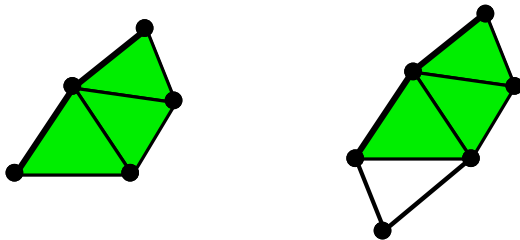


On Sequentially Cohen-Macaulay Complexes and Posets

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Joint work with Anders Björner and Volkmar Welker

Cohen-Macaulay complexes

A simplicial complex Δ is **Cohen-Macaulay** if its Stanley-Reisner ring is Cohen-Macaulay.

Theorem (Reisner, 1976)

Δ is CM \iff $\text{lk}_\Delta F$ is $(\dim \text{lk}_\Delta F - 1)$ -acyclic for all $F \in \Delta$.

$\text{lk}_\Delta F := \{G \in \Delta : F \cup G \in \Delta, F \cap G = \emptyset\}$

Δ is k -acyclic if $\tilde{H}_r(\Delta) = 0$ for all $r \leq k$

Cohen-Macaulay complexes

Quillen, 1978:

A simplicial complex Δ is said to be **homotopy Cohen-Macaulay** if $\mathrm{lk}_\Delta F$ is $(\dim \mathrm{lk}_\Delta F - 1)$ -connected for all $F \in \Delta$.

Δ is k -connected if $\pi_r(\Delta) = 0$ for all $r \leq k$

(pure) shellable \implies HCM \implies CM \implies pure



homotopy type of $\bigvee S^{\dim \Delta}$

Björner and MW (1996): nonpure shellability

Stanley (1996): extension of connection between shellability and commutative algebra to nonpure setting

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Sequentially Cohen-Macaulay Complexes

First Stanley defined **sequentially Cohen-Macaulay rings**.

A **sequentially Cohen-Macaulay complex** is a simplicial complex whose Stanley-Reisner ring is sequentially Cohen-Macaulay.

Stanley gave a characterization of SCM complexes in terms of relative homology and showed

$$\text{pure SCM} \iff \text{CM}$$

$$\text{(nonpure) shellable} \implies \text{SCM}$$

Sequentially Cohen-Macaulay Complexes

Theorem (Duval, 1996)

Let $\Delta^{[m]} :=$ subcomplex of Δ generated by faces of $\dim = m$

Δ is SCM iff $\Delta^{[m]}$ is CM, $\forall 0 \leq m \leq \dim \Delta$.

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Theorem (MW, 1999)

Let $\Delta^{\langle m \rangle} :=$ subcomplex of Δ generated by facets of $\dim \geq m$.

We say Δ is *sequentially acyclic* if

$$\Delta^{\langle m \rangle} \text{ is } (m - 1)\text{-acyclic } \forall 0 < m \leq \dim \Delta$$

Δ is SCM iff $\text{lk}_{\Delta}(F)$ is sequentially acyclic for all $F \in \Delta$

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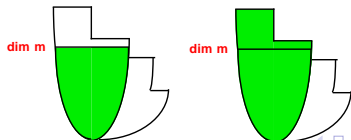
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Sequentially Cohen-Macaulay Complexes

Δ is **sequentially connected** if $\Delta^{\langle m \rangle}$ is $(m - 1)$ -connected for all $0 < m \leq \dim \Delta$

nonpure shellable \implies SHCM \implies SCM

Theorem (Björner, MW, Welker)

If Δ is sequentially connected then it has the homotopy type of a wedge of spheres of dimensions in $\{\dim F : F \text{ is a facet of } \Delta\}$.

The number of spheres of dimension m is equal to

$$(-1)^m \tilde{\chi}(\Delta^{\langle m \rangle}) + (-1)^{m+1} \tilde{\chi}(\Delta^{\langle m+1 \rangle})$$

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We say that a poset P is shellable (sequentially acyclic, sequentially connected, CM, HCM, SCM, SHCM) if its order complex $\Delta(P)$ is. Let

$$\hat{P} := P \cup \{\hat{0}, \hat{1}\}.$$

Proposition. A pure poset P is CM iff each open interval I of \hat{P} is $(\text{length}(I) - 1)$ -acyclic.

Proof.

- Links of $\Delta(P)$ are joins of order complexes of open intervals of \hat{P} .
- The join of an r -acyclic complex and an s -acyclic complex is $(r + s + 2)$ -acyclic.

The **join** of simplicial complexes Δ and Γ on disjoint vertex sets is

$$\Delta * \Gamma := \{F \cup G : F \in \Delta, G \in \Gamma\}$$

Theorem (Björner, MW, Welker)

The join of sequentially acyclic (connected) simplicial complexes is sequentially acyclic (connected)

Corollary

A poset P is $S(H)CM$ iff each open interval of \hat{P} is sequentially acyclic (connected).

Let Δ and Γ be sequentially connected. Need to show that $(\Delta * \Gamma)^{\langle m \rangle}$ is $(m - 1)$ -connected for $m \leq \dim \Delta + \dim \Gamma + 1$.

Pure case: $(\Delta * \Gamma)^{\langle m \rangle} = \Delta * \Gamma$ is $(\dim \Delta + \dim \Gamma)$ -connected.

