A 1-brush is a comb. # combs on leaf set $[n+1] = n!$.

2-brushes on leaf set $[2n+1] = (2n-1)!!^2$.



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Hanlon & MW (1995): The k-brushes form bases for

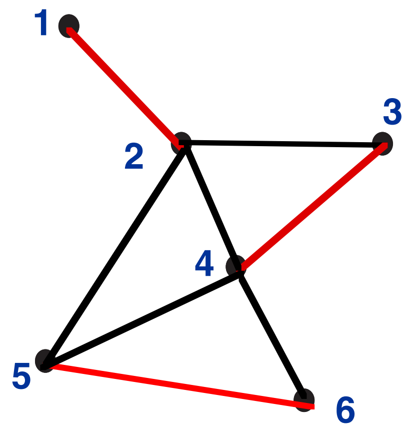
- $H^{n-2}(\bar{\Pi}_{kn+1}^{1 \bmod k})$
- lie_n^k

What about basis for homology?

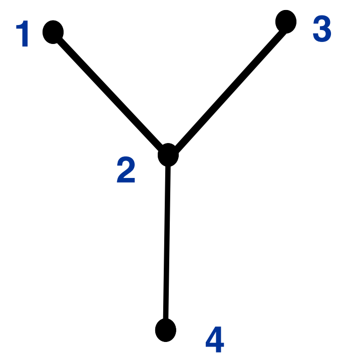
Are there nice k-analogs of the splitting basis or the nbc basis ?

The no-perfect matching complex - Linusson, Shareshian & Welker (2003)

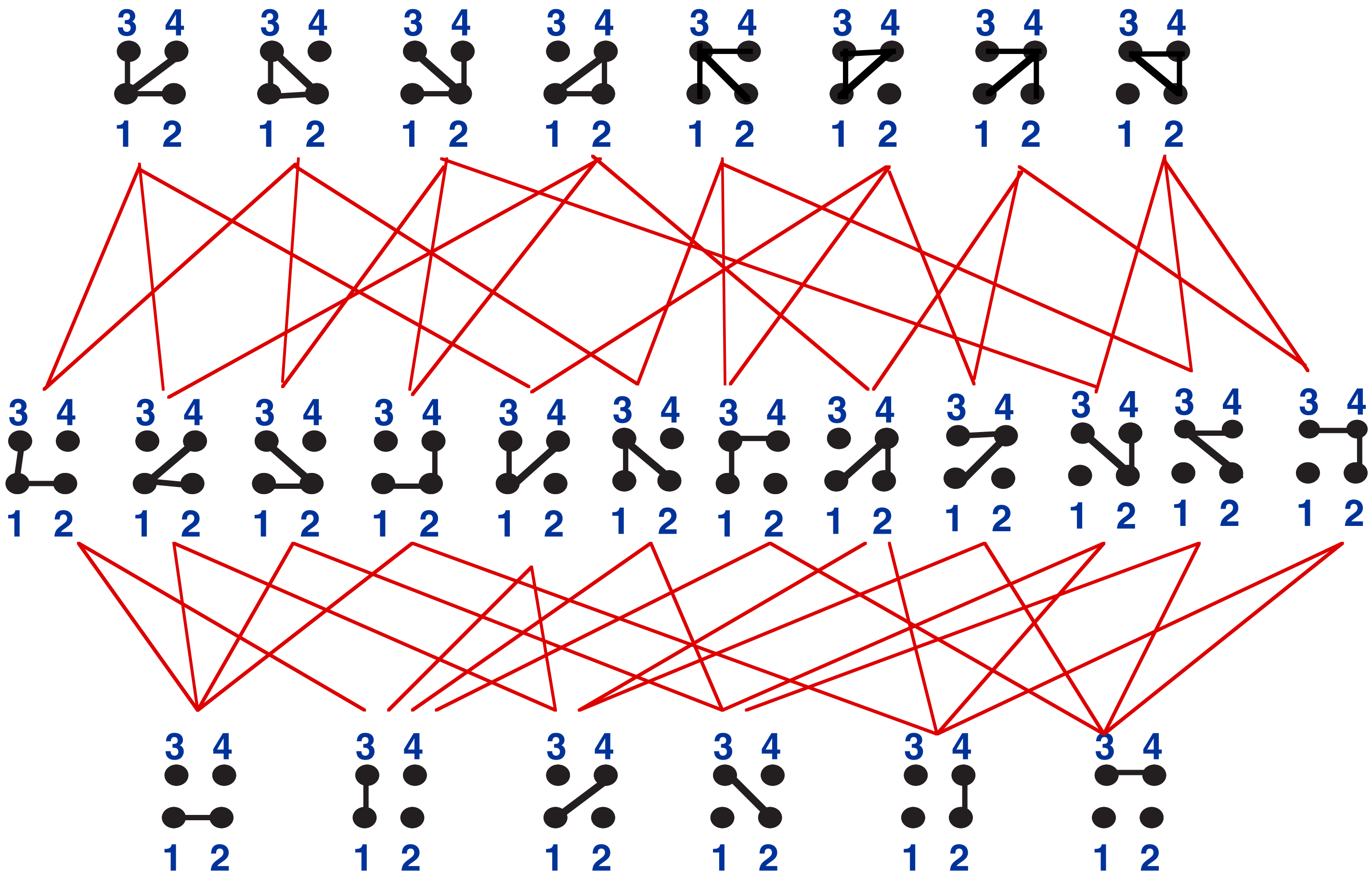
NPM_{2n} = simplicial complex of nonempty graphs on node set $[2n]$ with no perfect matching.



