

# Poset Topology and Permutation Statistics

John Shareshian and Michelle Wachs

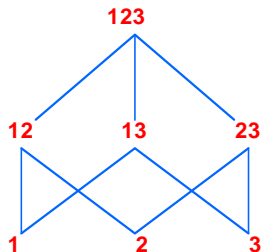
POSET TOPOLOGY  $\implies$  ENUMERATIVE COMBINATORICS

Let  $P$  and  $Q$  be pure (ranked) posets.

$$P * Q := \{(p, q) \in P \times Q : r(p) \geq r(q)\}$$

$(p_1, q_1) \leq (p_2, q_2)$  if the following holds

- $p_1 \leq_P p_2$
- $q_1 \leq_Q q_2$
- $r(p_2) - r(p_1) \geq r(q_2) - r(q_1)$



$B_3 - \{\emptyset\}$

\*



$C_3$

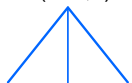
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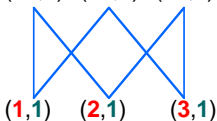
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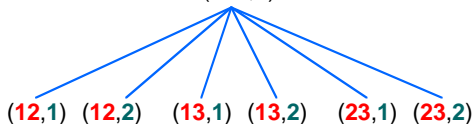
$(123, 1)$



$(12, 1)$   $(13, 1)$   $(23, 1)$



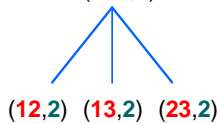
$(123, 2)$



$(12, 1)$   $(12, 2)$   $(13, 1)$   $(13, 2)$   $(23, 1)$   $(23, 2)$



$(123, 3)$



$(12, 2)$   $(13, 2)$   $(23, 2)$



## Theorem (Björner & Welker)

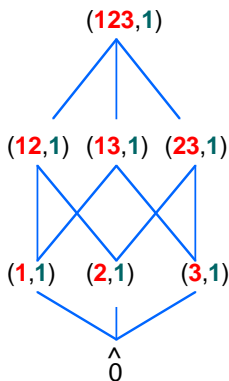
*The Rees product of two Cohen-Macaulay posets is Cohen-Macaulay. (CM means that homology of each interval vanishes below its top dimension.)*

## Conjecture (Björner & Welker)

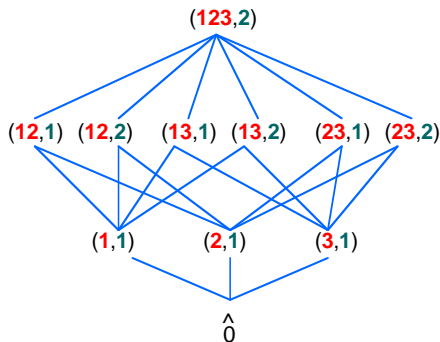
$$\dim \tilde{H}_{n-1}((B_n \setminus \{\emptyset\}) * C_n) = \# \text{ derangements in } \mathfrak{S}_n.$$

Proved by Jonsson.

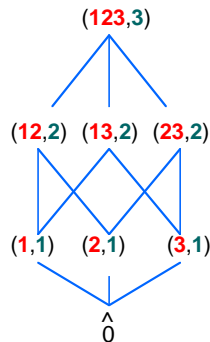
# Maximal Intervals of Rees Product



$$\beta(I_{3,1}) = 1$$



$$\beta(I_{3,2}) = 4$$

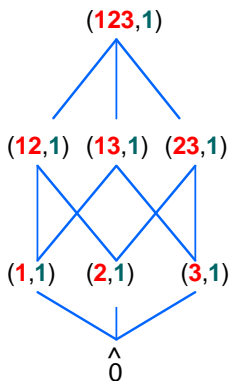


$$\beta(I_{3,3}) = 1$$

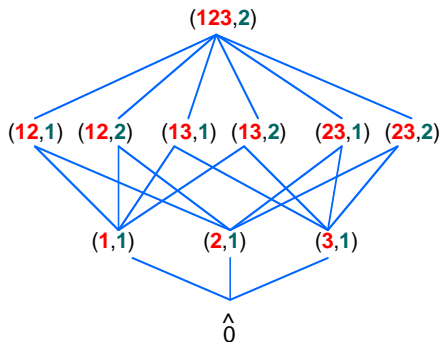
Eulerian polynomial:  $A_3(t) = 1 + 4t + t^2$

Björner-Welker-Jonsson derangement result follows.

