

Topology of Rees products of posets and q -Eulerian polynomials

Michelle Wachs

Joint work with John Shareshian

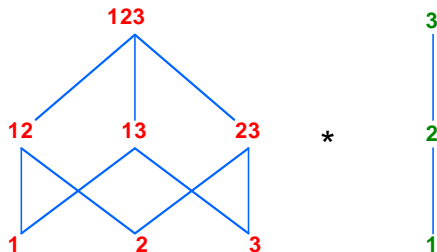
POSET TOPOLOGY \implies ENUMERATIVE COMBINATORICS

Let P and Q be pure (ranked) posets.

$$P * Q := \{(p, q) \in P \times Q : r(p) \geq r(q)\}$$

$(p_1, q_1) \leq (p_2, q_2)$ if the following holds

- $p_1 \leq_P p_2$
- $q_1 \leq_Q q_2$
- $r(p_2) - r(p_1) \geq r(q_2) - r(q_1)$



$B_3 - \{\emptyset\}$

C_3

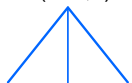
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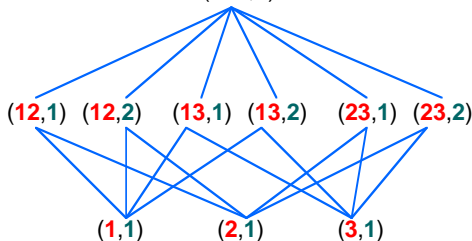
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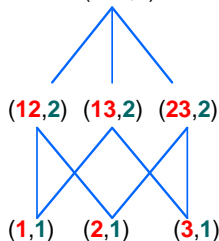
$(123, 1)$



$(123, 2)$



$(123, 3)$



Theorem (Björner & Welker)

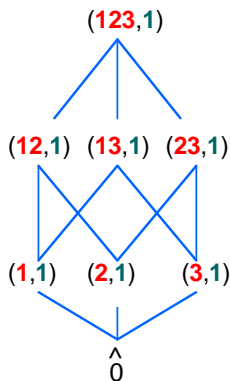
The Rees product of two Cohen-Macaulay posets is Cohen-Macaulay. (CM means that homology of each interval vanishes below its top dimension.)

Conjecture (Björner & Welker)

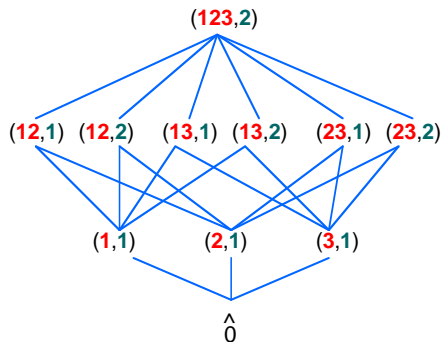
$$\dim \tilde{H}_{n-1}((B_n \setminus \{\emptyset\}) * C_n) = \# \text{ derangements in } \mathfrak{S}_n.$$

Proved by Jonsson.

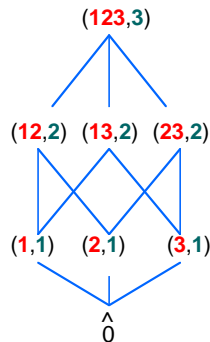
Maximal Intervals of Rees Product



$$\beta(I_{3,1}) = 1$$



$$\beta(I_{3,2}) = 4$$

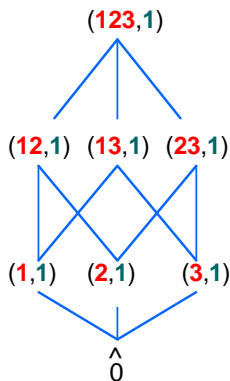


$$\beta(I_{3,3}) = 1$$

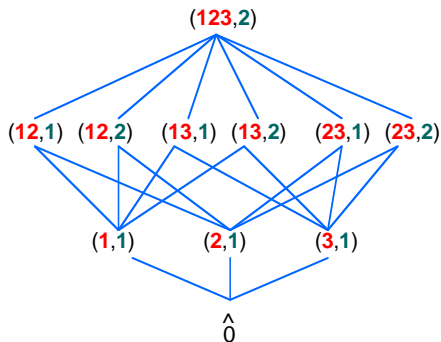
Eulerian polynomial: $A_3(t) = 1 + 4t + t^2$

Björner-Welker-Jonsson derangement result follows.

