Quillen Complexes, Coset Complexes and Matching Complexes

Michelle Wachs

Based on joint work with John Shareshian

The matching complex M_n is the simplicial complex of matchings on

$$[n] := \{1, 2, \ldots, n\}$$

vertices := 2 element subsets of $[n]$

faces := sets of pairwise disjoint 2 element subsets of $[n]$

Faces of M_4: $\{12, 34\}$, $\{14, 23\}$, $\{13, 24\}$,

$$\{12\},\ \{13\},\ \{14\},\ \{23\},\ \{24\},\ \{34\},\ \emptyset$$

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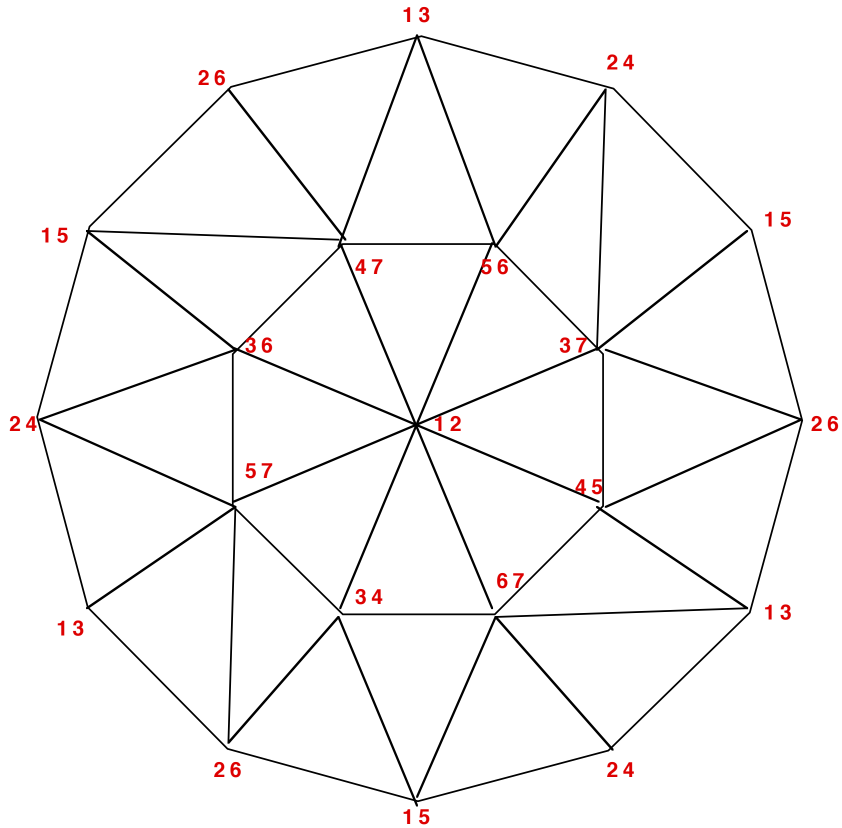
$\|M_4\| \quad =$

$$\simeq \quad \bullet \quad \bullet \quad \bullet \quad = \quad \mathbb{S}^0 \vee \mathbb{S}^0$$



Piece of M_7

Chessboard Complex: $M_{m,n}$

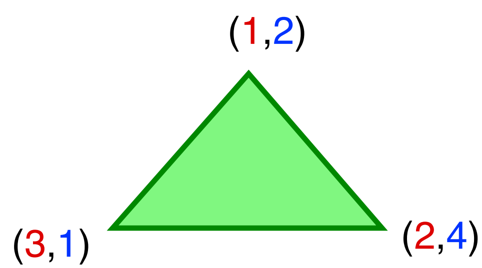
vertex set $:= \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}$