perfect matching



no perfect matching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GAP homology software package of Dumas, Heckenback, Saunders and Welker $\Rightarrow$ Betti numbers

$n = 2$: 0, 0, **1**

$n = 3$: 0, 0, 0, 0, 0, **9**, 0, 0, 0, 0

$n = 4$: 0, 0, 0, 0, 0, 0, 0, 0, **225**, 0, 0, 0, 0, 0, 0, 0, 0, . . .

$n = 5$: 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, **11025**, 0, 0, 0, 0, . . .

GAP homology software package of Dumas, Heckenback, Saunders and Welker $\Rightarrow$ Betti numbers

n = 2: 0, 0, $\mathbf{1^2}$

n = 3: 0, 0, 0, 0, 0, $\mathbf{(1 \cdot 3)^2}$, 0, 0, 0, 0

n = 4: 0, 0, 0, 0, 0, 0, 0, 0, $\mathbf{(1 \cdot 3 \cdot 5)^2}$, 0, 0, 0, 0, 0, 0, ...

n = 5: 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, $\mathbf{(1 \cdot 3 \cdot 5 \cdot 7)^2}$, 0, 0, ...

GAP homology software package of Dumas, Heckenback, Saunders and Welker $\Rightarrow$ Betti numbers

n = 2: 0, 0, $\mathbf{1^2}$
n = 3: 0, 0, 0, 0, 0, $\mathbf{(1 \cdot 3)^2}$, 0, 0, 0, 0
n = 4: 0, 0, 0, 0, 0, 0, 0, 0, $\mathbf{(1 \cdot 3 \cdot 5)^2}$, 0, 0, 0, 0, 0, 0, ...
n = 5: 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, $\mathbf{(1 \cdot 3 \cdot 5 \cdot 7)^2}$, 0, 0, ...

Linusson, Shareshian & Welker: $\Delta(\mathrm{NPM}_{2n})$ has the homotopy type of a wedge of $(2n-3)!!^2$ spheres of dimension $(3n-4)$.

Proof technique: discrete Morse theory and Gallai-Edmonds structure theorem

Björner-Stanley + Linusson-Shareshian-Welker implies

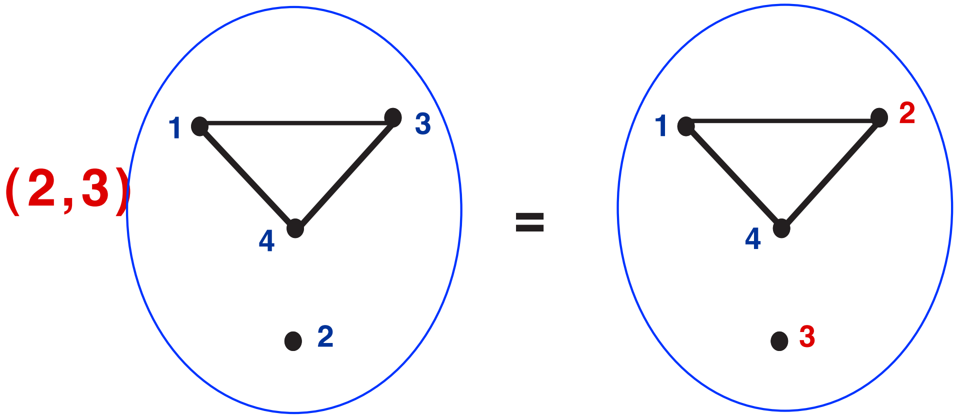
$$H_{n-3}(\Pi_{2n-1}^{1 \bmod 2}) \stackrel{\sim}{=} H_{3n-4}(\mathsf{NPM}_{2n})$$

Question: Is there an equivariant isomorphism?