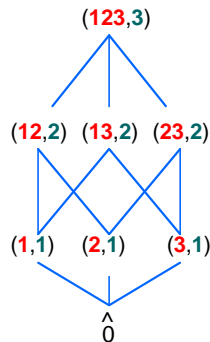
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q -Analog of $I_{n,j}$

$B_n(q) :=$ lattice of subspaces of \mathbb{F}_q^n .

$I_{n,j}(q) :=$ the open interval $(\hat{0}, (\mathbb{F}_q^n, j))$ of $((B_n(q) \setminus \{(0)\}) * C_n) \cup \hat{0}$

$$\beta(I_{3,1}(q)) = 1 \quad \beta(I_{3,2}(q)) = 2 + q + q^2 \quad \beta(I_{3,3}(q)) = 1$$

Betti number polynomial: $\beta_3(q, t) = 1 + (2 + q + q^2)t + t^2$

What are these polynomials?

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What are these polynomials?

Permutation Statistics

For σ in symmetric group \mathfrak{S}_n

- Eulerian statistics

Descent set: $\text{DES}(\sigma) := \{i \in [n] : \sigma(i) > \sigma(i+1)\}$

Excedance set: $\text{EXC}(\sigma) := \{i \in [n] : \sigma(i) > i\}$

$$\text{des}(\sigma) := |\text{DES}(\sigma)| \qquad \text{exc}(\sigma) := |\text{EXC}(\sigma)|$$

$$\sigma = 3.25.4.1 \qquad \text{DES}(\sigma) = \{1, 3, 4\} \qquad \text{des}(\sigma) = 3$$

$$\sigma = 32541 \qquad \text{EXC}(\sigma) = \{1, 3\} \qquad \text{exc}(\sigma) = 2$$

- Mahonian statistics

$$\text{inv}(\sigma) := |\{(i, j) : 1 \leq i < j \leq n \ \& \ \sigma(i) > \sigma(j)\}|$$

$$\text{maj}(\sigma) := \sum_{i \in \text{DES}(\sigma)} i$$

$$\text{inv}(32541) = 6 \qquad \text{maj}(32541) = 1 + 3 + 4 = 8$$

Eulerian polynomial

$$A_n(t) := \sum_{\sigma \in \mathfrak{S}_n} t^{\text{des}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{exc}(\sigma)}$$

\mathfrak{S}_3	des	exc
123	0	0
132	1	1
213	1	1
231	1	2
312	1	1
321	2	1

$$A_3(t) = 1 + 4t + t^2$$

Eulerian polynomial

$$A_n(t) := \sum_{\sigma \in \mathfrak{S}_n} t^{\text{des}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{exc}(\sigma)}$$

Exponential generating function:

$$\sum_{n \geq 0} A_n(t) \frac{z^n}{n!} = \frac{1-t}{e^{z(t-1)} - t}$$

Permutation Statistics

q-analog

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\text{inv}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}(\sigma)} = [n]_q!$$

where $[n]_q := 1 + q + \dots + q^{n-1}$ and $[n]_q! := [n]_q [n-1]_q \dots [1]_q$

\mathfrak{S}_3	inv	maj
123	0	0
132	1	2
213	1	1
231	2	2
312	2	1
321	3	3

$$1 + 2q + 2q^2 + q^3 = (1 + q + q^2)(1 + q)$$

q-Eulerian polynomials

$$A_n^{\text{inv,des}}(q, t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\text{inv}(\sigma)} t^{\text{des}(\sigma)}$$

$$A_n^{\text{maj,des}}(q, t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}(\sigma)} t^{\text{des}(\sigma)}$$

$$A_n^{\text{inv,exc}}(q, t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\text{inv}(\sigma)} t^{\text{exc}(\sigma)}$$

$$A_n^{\text{maj,exc}}(q, t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}(\sigma)} t^{\text{exc}(\sigma)}$$

