On the top homology of hypergraph matching complexes

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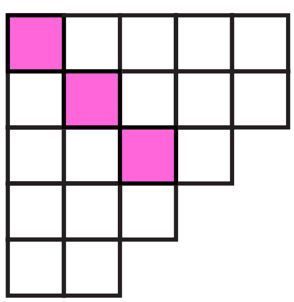
Littlewood:

$$\prod_{i\le j}(1-x_ix_j)\prod_i(1-x_i)^{-1} = \sum_{\lambda=\lambda'}(-1)^{(|\lambda|-d(\lambda))/2}\, s_\lambda$$

λ is a partition
λ' is the conjugate of λ
s_λ is a Schur function

$d(\lambda)$ is the size of the diagonal of λ.

λ =



d(λ) = 3

Macdonald: Weyl denominator formula for root system B_n

Burge: Combinatorial proof - sign reversing involution

Stanley: Representation theoretic interpretation?

- Jozefiak and Weyman: minimal free resolution of quotient of a polynomial ring

- Sigg: homology of free two-step nilpotent Lie algebra

Extract degree n terms

$$\sum_k (-1)^k \, e_k[h_2] \, h_{n-2k} = \sum_k (-1)^k \sum_{\substack{\lambda :\ \lambda \vdash n \\ \lambda = \lambda' \\ d(\lambda) = n-2k}} s_\lambda$$

Hopf trace formula for the **matching complex**

The matching complex M_n is the simplicial complex of matchings on

$$[n] := \{1, 2, \ldots, n\}$$

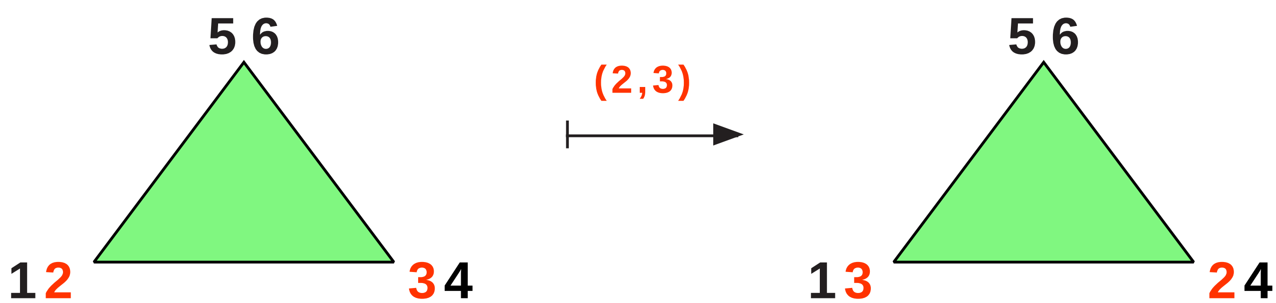
vertices := 2 element subsets of $[n]$

faces := sets of pairwise disjoint 2 element subsets of $[n]$

$M_4 =$

Homology of the Matching Complex

The symmetric group $\mathfrak{S}_n$ acts on M_n by permuting node labels.



This induces a representation of $\mathfrak{S}_n$ on $C_k(M_n;\mathbb{C})$ and $\tilde{H}_k(M_n;\mathbb{C})$.

$$C_{k-1}(M_n) \cong (S^{1^k}[S^2] \otimes S^{n-2k}) \uparrow_{\mathfrak{S}_k[\mathfrak{S}_2]\times\mathfrak{S}_{n-2k}}^{\mathfrak{S}_n}$$

where S^λ is the irreducible $\mathfrak{S}_n$-module indexed by λ.

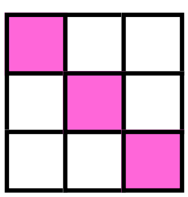
$$\text{ch } C_{k-1}(M_n) = e_k[h_2]h_{n-2k}$$

Theorem (Bouc, 1990). As $\mathfrak{S}_n$-modules,

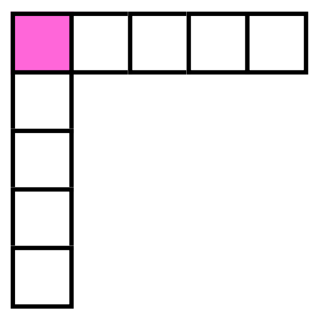
$$\tilde{H}_{k-1}(M_n;\mathbb{C}) \cong \bigoplus_{\substack{\lambda:\ \lambda \vdash n \\ \lambda = \lambda' \\ d(\lambda) = n-2k}} S^{\lambda}$$