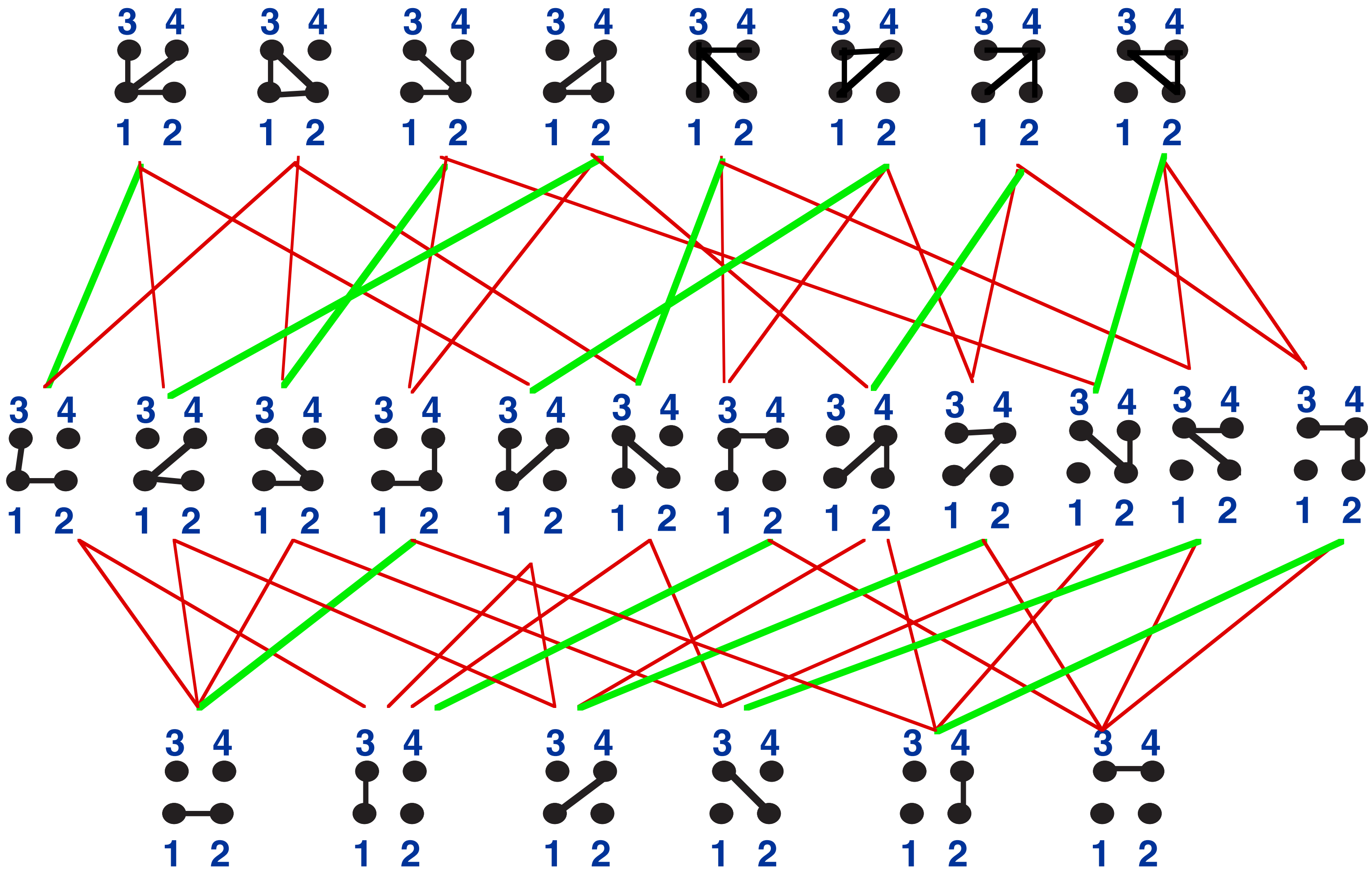
Shareshian & MW: As $\mathfrak{S}_{2n-1}$-modules

$$H_{3n-4}(\mathrm{NPM}_{2n}) \downarrow^{\mathfrak{S}_{2n}}_{\mathfrak{S}_{2n-1}} \cong \mathrm{sgn} \otimes H_{n-3}(\Pi^{1 \bmod 2}_{2n-1})$$

Symmetric group $\mathfrak{S}_{2n}$ acts on the poset NPM_{2n} by relabeling graph nodes.