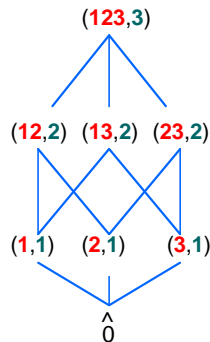
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# $q$ -Analog of $I_{n,j}$

$B_n(q) :=$  lattice of subspaces of  $\mathbb{F}_q^n$ .

$I_{n,j}(q) :=$  the open interval  $(\hat{0}, (\mathbb{F}_q^n, j))$  of  $((B_n(q) \setminus \{\emptyset\}) * C_n) \cup \hat{0}$

$$\beta(I_{3,1}(q)) = 1 \quad \beta(I_{3,2}(q)) = 2 + q + q^2 \quad \beta(I_{3,3}(q)) = 1$$

Betti number polynomial:  $\beta_3(q, t) = 1 + (2 + q + q^2)t + t^2$

What are these polynomials?

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# Permutation Statistics

For  $\sigma$  in symmetric group  $\mathfrak{S}_n$

- Eulerian statistics

Descent set:  $\text{DES}(\sigma) := \{i \in [n] : \sigma(i) > \sigma(i+1)\}$

Excedance set:  $\text{EXC}(\sigma) := \{i \in [n] : \sigma(i) > i\}$

$$\text{des}(\sigma) := |\text{DES}(\sigma)| \qquad \text{exc}(\sigma) := |\text{EXC}(\sigma)|$$

$$\sigma = 3.25.4.1 \qquad \text{DES}(\sigma) = \{1, 3, 4\} \qquad \text{des}(\sigma) = 3$$

$$\sigma = 32541 \qquad \text{EXC}(\sigma) = \{1, 3\} \qquad \text{exc}(\sigma) = 2$$

- Mahonian statistics

$$\text{inv}(\sigma) := |\{(i, j) : 1 \leq i < j \leq n \ \& \ \sigma(i) > \sigma(j)\}|$$

$$\text{maj}(\sigma) := \sum_{i \in \text{DES}(\sigma)} i$$

$$\text{inv}(32541) = 6 \qquad \text{maj}(32541) = 1 + 3 + 4 = 8$$

## Eulerian polynomial

$$A_n(t) := \sum_{\sigma \in \mathfrak{S}_n} t^{\text{des}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{exc}(\sigma)}$$

$\mathfrak{S}_3$	des	exc
123	0	0
132	1	1
213	1	1
231	1	2
312	1	1
321	2	1

$$A_3(t) = 1 + 4t + t^2$$

## Eulerian polynomial

$$A_n(t) := \sum_{\sigma \in \mathfrak{S}_n} t^{\text{des}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{exc}(\sigma)}$$

Exponential generating function:

$$\sum_{n \geq 0} A_n(t) \frac{z^n}{n!} = \frac{1-t}{e^{z(t-1)} - t}$$

# Permutation Statistics

*q*-analog

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\text{inv}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}(\sigma)} = [n]_q!$$

where  $[n]_q := 1 + q + \dots + q^{n-1}$  and  $[n]_q! := [n]_q [n-1]_q \dots [1]_q$

$\mathfrak{S}_3$	inv	maj
123	0	0
132	1	2
213	1	1
231	2	2
312	2	1
321	3	3

$$1 + 2q + 2q^2 + q^3 = (1 + q + q^2)(1 + q)$$

# q-Eulerian polynomials

$$A_n^{\text{inv,des}}(q, t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\text{inv}(\sigma)} t^{\text{des}(\sigma)}$$

$$A_n^{\text{maj,des}}(q, t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}(\sigma)} t^{\text{des}(\sigma)}$$

$$A_n^{\text{inv,exc}}(q, t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\text{inv}(\sigma)} t^{\text{exc}(\sigma)}$$

$$A_n^{\text{maj,exc}}(q, t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}(\sigma)} t^{\text{exc}(\sigma)}$$