Theorem (q-exponential generating function, Stanley 1976)

$$\sum_{n \geq 0} A_n^{\text{inv, des}}(q, t) \frac{z^n}{[n]_q!} = \frac{1 - t}{\text{Exp}_q(z(t - 1)) - t}$$

where

$$\text{Exp}_q(z) := \sum_{n \geq 0} \frac{q^{\binom{n}{2}} z^n}{[n]_q!}$$

Betti number polynomial

$$\begin{aligned}\beta_3(q, t) &= 1 + (2 + q + q^2)t + t^2 \\ &= 1 + 2t + qt + q^2t + t^2\end{aligned}$$

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$$A_3^{\text{maj.exc}}(q, t) = 1 + 2qt + q^2t + q^3t + q^2t^2$$

Betti number polynomial

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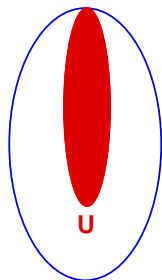
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Computing the Betti number polynomial

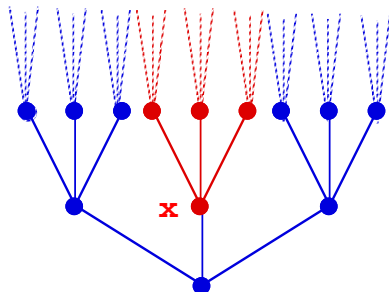
Let $P_{n,t}(q) := (B_n(q) * T_{n,t}) \cup \{\hat{1}\}$.

Upper interval

$$[(U, x), \hat{1}] \cong P_{n-\dim U, t}(q)$$



$B_n(q)$



$T_{n,t}$

Lemma: $\beta_n(q, t) = (-1)^n \mu(P_{n,t}(q))$

Computing the Betti number polynomial

$$\begin{aligned}\sum_{n \geq 0} \beta_n(q, t) \frac{z^n}{[n]_q!} &= \sum_{n \geq 0} (-1)^n \mu(P_{n,t}(q)) \frac{z^n}{[n]_q!} \\ &= \frac{(1-t) \exp_q(z)}{\exp_q(zt) - t \exp_q(z)}\end{aligned}$$

where $\exp_q(z) = \sum_{n \geq 0} \frac{z^n}{[n]_q!}$

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The maj-exc distribution

$$A_n^{\text{inv,des}}(q, t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\text{inv}(\sigma)} t^{\text{des}(\sigma)}$$

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$$A_n^{\text{maj,exc}}(q, t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}(\sigma)} t^{\text{exc}(\sigma)}$$

[Clarke-Steingrimsson-Zeng, 1995] It is of course possible to define scores of different families of Eulerian-Mahonian statistics by arbitrarily combining an Eulerian statistic and a Mahonian one. Although some needles are sure to be found in that haystack, most of the possible such statistics seem rather unattractive and unlikely to possess particular interesting properties.

Symmetric Function Generalization

$$\sum_{n \geq 0} A_n^{\text{maj,exc}}(q, t) \frac{z^n}{[n]_q!} = \frac{(1 - tq) \exp_q(z)}{\exp_q(z tq) - tq \exp_q(z)}$$

Symmetric Function Generalization

$H(z) := \sum_{n \geq 0} h_n z^n$, where h_n is the n th complete homogeneous symmetric function in x_1, x_2, \dots .

$$\frac{(1-t)H(z)}{H(zt) - tH(z)}$$

$$\begin{array}{l} \downarrow \\ x_i := q^{i-1} \\ z := z(1-q) \\ t := qt \end{array}$$

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Symmetric Function Generalization

For $\sigma \in \mathfrak{S}_n$, let $\bar{\sigma}$ be obtained by placing bars above each **excedance**.