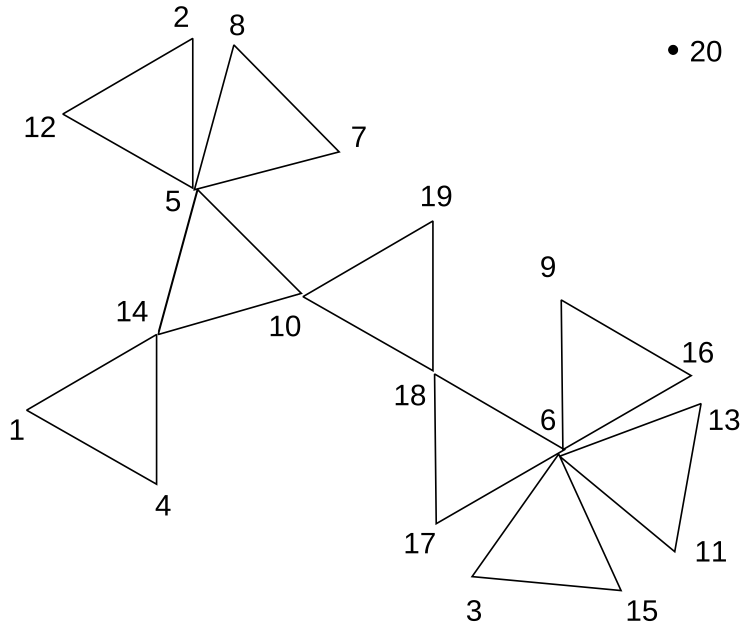
Proof technique: discrete Morse theory and bases



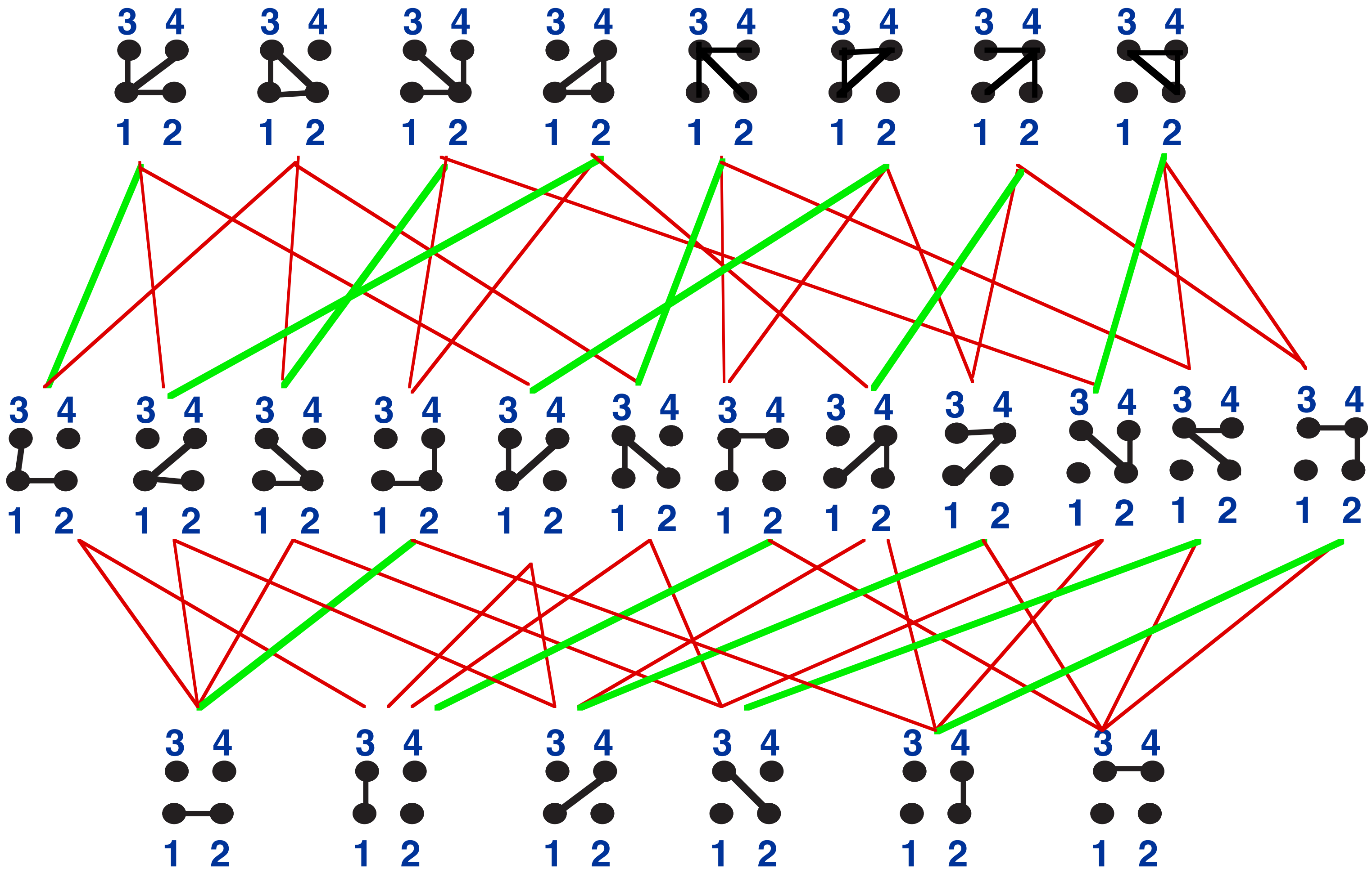
Discrete Morse Theory - Forman (1998)