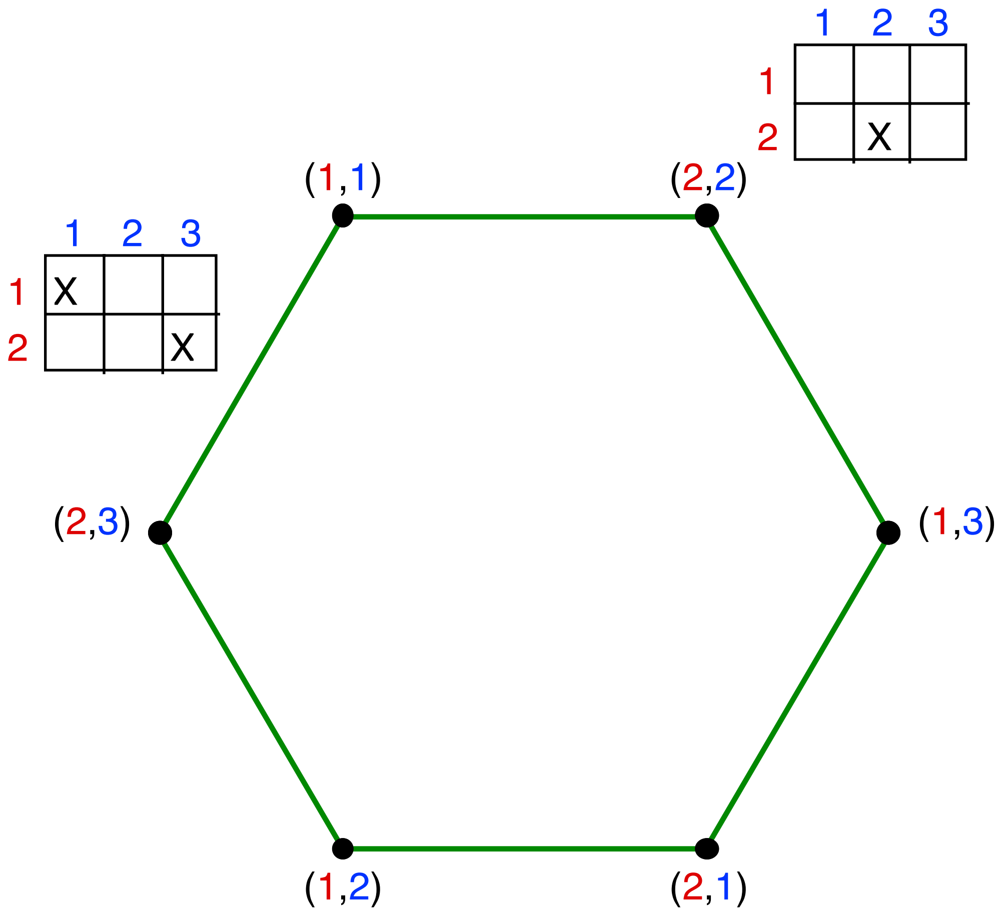
face $:= \{(i_1, j_1), \ldots, (i_k, j_k)\} \quad i_s \neq i_t, \; j_s \neq j_t \; \forall s, t$

facet of $M_{3,4}$:

	1	2	3	4
1		×		
2				×
3	×			

$\leftrightarrow \quad \{(1, 2), (2, 4), (3, 1)\}$





$\|M_{2,3}\| \simeq \mathbb{S}^1$

- Tits coset complexes: Garst(1979), MW(2000), Babson & Reiner(2001),

- Quillen complexes: Bouc(1990), Ksontini(2000), Shareshian(2004), Shareshian & MW(2005)

- computational geometry (colored Tverberg problem): Vrécica & Živaljević (1992), Björner, Lovász, Vrécica, Živaljević (1994)

- minimal free resolutions: Reiner & Roberts (1997)

- topological combinatorics (bounded degree graph complexes): Ziegler (1994), Björner & Welker(1998), Friedman & Hanlon (1998), Karaguezian, Reiner & MW(1999), Dong(1999), Dong & MW(2000), MW(2000), Athanasiadis(2002), Shareshian & MW(2004), Jonsson(2005).