$$\bar{5}\bar{3}14\bar{6}2$$

View $\bar{\sigma}$ as a word over ordered alphabet

$$\{\bar{1} < \bar{2} < \dots < \bar{n} < 1 < 2 < \dots < n\}.$$

Define

$$\text{DEX}(\sigma) := \text{DES}(\bar{\sigma})$$

$$\text{DEX}(531462) = \text{DES}(\bar{5}\bar{3}14\bar{6}2) = \{1, 4\}$$

$$\sum_{i \in \text{DEX}(\sigma)} i = \text{maj}(\sigma) - \text{exc}(\sigma)$$

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Symmetric Function Generalization

For $T \subseteq [n - 1]$, quasisymmetric function

$$F_T(x_1, x_2, \dots) := \sum_{\substack{s_1 \geq \dots \geq s_n \\ i \in T \Rightarrow s_i > s_{i+1}}} x_{s_1} \dots x_{s_n}$$

From theory of quasisymmetric functions we have

$$F_T(1, q, q^2, \dots) = \frac{q^{\sum T}}{(1 - q)(1 - q^2) \dots (1 - q^n)}$$

Hence

$$F_{\text{DEX}(\sigma)}(1, q, q^2, \dots) = \frac{q^{\text{maj}(\sigma) - \text{exc}(\sigma)}}{(1 - q)(1 - q^2) \dots (1 - q^n)}$$

Symmetric Function Generalization

By setting $x_i := q^{i-1}$ and $z := z(1 - q)$ in

$$\sum_{n \geq 0} \sum_{\sigma \in \mathfrak{S}_n} F_{\text{DEX}(\sigma)} t^{\text{exc}(\sigma)} z^n$$

we get

$$\sum_{n \geq 0} \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}(\sigma) - \text{exc}(\sigma)} t^{\text{exc}(\sigma)} \frac{z^n}{[n]_q!}$$

Now set $t := qt$ to get

$$\sum_{n \geq 0} A_n^{\text{maj,exc}}(q, t) \frac{z^n}{[n]_q!}$$

Symmetric Function Generalization

$H(z) := \sum_{n \geq 0} h_n z^n$, where h_n is the n th complete homogeneous symmetric function in x_1, x_2, \dots .

$$\begin{array}{ccc} ? & = & \frac{(1-t)H(rz)}{H(zt) - tH(z)} \\ \downarrow \begin{array}{l} x_i := q^{i-1} \\ z := z(1-q) \\ t := qt \end{array} & & \downarrow \begin{array}{l} x_i := q^{i-1} \\ z := z(1-q) \\ t := qt \end{array} \\ \sum_{n \geq 0} A_n^{\text{maj, exc, fix}}(q, t, r) \frac{z^n}{[n]_q!} & = & \frac{(1-tq) \exp_q(rz)}{\exp_q(ztq) - tq \exp_q(z)} \end{array}$$

Symmetric Function Generalization

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$$\sum_{n \geq 0} \sum_{\sigma \in \mathfrak{S}_n} F_{\text{DEX}(\sigma)} t^{\text{exc}(\sigma)} r^{\text{fix}(\sigma)} z^n = \frac{(1-t)H(rz)}{H(zt) - tH(z)}$$

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Symmetric Function Generalization

Theorem (Shareshian and MW)

Let

$$Q_{n,j,k} := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{exc}(\sigma) = j \\ \text{fix}(\sigma) = k}} F_{\text{DEX}(\sigma)}$$

Then

$$\sum_{n \geq 0} \sum_{j=0}^{n-1} \sum_{k=0}^n Q_{n,j,k} t^j r^k z^n = \frac{(1-t)H(rz)}{H(zt) - tH(z)}$$

Steps of the Proof of $\sum Q_{n,j,k} t^j r^k z^n = \frac{(1-t)H(rz)}{H(zt)-tH(z)}$

- 1 Modification of bijection of Gessel and Reutenauer, which takes **compatible pairs** to **ornaments**, is used to give alternative characterization of $Q_{n,j,k}$
- 2 Bijection from **ornaments** to **banners**, using Lyndon decomposition, is used to give another alternative characterization of $Q_{n,j,k}$
- 3 Generalization of bijection of Stembridge is used to show $Q_{n,j,k}$ satisfies **recurrence relation**, which yields generating function.