Let Δ have a Morse matching with c_i critical elements of dimension i for each i. Then Δ has the homotopy type of a CW-complex with c_i cells of dimension i.

Linusson, Shareshian, Welker: nontrivial critical elements are the increasing triangle trees on leaf set $[2n-1]$ together with isolated node $2n$.



The number of increasing triangle trees on node set $[2n-1]$ is $(2n-3)!!^2$.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A graph is called factor critical if removal of any node results in a graph with a perfect matching.

Example: triangle trees

Let FC_{2n-1} be the poset of factor critical graphs on node set $[2n-1]$.

Shareshian and MW:

(a) $H_{3n-4}(\mathsf{NPM}_{2n}) \downarrow^{\mathfrak{S}_{2n}}_{\mathfrak{S}_{2n-1}} \cong_{\mathfrak{S}_{2n-1}} H^*(\mathsf{FC}_{2n-1}) \otimes \mathrm{sgn}$

(b) $H^*(\mathsf{FC}_{2n-1}) \cong_{\mathfrak{S}_{2n-1}} H_{n-3}(\Pi^{1 \bmod 2}_{2n-1})$

(a) $H_{3n-4}(\mathsf{NPM}_{2n}) \downarrow^{\mathfrak{S}_{2n}}_{\mathfrak{S}_{2n-1}} \cong_{\mathfrak{S}_{2n-1}} H^*(\mathsf{FC}_{2n-1}) \otimes \mathsf{sgn}$

Lemma: Let Δ be a G-simplicial complex with a Morse matching. If the (nontrivial) critical elements form a G-invariant upper order ideal Q of $F(\Delta)$ with a maximum element m then for all i

$$H_i(\Delta) \cong_G H^{r(m)-i}(Q) \otimes \mathsf{sgn}_{G,m}$$

(b) $H^*(\mathsf{FC}_{2n-1}) \stackrel{\sim}{=}_{\mathfrak{S}_{2n-1}} H_{n-3}(\Pi_{2n-1}^{1 \bmod 2})$

Linusson, Shareshian & Welker: Basis for $H^*(\mathsf{FC}_{2n-1})$,

$$\{\gamma_T \mid T \in \mathsf{IT}_{2n-1}\},$$

IT_{2n-1} := set of increasing triangle trees on node set $[2n-1]$.