Homology of the Matching Complex

The symmetric group $\mathfrak{S}_n$ acts on M_n in a natural way.

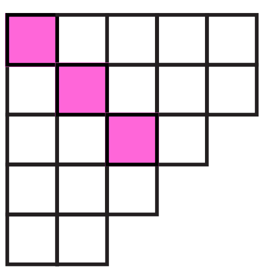
$$(2,3)\{12,34,56\} = \{13,24,56\}$$

This induces a representation of $\mathfrak{S}_n$ on $C_i(M_n;\mathbb{C})$ and $H_i(M_n;\mathbb{C})$.

Theorem (Bouc, 1990). As $\mathfrak{S}_n$-modules,

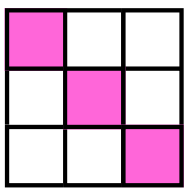
$$H_{i-1}(M_n;\mathbb{C}) \cong \bigoplus_{\substack{\lambda:\ \lambda \vdash n \\ \lambda = \lambda' \\ d(\lambda) = n-2i}} S^\lambda$$

where S^λ is the irreducible $\mathfrak{S}_n$-module indexed by partition λ, λ' is the conjugate of λ and $d(\lambda)$ is the size of the diagonal of λ.

λ =  d(λ) = 3

Example: $H_{i-1}(M_9;\mathbb{C})$

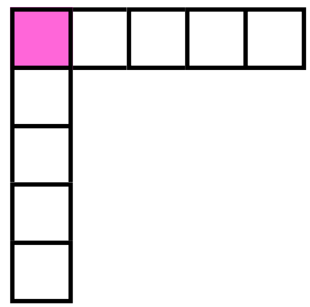
Self-conjugate shapes with 9 cells:



i = $\frac{9-3}{2}$ = 3

$H_2(M_9;\mathbb{C}) = S^{333}$

$$\beta_2(M_9) = \frac{9!}{5\cdot 4\cdot 3\cdot 4\cdot 3\cdot 2\cdot 3\cdot 2} = 42$$



i = $\frac{9-1}{2}$ = 4

$H_3(M_9;\mathbb{C}) = S^{51111}$

$$\beta_3(M_9) = \frac{9!}{9\cdot 4\cdot 3\cdot 2\cdot 4\cdot 3\cdot 2\cdot} = 70$$

Quillen Complexes

Order complex ΔP of a partially ordered set P is the simplicial complex whose faces are the chains of the poset.

$\mathcal{S}_p(G) =$ poset of nontrivial p-subgroups of finite group G ordered by inclusion

$\mathcal{A}_p(G) =$ subposet of nontrivial elementary abelian p-subgroups of G

Brown complex: $\Delta\mathcal{S}_p(G)$ Quillen complex: $\Delta\mathcal{A}_p(G)$

Quillen (1978): $\Delta\mathcal{S}_p(G) \simeq \Delta\mathcal{A}_p(G)$

$$H_i(\Delta\mathcal{S}_p(G);\mathbb{C}) \stackrel{\sim}{=}_G H_i(\Delta\mathcal{A}_p(G);\mathbb{C})$$

where G acts by conjugation.

Quillen Conjecture (1978): $\Delta\mathcal{A}_p(G)$ is contractible if and only if G has a nontrivial normal p-subgroup.

$\Leftarrow$ Easy to show $\Delta\mathcal{S}_p(G)$ contractible

True for solvable G

Aschbacher and Smith (1993); true for large class of G

Quillen (1978): If G is a group of Lie type in characteristic p then $\Delta\mathcal{A}_p(G)$ is homotopy equivalent to the building for G (which is homotopy equivalent to a wedge of spheres)

Webb (1987): The representation of G on $H_*(\Delta\mathcal{A}_p(G);\mathbb{C})$ is isomorphic to the representation on the building (the Steinberg representation).