Theorem (q-exponential generating function, Stanley 1976)

$$\sum_{n \geq 0} A_n^{\text{inv, des}}(q, t) \frac{z^n}{[n]_q!} = \frac{1 - t}{\text{Exp}_q(z(t - 1)) - t}$$

where

$$\text{Exp}_q(z) := \sum_{n \geq 0} \frac{q^{\binom{n}{2}} z^n}{[n]_q!}$$

# Betti number polynomial

$$\begin{aligned}\beta_3(q, t) &= 1 + (2 + q + q^2)t + t^2 \\ &= 1 + 2t + qt + q^2t + t^2\end{aligned}$$

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$$A_3^{\text{maj.exc}}(q, t) = 1 + 2qt + q^2t + q^3t + q^2t^2$$

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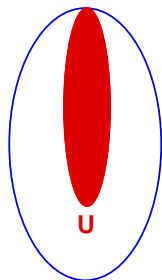
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# Computing the Betti number polynomial

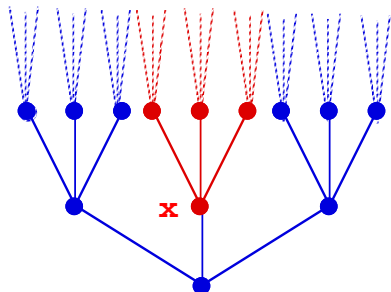
Let  $P_{n,t}(q) := (B_n(q) * T_{n,t}) \cup \{\hat{1}\}$ .

Upper interval

$$[(U, x), \hat{1}] \cong P_{n-\dim U, t}(q)$$



$B_n(q)$



$T_{n,t}$

Lemma:  $\beta_n(q, t) = (-1)^n \mu(P_{n,t}(q))$

# Computing the Betti number polynomial

$$\begin{aligned}\sum_{n \geq 0} \beta_n(q, t) \frac{z^n}{[n]_q!} &= \sum_{n \geq 0} (-1)^n \mu(P_{n,t}(q)) \frac{z^n}{[n]_q!} \\ &= \frac{(1-t) \exp_q(z)}{\exp_q(zt) - t \exp_q(z)}\end{aligned}$$

where  $\exp_q(z) = \sum_{n \geq 0} \frac{z^n}{[n]_q!}$

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# The maj-exc distribution

$$A_n^{\text{inv,des}}(q, t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\text{inv}(\sigma)} t^{\text{des}(\sigma)}$$

$$A_n^{\text{maj,des}}(q, t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}(\sigma)} t^{\text{des}(\sigma)}$$

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*[Clarke-Steingrimsson-Zeng, 1995] It is of course possible to define scores of different families of Eulerian-Mahonian statistics by arbitrarily combining an Eulerian statistic and a Mahonian one. Although some needles are sure to be found in that haystack, most of the possible such statistics seem rather unattractive and unlikely to possess particular interesting properties.*

# Symmetric Function Generalization

$$\sum_{n \geq 0} A_n^{\text{maj,exc}}(q, t) \frac{z^n}{[n]_q!} = \frac{(1 - tq) \exp_q(z)}{\exp_q(z tq) - tq \exp_q(z)}$$

# Symmetric Function Generalization

$$\frac{(1-t)H(z)}{H(zt) - tH(z)}$$

$$\begin{array}{l} \downarrow \\ x_i := q^{i-1} \\ z := z(1-q) \\ t := qt \end{array}$$

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$H(z) := \sum_{n \geq 0} h_n z^n$ , where  $h_n$  is the  $n$ th complete homogeneous symmetric function in  $x_1, x_2, \dots$ .

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For  $\sigma \in \mathfrak{S}_n$ , let  $\bar{\sigma}$  be obtained by placing bars above each **excedance**.

$$\bar{5}\bar{3}14\bar{6}\bar{2}$$

View  $\bar{\sigma}$  as a word over ordered alphabet

$$\{\bar{1} < \bar{2} < \dots < \bar{n} < 1 < 2 < \dots < n\}.$$

Define

$$\text{EXD}(\sigma) := \text{DES}(\bar{\sigma})$$

$$\text{EXD}(531462) = \text{DES}(\bar{5}.\bar{3}14.\bar{6}\bar{2}) = \{1, 4\}$$

$$\sum_{i \in \text{EXD}(\sigma)} i = \text{maj}(\sigma) - \text{exc}(\sigma)$$

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# Symmetric Function Generalization