Shareshian & MW: Bijection

$$\mathsf{IT}_{2n-1} \quad \rightarrow \quad \{2-\text{brushes on } [2n-1]\}$$

(b) $H^*(\mathsf{FC}_{2n-1}) \cong_{\mathfrak{S}_{2n-1}} H_{n-3}(\Pi^{\mathsf{odd}}_{2n-1})$

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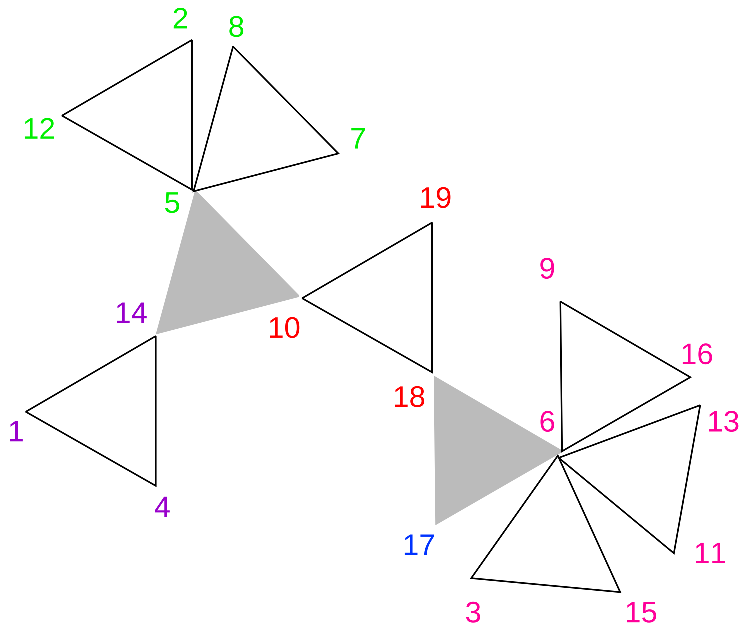
not equivariant

Triangle tree splitting basis for $H_{n-3}(\Pi_{2n-1}^{1 \bmod 2})$

For triangle tree T on node set $[2n-1]$, split T by choosing some triangles of T and removing all the edges of these triangles.



$1, 4, 14$ / $12, 5, 2, 8, 7$ / $10, 18, 19$ / 17 / $3, 15, 11, 13, 16, 9, 6$

Let Π_T be the subposet consisting of all the partitions obtained by splitting T.

Π_T is isomorphic to the lattice of subsets of $[n-1]$.
$\rho_T :=$ fundamental cycle of $\Delta(\bar{\Pi}_T)$

- $\{\rho_T \mid T \in \mathrm{IT}_{2n-1}\}$ is a basis for $H_{n-3}(\Pi_{2n-1}^{1 \bmod 2})$
- The bijection $\rho_T \mapsto \gamma_T$ determines an $\mathfrak{S}_{2n-1}$-module isomorphism

$$H_{n-3}(\Pi_{2n-1}^{1 \bmod 2}) \to H^*(\mathrm{FC}_{2n-1})$$

$(k+1)$-clique tree splitting basis for $H_{n-2}(\Pi_{nk+1}^{1 \bmod k})$

A connected graph T is a k-clique tree if either

- T is a single node, or
- T contains a k-clique, the removal of whose edges disconnects the graph into k connected components that are all k-clique trees.

2-clique tree = ordinary tree, 3-clique tree = triangle tree

$(k+1)$-clique tree splitting basis for $H_{n-2}(\Pi_{nk+1}^{1 \bmod k})$

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For $(k+1)$-clique tree T on node set $[nk+1]$, let Π_T be subposet of $\Pi_{nk+1}^{1 \bmod k}$ - remove edges of $(k+1)$-cliques.

fundamental cycle ρ_T

basis: $\{\rho_T : T$ increasing $(k+1)$-clique tree$\}$

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$k = 1$: Björner s nbc basis

Problem: Is there a generalization of the nonperfect matching poset or the factor critical graph poset whose homology is isomorphic to the $\mathfrak{S}_{nk+1}$-module $H_{n-2}(\Pi_{nk+1}^{1 \bmod k})$?

The complex of not k-edge connected graphs - Jakob Jonsson s Thesis (KTH, 2005) page 298.