What about the symmetric group $\mathfrak{S}_n$? What can we say about the homotopy type and homology of $\Delta\mathcal{A}_p(\mathfrak{S}_n)$?

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Bouc (1990): Subposet $\mathcal{T}_2(\mathfrak{S}_n)$ of $\mathcal{A}_2(\mathfrak{S}_n)$ consisting of elementary abelian 2-subgroups generated by transpositions.

Poset isomorphism $\mathcal{T}_2(\mathfrak{S}_n) \to P(M_n)$

$$\langle (1,3),(2,5),(4,6) \rangle \mapsto \{13, 25, 46\},$$

where $P(M_n)$ is the face poset of the matching complex.

$$\Delta\mathcal{T}_2(\mathfrak{S}_n) \simeq M_n$$

$$H_i(\Delta\mathcal{T}_2(\mathfrak{S}_n);\mathbb{C}) \cong_{\mathfrak{S}_n} H_i(M_n;\mathbb{C})$$

Bouc (1990): M_n is $(\nu(n) - 1)$-connected, where $\nu(n) = \lfloor \frac{n-2}{3} \rfloor$.
So connected for $n \geq 5$ and simply connected for $n \geq 8$.

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Ksontini (2000, 2003): If $n \geq p^2 + p$ then the inclusion map induces a surjective homomorphism.

$$f : \pi_i(\Delta \mathcal{T}_p(\mathfrak{S}_n)) \to \pi_i(\Delta \mathcal{A}_p(\mathfrak{S}_n))$$

So $\Delta \mathcal{A}_2(\mathfrak{S}_n)$ is simply connected for $n \geq 8$.

Ksontini uses p-hypergraph matching complex M_n^p and p-cycle poset $\mathcal{T}_p(\mathfrak{S}_n)$ to obtain homotopy information about $\Delta \mathcal{A}_p(\mathfrak{S}_n)$

When p is odd, $\Delta \mathcal{A}_p(\mathfrak{S}_n)$ is simply connected for $n \geq p^2 + p$

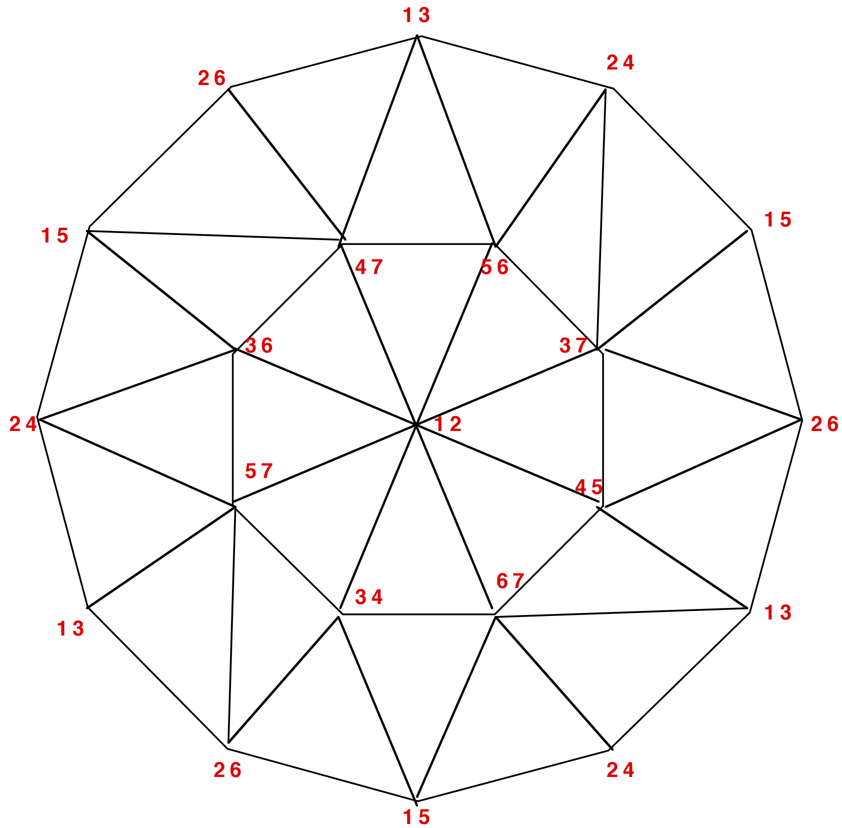
Shareshian (2004): $H_2(\Delta \mathcal{A}_3(\mathfrak{S}_{13}))$ has torsion.