For  $S \subseteq [n - 1]$ , quasisymmetric function

$$F_S(x_1, x_2, \dots) := \sum_{\substack{i_1 \geq \dots \geq i_n \\ j \in S \Rightarrow i_j > i_{j+1}}} x_{i_1} \dots x_{i_n}$$

From theory of quasisymmetric functions we have

$$F_S(1, q, q^2, \dots) = \frac{q^{\sum S}}{(1 - q)(1 - q^2) \dots (1 - q^n)}$$

Hence

$$F_{\text{EXD}(\sigma)}(1, q, q^2, \dots) = \frac{q^{\text{maj}(\sigma) - \text{exc}(\sigma)}}{(1 - q)(1 - q^2) \dots (1 - q^n)}$$

# Symmetric Function Generalization

By setting  $x_i := q^{i-1}$  and  $z := z(1 - q)$  in

$$\sum_{n \geq 0} \sum_{\sigma \in \mathfrak{S}_n} F_{\text{EXD}(\sigma)} t^{\text{exc}(\sigma)} z^n$$

we get

$$\sum_{n \geq 0} \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}(\sigma) - \text{exc}(\sigma)} t^{\text{exc}(\sigma)} \frac{z^n}{[n]_q!}$$

Now set  $t := qt$  to get

$$\sum_{n \geq 0} A_n^{\text{maj,exc}}(q, t) \frac{z^n}{[n]_q!}$$

# Symmetric Function Generalization

## Theorem (Shareshian and MW)

Let

$$Q_{n,j} := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{exc}(\sigma) = j}} F_{\text{EXD}(\sigma)}$$

Then

$$\sum_{n \geq 0} \sum_{j=0}^{n-1} Q_{n,j} t^j z^n = \frac{(1-t)H(z)}{H(zt) - tH(z)}$$

# The Steps of the Proof

For  $\lambda \vdash n$  define

$$Q_{\lambda,j} := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{exc}(\sigma) = j \\ \lambda(\sigma) = \lambda}} F_{\text{EXD}(\sigma)}$$

- ① Gessel-Reutenauer theory of necklaces  $\implies$

$$Q_{\lambda,j} = \sum_{R \in \mathcal{R}_{\lambda,j}} \text{wt}(R),$$

where  $\mathcal{R}_{\lambda,j}$  is the set of **ornaments** of type  $\lambda$  with  $j$  blue letters.

- ② Bijection from **ornaments** to **banners** using Lyndon decomposition.
- ③ Generalization of bijection of Stembridge takes **banners** to **marked sequences**
- ④ **Marked sequences** satisfy recurrence relation, which yields generating function

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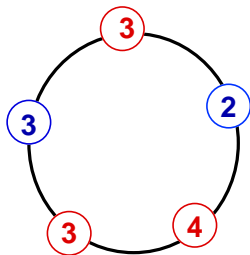
# Step 1

A **necklace** is a primitive circular word over alphabet

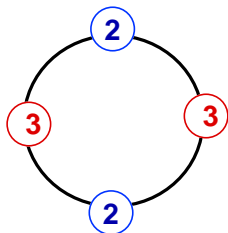
$$\{1, 1, 2, 2, \dots\}$$

such that

- a **blue** letter is followed by letter greater than or equal in value
- a **red** letter is followed by a letter less than or equal in value



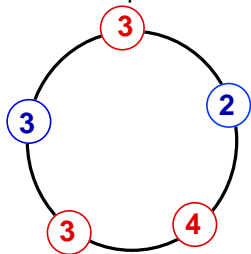
necklace



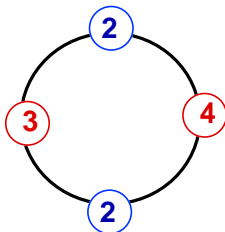
not a necklace

# Step 1

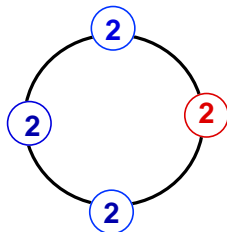
An **ornament** of type  $\lambda$  is a multiset of necklaces whose necklace sizes form partition  $\lambda$



type =  $(5, 4, 4)$



weight =  $x_2^7 x_3^4 x_4^2$



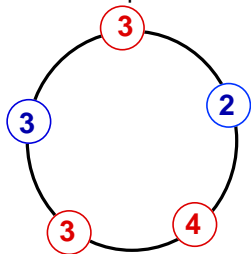
Theorem (Shareshian and MW)