Step 1 - Ornaments

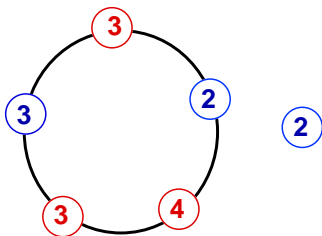
A **necklace** is a primitive circular word over alphabet

$$\{1, 1, 2, 2, \dots\}$$

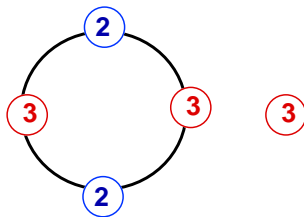
such that if size > 1

- a **blue** letter is followed by letter greater than or equal in value
- a **red** letter is followed by a letter less than or equal in value

Necklaces of size 1 are **blue**.



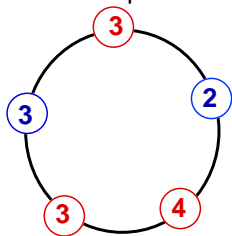
necklaces



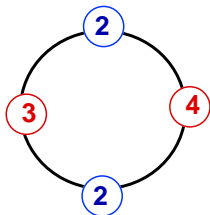
not necklaces

Step 1 - Ornaments

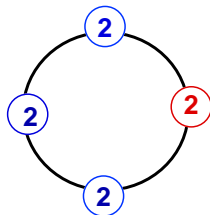
An **ornament** of type λ is a multiset of necklaces whose necklace sizes form partition λ



type = (5, 4, 4)



weight = $x_2^7 x_3^4 x_4^2$



Let $\mathcal{R}_{\lambda,j}$ = set of ornaments of type λ with j red letters.

Step 1 - Ornaments

Theorem (Sharehian and MW)

For $\lambda \vdash n$, define

$$Q_{\lambda,j} := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{exc}(\sigma) = j \\ \lambda(\sigma) = \lambda}} F_{\text{DEX}(\sigma)}$$

Then

$$Q_{\lambda,j} = \sum_{R \in \mathcal{R}_{\lambda,j}} \text{wt}(R)$$

$$F_{\text{DEX}(\sigma)} = \sum_{\substack{s_1 \geq \dots \geq s_n \\ i \in \text{DEX}(\sigma) \Rightarrow s_i > s_{i+1}}} x_{s_1} \dots x_{s_n}$$

Step 1 - Ornaments

Theorem (Sharehian and MW)

For $\lambda \vdash n$, define

$$Q_{\lambda,j} := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{exc}(\sigma) = j \\ \lambda(\sigma) = \lambda}} F_{\text{DEX}(\sigma)}$$

Then

$$Q_{\lambda,j} = \sum_{R \in \mathcal{R}_{\lambda,j}} \text{wt}(R)$$

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Step 1 - Ornaments

For $\lambda \vdash n$ and $j = 0, \dots, n-1$, let

$\text{Com}_{\lambda,j} := \{(\sigma, s) : \sigma \in \mathfrak{S}_n, \lambda(\sigma) = \lambda, \text{exc}(\sigma) = j, s \text{ is } \sigma\text{-compat}\}$.

Bijection $\phi : \text{Com}_{\lambda,j} \rightarrow \mathcal{R}_{\lambda,j}$

Let $\sigma = 45162387$ and $s = (7, 7, 7, 5, 5, 4, 2, 2)$.