Shareshian and MW (2005): results on top homology of $\Delta \mathcal{A}_p(\mathfrak{S}_n)$

Bouc (1990): For $n \equiv 1 \bmod 3$ and $n \geq 7$, the bottom nonvanishing reduced integral homology of M_n is

$$H_{\nu(n)}(M_n) = \mathbb{Z}_3.$$

For $n \equiv 0 \bmod 3$ and $n \geq 12$, $H_{\nu(n)}(M_n)$ is a nonvanishing 3-group of exponent at most 9.



Piece of M_7

Bouc (1990):

If $n \equiv 1 \bmod 3,\ n \geq 7$ then $H_{\nu(n)}(M_n) \stackrel{\sim}{=} \mathbb{Z}_3$

If $n \equiv 0 \bmod 3,\ n \geq 12$ then $H_{\nu(n)}(M_n)$ is a nontrivial finite 3-group with exponent ≤ 9.

Proof idea: Bouc shows for $n = 0, 1 \bmod 3$, $H_{\nu(n)}(M_n)$ is generated by cycles of the form $\alpha \wedge \beta$ where

$$\alpha \in H_0(M_{\{a,b,c\}}), \quad \beta \in H_{\nu(n-3)}(M_{[n]\setminus\{a,b,c\}}).$$

Since

$$3(\alpha \wedge \beta) = \alpha \wedge 3\beta$$

induction can be used.

Base step of induction for 1 mod 3:

$$H_{\nu(7)}(M_7) = \mathbb{Z}_3$$

Shareshian & MW (2004): For $n \geq 12$ (except possibly $n = 14$), $H_{\nu(n)}(M_n)$ has exponent 3.

Our improvement:

Base step for $n \equiv 0 \bmod 3$: Computer calculation using homology software of Heckenbach, Welker, Dumas, Saunders:

$$H_{\nu(12)}(M_{12}) = \mathbb{Z}_3^{56}$$

$n \equiv 2 \bmod 3$: We show that $H_{\nu(n)}(M_n)$ is generated by cycles of the form

$$\alpha \wedge \beta,$$

$$\alpha \in H_0(M_{\{a,b,c,d,e\}}), \;\; \beta \in H_{\nu(n-5)}(M_{[n]\setminus\{a,b,c,d,e\}})$$

Since $3(\alpha \wedge \beta) = \alpha \wedge 3\beta$ and $n - 5 \equiv 0 \bmod 3$, we can apply the 0 mod 3 case when $n - 5 \geq 12$.

Tits Coset Complexes

Let G be a group and $G^1, \ldots, G^m$ a family of subgroups. Form a simplicial complex

$$\Delta(G; G^1, \ldots, G^m)$$

whose vertices are the cosets of the subgroups and whose facets are of the form $\{gG^1, \ldots, gG^m\}$ where $g \in G$.

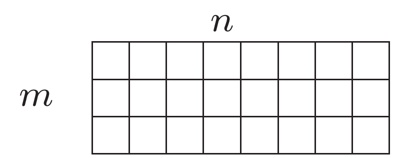
Examples: Coxeter complexes, Tits buildings, chessboard complexes (Garst)

$M_{m,n} = \Delta(G; G^1, \ldots, G^m)$ where

$G = \mathfrak{S}_n$ and $G^i = \{\sigma \in \mathfrak{S}_n \mid \sigma(i) = i\}$ for $i = 1, \ldots, m$.