Let  $\mathcal{R}_{\lambda,j}$  = set of ornaments of type  $\lambda$  with  $j$  blue letters. Then

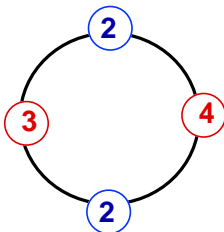
$$Q_{\lambda,j} = \sum_{R \in \mathcal{R}_{\lambda,j}} \text{wt}(R)$$

# Step 1

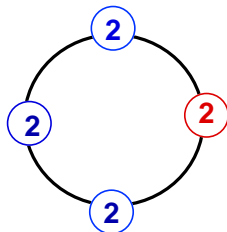
An **ornament** of type  $\lambda$  is a multiset of necklaces whose necklace sizes form partition  $\lambda$



type =  $(5, 4, 4)$



weight =  $x_2^7 x_3^4 x_4^2$



## Theorem (Shareshian and MW)

Let  $\mathcal{R}_{\lambda,j}$  = set of ornaments of type  $\lambda$  with  $j$  blue letters. Then

$$Q_{\lambda,j} = \sum_{R \in \mathcal{R}_{\lambda,j}} \text{wt}(R)$$

# The Steps of the Proof

For  $\lambda \vdash n$  define

$$Q_{\lambda,j} := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{exc}(\sigma) = j \\ \lambda(\sigma) = \lambda}} F_{\text{EXD}(\sigma)}$$

- ① Gessel-Reutenauer theory of necklaces  $\implies$

$$Q_{\lambda,j} = \sum_{R \in \mathcal{R}_{\lambda,j}} \text{wt}(R),$$

where  $\mathcal{R}_{\lambda,j}$  is the set of **ornaments** of type  $\lambda$  with  $j$  blue letters.

- ② Bijection from **ornaments** to **banners** using Lyndon decomposition.
- ③ Generalization of bijection of Stembridge takes **banners** to **marked sequences**
- ④ **Marked sequences** satisfy recurrence relation, which yields generating function

Action of  $\mathfrak{S}_n$  on  $B_n$  induces an action of  $\mathfrak{S}_n$  on  $I_{n,j}$  which induces a representation of  $\mathfrak{S}_n$  on  $\tilde{H}_{n-2}(I_{n,j})$

## Theorem (Shareshian and MW)

$$1 + \sum_{n \geq 1} \sum_{j=1}^n \text{ch} \tilde{H}_{n-2}(I_{n,j}) t^{j-1} z^n = \frac{(1-t)E(z)}{E(zt) - tE(z)},$$

where  $\text{ch}$  is the Frobenius characteristic,  $E(z) = \sum_{n \geq 0} e_n z^n$  and  $e_n$  is the  $n$ th elementary symmetric function.

# Connection with Toric Varieties

## Theorem (Procesi, Stanley 1989)

Let  $X_n$  be the toric variety associated with the Coxeter complex of  $\mathfrak{S}_n$ . The action of  $\mathfrak{S}_n$  on  $X_n$  induces a representation of  $\mathfrak{S}_n$  on  $H^{2j}(X_n)$ .

$$\sum_{n \geq 0} \sum_{j=0}^{n-1} \text{ch} H^{2j}(X_n) t^j z^n = \frac{(1-t)H(z)}{H(zt) - tH(z)}.$$

## Corollary (Shareshian and MW)

$$H^{2j}(X_n) \cong_{\mathfrak{S}_n} \tilde{H}_{n-2}(I_{n,j+1}) \otimes \text{sgn}$$

# Open Problems

- Show that  $Q_{\lambda,j}$  is Schur positive. In other words show it is the Frobenius characteristic of some representation.
- Explain the isomorphism between the homology of  $I_{n,j}$  and the cohomology of  $X_n$ .
- Use Rees product to discover other enumerative results\*.
- Find analogs for other Coxeter groups\*.