- 1 write σ in cycle form,

$$\sigma = (1, 4, 6, 3)(2, 5), (7, 8).$$

- 2 color letters that are followed (cyclicly) by larger letters **red** and letters that are singletons or are followed by smaller letters **blue**,

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Steps of the Proof of $\sum Q_{n,j,k} t^j r^k z^n = \frac{(1-t)H(rz)}{H(zt)-tH(z)}$

- 1 Modification of bijection of Gessel and Reutenauer which takes **compatible pairs** to **ornaments** is used to give alternative characterization of $Q_{n,j,k}$
- 2 Bijection from **ornaments** to **banners** using Lyndon decomposition is used to give another alternative characterization of $Q_{n,j,k}$
- 3 Generalization of bijection of Stembridge is used to show $Q_{n,j,k}$ satisfies **recurrence relation**, which yields generating function.

Step 2 - Banners

A *banner* is a word over alphabet

$$\{1, 1, 2, 2, \dots\}$$

such that

- blue letter is followed by letter greater than or equal in value or is last
- red letter is followed by a letter less than or equal in value

Example:

22757547

Step 2 - Banners

A **Lyndon word** over an ordered alphabet is a word that is strictly lexicographically larger than all its circular rearrangements.

The **Lyndon factorization** of a word over an ordered alphabet is a factorization into a weakly lexicographically increasing sequence of Lyndon words.

The **Lyndon type** $\lambda(w)$ of a word w is the partition whose parts are the lengths of the words in its Lyndon factorization.

Use the ordering $1 < 1 < 2 < 2 < \dots$ for our alphabet.
Example:

$$\lambda(22757547) = \lambda(22 \cdot 75 \cdot 7547) = 4, 2, 2$$

Step 2 - Banners

Theorem (Shareshian and MW)

Let $\mathcal{B}_{\lambda,j}$ = the set of banners of Lyndon type λ with j red letters.
Then there is a weight-preserving bijection

$$\mathcal{B}_{\lambda,j} \rightarrow \mathcal{R}_{\lambda,j}.$$

Consequently

$$Q_{\lambda,j} = \sum_{b \in \mathcal{B}_{\lambda,j}} \text{wt}(b)$$

$$22757547 \mapsto 22 \cdot 75 \cdot 7547 \mapsto (2, 2)(7, 5)(7, 5, 4, 7)$$

Steps of the Proof of $\sum Q_{n,j,k} t^j r^k z^n = \frac{(1-t)H(rz)}{H(zt)-tH(z)}$

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Step 3 - Recurrence Relation

Using ornaments one can easily show that the formula

$$\sum_{n \geq 0} \sum_{j=0}^{n-1} \sum_{k=0}^n Q_{n,j,k} t^j r^k z^n = \frac{(1-t)H(rz)}{H(zt) - tH(z)}$$

is equivalent to

$$\sum_{n \geq 0} \sum_{j=0}^{n-1} Q_{n,j,0} t^j z^n = \frac{1-t}{H(zt) - tH(z)}$$

which is equivalent to recurrence relation

$$Q_{n,j,0} = \sum_{\substack{0 \leq m \leq n-2 \\ j+m-n < i < j}} Q_{m,i,0} h_{n-m}$$

Step 3 - Recurrence Relation

From Steps 1 and 2 we have

$$Q_{n,j,0} = \sum_{b \in \mathcal{B}_{n,j}} wt(b)$$

where

$$\mathcal{B}_{n,j} := \bigcup_{\substack{\lambda \vdash n \\ \lambda \text{ has no parts of size } 1}} \mathcal{B}_{\lambda,j}$$

Theorem (Shareshian and MW)

For all $n \geq 2$, there is a bijection

$$\gamma : \mathcal{B}_{n,j} \rightarrow \bigcup_{\substack{0 \leq m \leq n-2 \\ j+m-n < i < j}} \mathcal{B}_{m,i} \times \mathcal{M}_{n-m},$$

where

$$\mathcal{M}_n := \{(a_1 \leq \dots \leq a_n) : a_i \in \mathbb{Z}^+\}$$

Connection with Toric Varieties

Let X_n be the toric variety associated with the Coxeter complex of \mathfrak{S}_n . The action of \mathfrak{S}_n on X_n induces a representation of \mathfrak{S}_n on $H^{2j}(X_n)$.