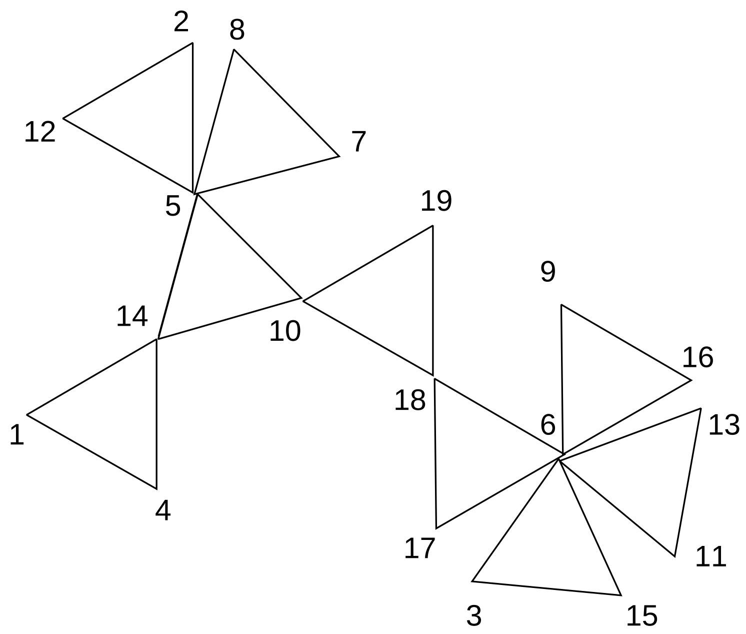
A graph is said to be k-edge connected if removal of any set of at most $k-1$ edges leaves the graph connected.

1-edge connected = connected

trees are 1-edge connected but not 2-edge connected

triangle trees are 2-edge connected but not 3-edge connected.

k-clique trees are $(k-1)$-edge connected but not k-edge connected.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NEC_n^k = complex of not k-edge connected graphs on $[n]$.

Jonsson: Bottom nonvanishing integral homology of NEC^2_{2n+1} occurs in dimension $3n-2$ and has rank $((2n-1)!!)^2$.

Proof: Discrete Morse theory

basis: $\{\partial T : T$ is an increasing triangle tree$\}$

NEC_n^k = complex of not k-edge connected graphs on $[n]$.

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Proof: Discrete Morse theory

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Shareshian & MW (2005):

$$H_{3n-2}(\mathsf{NEC}_{2n+1}^2) \cong_{\mathfrak{S}_{2n-1}} H_{n-2}(\Pi_{2n+1}^{1 \bmod 2}) \otimes \mathsf{sgn}_{2n+1}$$

Conjecture (Shareshian and MW): Bottom nonvanishing homology of NEC^k_{kn+1} occurs in dimension $\binom{k+1}{2}n - 2$ and

$$H_{\binom{k+1}{2}n-2}(\mathsf{NEC}^k_{kn+1}) \cong_{\mathfrak{S}_{kn+1}} H_{n-2}(\Pi^{1 \bmod k}_{kn+1}) \otimes \mathsf{sgn}^{\otimes k+1}_{kn+1}$$