Example: $\tilde{H}_{k-1}(M_9;\mathbb{C})$

Self-conjugate shapes with 9 cells:



$k = \frac{9-3}{2} = 3$



$k = \frac{9-1}{2} = 4$

$\tilde{H}_2(M_9;\mathbb{C}) = S^{333}$

$\tilde{H}_3(M_9;\mathbb{C}) = S^{51111}$

The r-matching complex $M_n(r)$ is the simplicial complex of r-matchings on $[n]$

$$\begin{aligned} \text{vertices} &:= r-\text{subsets of } [n] \\ \text{faces} &:= \text{sets of pairwise disjoint} \\ & \quad\; r-\text{subsets of } [n] \end{aligned}$$

Generalization of Littlewood identity?
Generalization of Bouc's Theorem?

$$\sum_k (-1)^k \, e_k[h_r] \, h_{n-rk} = \sum_k (-1)^k \mathrm{ch} \tilde{H}_{k-1}(M_n(r))$$

$$\prod_{a \le b \le c} (1 - x_a x_b x_c) \prod_i (1 - x_i)^{-1} = \sum_{n,k} (-1)^k \mathrm{ch} \tilde{H}_{k-1}(M_n(3))$$

$$\mathrm{ch} \tilde{H}_{k-1}(M_n(r)) \le_{Schur} e_k[h_r] h_{n-rk}$$

r= 3

n	$\tilde{H}_{\lfloor n/3 \rfloor -1}(M_n(3))$
4	$(3,1)$
5	$(4,1)+(3,2)$
6	0
7	$(5,1,1)+(3,3,1)$
8	$(6,1,1)+(5,3)+(5,2,1)+(4,3,1)+(3,3,2)$
9	0
10	$(7,1,1,1)+(5,3,1,1)+(3,3,3,1)$
11	$(8,1,1,1)+(7,3,1)+(7,2,1,1)+(6,4,1)$ $+(6,3,2)+(6,3,1,1)+(5,4,3)+(5,4,1,1)$ $+(5,3,3)+(5,3,2,1)+(4,3,3,1)+(3,3,3,2)$
12	0
13	$(9,1,1,1,1)+(7,3,1,1,1)+(5,5,1,1,1)$ $+(5,3,3,1,1)+(3,3,3,3,1)$

Conjecture:

$$\text{ch}\tilde{H}_{k-1}(M_{rk+1}(r)) = e_k[h_r]h_1|_{length=k+1}$$

For any partition λ, let $\lambda^{(k)}$ be the partition obtained from λ by adding a column of size k. Homomorphism $\gamma_k : \Lambda \to \Lambda$ defined by

$$s_\lambda \mapsto s_{\lambda^{(k)}}$$

Plethysm Lemma:

- $length\ e_k[h_r] \leq k$
- $e_k[h_r]|_{length=k} = \gamma_k(h_k[h_{r-1}])$

So the conjecture is equivalent to

$$\text{ch}\tilde{H}_{k-1}(M_{rk+1}(r)) = \gamma_{k+1}(h_k[h_{r-1}])$$

Thm: For $r = 2, 3$ the conjecture holds.

- $r = 2$:

$$\mathrm{ch}\tilde{H}_{k-1}(M_{2k+1}(2)) = s_{(k+1,1,\ldots,1)}$$

- $r = 3$:

$$\mathrm{ch}\tilde{H}_{k-1}(M_{3k+1}(3)) = \sum_{\lambda \in A_k} s_\lambda$$

where A_k consists of partitions $(\lambda_1, \ldots, \lambda_k, 1)$ of $3k + 1$ with each λ_i odd.

Proof uses the Plethysm Lemma and a long exact sequence of Ksontini.

Ksontini's long exact sequence for $r = 3$

$$0 \to \tilde{H}_{k-1}(M_{3k+1}(3)) \downarrow_{\mathfrak{S}_{3k}} \to \tilde{H}_{k-2}(M_{3k-2}(3)) \uparrow^{\mathfrak{S}_{3k}}$$

$$\to \tilde{H}_{k-2}(M_{3k}(3)) \to \tilde{H}_{k-2}(M_{3k+1}(3)) \downarrow_{\mathfrak{S}_{3k}} \to \cdots$$

This is used to eliminate partitions of length less than $k + 1$.