Nonpure case: $(\Delta * \Gamma)^{\langle m \rangle} = \bigcup_{0 \leq i \leq m} \Delta^{\langle i \rangle} * \Gamma^{\langle m-i-1 \rangle}$

Lemma (Quillen, 1978)

Let $f : P \rightarrow Q$ be a poset map and suppose that the fiber $f^{-1}(Q_{\leq q})$ is t -connected for all $q \in Q$. Then P is t -connected iff Q is t -connected.

$P := P((\Delta * \Gamma)^{\langle m \rangle})$ poset of nonempty faces.

$Q :=$ poset of closed intervals of totally ordered set

$$m - \dim \Delta - 1 < m - \dim \Delta < \cdots < \dim \Gamma$$

ordered by reverse inclusion.

$f : P \rightarrow Q$, $f(F \cup G) = [m - \deg F - 1, \deg G]$, where
 $\deg F := \dim F + \dim \text{lk}_{\Delta} F + 1$.

$f^{-1}(Q_{\leq [a,b]}) = P(\Delta^{\langle m-a-1 \rangle} * \Gamma^{\langle b \rangle})$, $(m - 1 + b - a)$ -connected.

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- intervals and order ideals
- ordinal sums
- products
- rank selection

Shellability (pure and nonpure) and CM are preserved.
What about sequential Cohen-Macaulayness?

Poset Operations - products

$$\hat{P} := P \cup \{\hat{0}, \hat{1}\} \quad \bar{P} := P \setminus \{\hat{0}, \hat{1}\}$$

Theorem (Björner, MW, Welker)

If P and Q are sequentially (connected) acyclic then so is $\overline{\hat{P} \times \hat{Q}}$

Proof. Use Walker (1988):

$$\Delta(\overline{\hat{P} \times \hat{Q}}) \simeq \Delta(P) * \Delta(Q) * \{a, b\}$$

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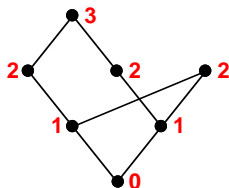
Corollary

Let P and Q have minimum elements. Then $P \times Q$ is $S(H)$ CM iff P and Q are.

Poset Operations - Rank Selection

Let P be **semipure**, i.e. P has a minimum element and all of its intervals are pure.

Define **rank** of $x \in P$ by $r(x) = \ell([\hat{0}, x])$.



For $S \subseteq \{0, 1, \dots, \ell(P)\}$, define the rank-selected subposet

$$P_S := \{x \in P \mid r(x) \in S\}$$

Poset Operations - Rank Selection

Theorem (Björner, MW, Welker)

Let P be semipure. If P is $S(H)$ CM then so is P_S for all $S \subseteq \{0, 1, \dots, \ell(P)\}$.

Theorem (Baclawski-Garsia, 1981; Walker 1981)

Let P be pure. Then P is CM iff P_S is $(|S| - 1)$ -acyclic for all $S \subseteq \{0, 1, \dots, \ell(P)\}$.

The analogous result holds for HCM.

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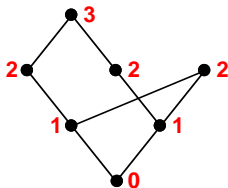
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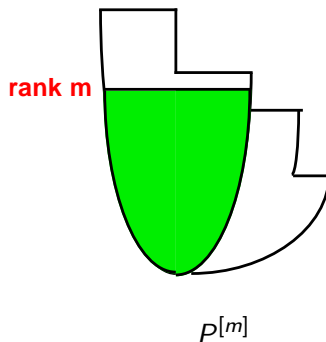
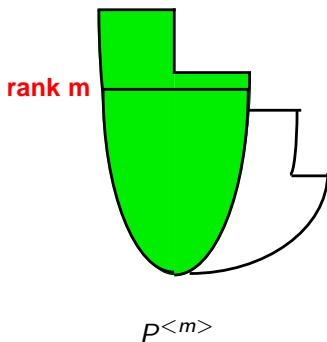


Poset analog of Duval's characterization

Let P be semipure and $m \leq \ell(P)$. Define

$P^{<m>}$:= lower order ideal generated by **maximal** elts of rank $\geq m$.

$P^{[m]}$:= lower order ideal generated by elts of rank $= m$



$$\Delta(P)^{<m>} = \Delta(P^{<m>}) \quad \text{but} \quad \Delta(P)^{[m]} \neq \Delta(P^{[m]})$$

Poset analog of Duval's characterization

Theorem (Björner, MW, Welker)

Let P be semipure. Then P is $S(H)CM$ iff $P^{[m]}$ is $(H)CM$ for all $m = 1, \dots, \ell(P)$.

Proof uses basic algebraic topology techniques, e.g. Meyer-Vietoris, Seifert-Van Kampen, Hurewicz.

Corollary (Generalization of Baclawski-Garsia and Walker)

Let P be semipure. Then P is SCM iff $P_S^{[m]}$ is $(|S| - 1)$ -acyclic for all $m \leq \ell(P)$ and S such that $S \subseteq \{1, \dots, m\}$.

The analogous result holds for $SHCM$.

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Theorem (Björner, MW, Welker)

Let P and Q be semipure posets and let $f : P \rightarrow Q$ be a surjective rank-preserving poset map. Assume that for all $q \in Q$ the fiber $\Delta(f^{-1}(Q_{\leq q}))$ is (H)CM. If Q is S(H)CM, then so is P .

Specializes to results of Baclawski and Quillen in pure case.