Theorem (Procesi, Stanley 1989)

$$\sum_{n \geq 0} \sum_{j=0}^{n-1} \text{ch} H^{2j}(X_n) t^j z^n = \frac{(1-t)H(z)}{H(zt) - tH(z)},$$

where ch is the Frobenius characteristic,

Theorem (Shareshian and MW)

$$\text{ch} H^{2j}(X_n) = Q_{n,j} := \sum_{k=0}^n Q_{n,j,k}$$

Representation on Rees product

Action of \mathfrak{S}_n on B_n induces an action of \mathfrak{S}_n on maximal open intervals $I_{n,j} := (\hat{0}, ([n], j))$ of Rees product $(B_n - \{\hat{0}\}) * C_n$, which induces a representation of \mathfrak{S}_n on $\tilde{H}_{n-2}(I_{n,j})$

Theorem (Shareshian and MW)

$$1 + \sum_{n \geq 1} \sum_{j=0}^{n-1} \text{ch} \tilde{H}_{n-2}(I_{n,j+1}) t^j z^n = \frac{(1-t)E(z)}{E(zt) - tE(z)},$$

where $E(z) = \sum_{n \geq 0} e_n z^n$ and e_n is the n th elementary symmetric function.

Corollary

$$\text{ch} \tilde{H}_{n-2}(I_{n,j+1}) = \omega Q_{n,j}$$

and

$$H^{2j}(X_n) \cong_{\mathfrak{S}_n} \tilde{H}_{n-2}(I_{n,j+1}) \otimes \text{sgn}$$

Another specialization of $\sum Q_{n,j,k} t^j r^k z^n = \frac{(1-t)H(rz)}{H(zt)-tH(z)}$

Two ways to specialize the quasisymmetric functions:

$$F_T(1, q, q^2, \dots) = \frac{q^{\sum T}}{(q; q)_n}$$

$$\sum_{m \geq 0} F_T(1, q, \dots, q^{m-1}) p^m = \frac{p^{|T|+1} q^{\sum T}}{(p; q)_{n+1}}.$$

where

$$(a; q)_n := \begin{cases} 1 & \text{if } n = 0 \\ (1-a)(1-aq) \cdots (1-aq^{n-1}) & \text{if } n \geq 1. \end{cases}$$

Another specialization of $\sum Q_{n,j,k} t^j r^k z^n = \frac{(1-t)H(rz)}{H(zt)-tH(z)}$

First specialization: $\sum \text{DEX}(\sigma) = \text{maj}(\sigma) - \text{exc}(\sigma)$.

Theorem (Shareshian and MW)

$$\sum_{n \geq 0} A_n^{\text{maj,exc,fix}}(q, t, r) \frac{z^n}{[n]_q!} = \frac{(1-tq) \exp_q(rz)}{\exp_q(ztq) - tq \exp_q(z)}.$$

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Second specialization:

$$|\text{DEX}(\sigma)| = \begin{cases} \text{des}(\sigma) & \text{if } \sigma(1) = 1 \\ \text{des}(\sigma) - 1 & \text{otherwise} \end{cases}$$

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Theorem (Foata-Han (2007), Gessel-Reutenauer (1993) t=1)

$$\begin{aligned} \sum_{n \geq 0} A_n^{\text{maj,exc,fix,des}}(q, t, r, p) \frac{z^n}{(p; q)_{n+1}} &= \\ &= \sum_{m \geq 0} p^m \frac{(1-tq)(z; q)_m (ztq; q)_m}{((z; q)_m - tq(ztq; q)_m)(zr; q)_{m+1}}, \end{aligned}$$

Open Problems

- Show that $Q_{\lambda,j}$ is Schur positive. In other words show it is the Frobenius characteristic of some representation.
- Find an explicit isomorphism between the homology of $I_{n,j+1}$ and the $2j$ th cohomology of X_n .
- Use Rees product to discover other enumerative results*.
- Find analogs for other Coxeter groups*.