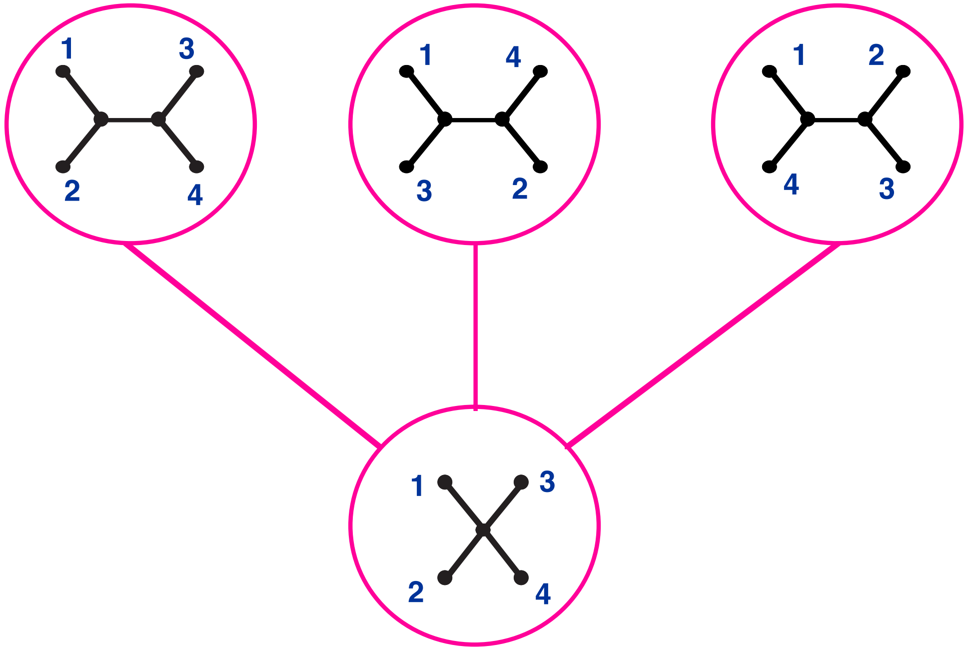
$k = 1$: well-known

$k = 2$: Jonsson, Shareshian-MW

Tree poset Vogtmann (1990)

$\mathcal{T}_n$ = poset of trees on leaf set $[n]$ where all internal nodes have degree ≥ 3.

$T < T'$ if T can be obtained from T' by contracting internal edges.



$\Delta(\mathcal{T}_n)$ has the homotopy type of a wedge of $(n-2)!$ $(n-4)$-spheres.

So $\Delta(\mathcal{T}_{n+1}) \simeq \Delta(\Pi_n)$

Robinson (2004), Ardila and Klivans (2004): $\Delta(\mathcal{T}_{n+1})$ and $\Delta(\Pi_n)$ are homeomorphic

Robinson and Whitehouse (1996):
As $\mathfrak{S}_n$-modules

$$H_{n-3}(\mathcal{T}_{n+1}) \downarrow_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}} \cong H_{n-3}(\Pi_n)$$

As $\mathfrak{S}_{n+1}$-modules

$$H_{n-3}(\mathcal{T}_{n+1}) \;\cong\; H_{n-3}(\Pi_n) \uparrow_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}} - H_{n-2}(\Pi_{n+1})$$

Other occurrences of Vogtmann complex and Whitehouse module:

- Cyclic Lie operad - Kontsevich (1993), Getzler, Kapranov (1995)
- Not 2-connected graph complex - Vassiliev (1997), Babson, Björner, Linusson, Shareshian, Welker (1997), Turchin (1997)
- Kernel of Varchenko operator - Hanlon & Stanley (1998)
- Bounded block size partition poset - Sundaram (1998)
- Nonmodular partition poset - Sundaram (1999)
- Phylogenetic tree space -Billera, Holms, Vogtmann (2003)
- Bergman complexes- Sturmfels, Ardila, Klivans (2004)

k-analog of Whitehouse Module - Hanlon (1996)

$\mathcal{T}^k_{kn+2}$ = subposet of $\mathcal{T}_{kn+2}$ consisting of trees in which all internal nodes have degree $\equiv 2 \bmod k$.

- As $\mathfrak{S}_{kn+1}$-modules

$$H_{n-2}(\mathcal{T}^k_{kn+2}) \downarrow^{\mathfrak{S}_{kn+1}}_{\mathfrak{S}_{kn+2}} \cong H_{n-2}(\Pi^{1 \bmod k}_{kn+1})$$

- As $\mathfrak{S}_{kn+2}$-modules

$$H_{n-2}(\mathcal{T}^k_{kn+2}) \cong H_{n-2}(\Pi^{1 \bmod k}_{kn+1}) \uparrow^{\mathfrak{S}_{kn+2}}_{\mathfrak{S}_{kn+1}} - H_{n-2}(\Pi^{1 \bmod k}_{kn+2})$$

Trappmann and Ziegler, MW: $\mathcal{T}^k_{kn+2}$ is shellable, brush basis

Robinson s homeomorphism restricts: $\Delta(\mathcal{T}^k_{kn+2})$ and $\Delta(\Pi^{1 \bmod k}_{kn+1})$ are homeomorphic.

Shareshian & MW: As $\mathfrak{S}_{2n-1}$-modules

$$H_{3n-4}(\mathsf{NPM}_{2n}) \downarrow^{\mathfrak{S}_{2n}}_{\mathfrak{S}_{2n-1}} \cong H_{n-3}(\mathcal{T}^2_{2n}) \downarrow^{\mathfrak{S}_{2n}}_{\mathfrak{S}_{2n-1}}$$

Conjecture (Linusson, Shareshian, Welker): As $\mathfrak{S}_{2n}$-modules

$$H_{3n-4}(\mathsf{NPM}_{2n}) \cong H_{n-3}(\mathcal{T}^2_{2n})$$