Garst (1979): $M_{m.n}$ has the homotopy type of a wedge of $(m-1)$-spheres if and only if $2m - 1 \leq n$.

Assume $m \leq n$. dim $M_{m,n} = m - 1$.



Björner, Lovász, Vrécica, Živaljević (1994): Let

$$\nu(m,n) = \min\{m, \lfloor \frac{m+n+2}{3} \rfloor\} - 1.$$

Then $M_{m,n}$ is $(\nu(m,n) - 1)$-connected. Consequently

$$H_t(M_{m,n}) = 0 \quad \forall\, t < \nu(m,n).$$

BLVZ Conjecture: $H_{\nu(m,n)}(M_{m,n}) \neq 0$.

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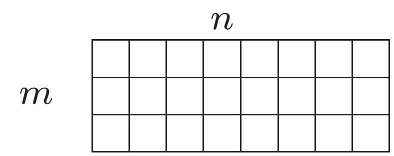

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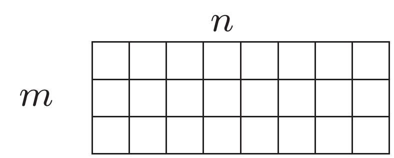

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Friedman & Hanlon (1998): $H_{\nu(m,n)}(M_{m,n})$ infinite iff $n \geq 2m - 5$ or $(m,n) = (6,6), (7,7), (8,9)$

Chessboard complex - homology groups $H_{\nu(m,n)}(M_{m,n})$

$m \backslash n$	2	3	4	5	6	7
1	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^3$	$\mathbb{Z}^4$	$\mathbb{Z}^5$	$\mathbb{Z}^6$
2	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^5$	$\mathbb{Z}^{11}$	$\mathbb{Z}^{19}$	$\mathbb{Z}^{29}$
3		$\mathbb{Z}^4$	$\mathbb{Z}^2$	$\mathbb{Z}^{14}$	$\mathbb{Z}^{47}$	$\mathbb{Z}^{104}$
4			$\mathbb{Z}^{15}$	$\mathbb{Z}^{20}$	$\mathbb{Z}^5$	$\mathbb{Z}^{225}$
5				$\mathbb{Z}_3$	$\mathbb{Z}^{152}$	$\mathbb{Z}^{98}$
6					$\mathbb{Z}^{25} \oplus \mathbb{Z}_3^{10}$	$\mathbb{Z}_3$
7						$\mathbb{Z}^{588} \oplus \mathbb{Z}_3^{66}$

Shareshian & MW (2004): BLVZ conjecture true for all m, n.
Finite $H_{\nu(m,n)}(M_{m,n})$ always has 3-torsion.

Moreover, if $m + n \equiv 1 \bmod 3$ and $n \leq 2m - 5$ then

$$H_{\nu(m,n)}(M_{m,n}) \cong \mathbb{Z}_3$$

Analog of Bouc s long exact sequence $\Longrightarrow H_{\nu(m,n)}(M_{m,n})$ is generated by cycles of the form

$$\alpha \wedge \beta$$

where

$$\alpha \in H_0(M_{2,1}), \qquad \beta \in H_{\nu(m-2,n-1)}(M_{m-2,n-1})$$

or

$$\alpha \in H_0(M_{1,2}), \qquad \beta \in H_{\nu(m-1,n-2)}(M_{m-1,n-2})$$

.

We must factor cycles in the top homology in order to apply induction.

Top Homology

The symmetric group $\mathfrak{S}_n$ acts on $M_{m,n}$ by permuting the columns of the chessboard. This induces a representation of $\mathfrak{S}_n$ on $H_*(M_{m,n})$.

Garst (1979): As $\mathfrak{S}_n$-modules

$$H_{m-1}(M_{m,n};\mathbb{C}) \stackrel{\sim}{=} \bigoplus_{\substack{\lambda \vdash n \\ \lambda_1 = n-m}} f^{\lambda^*} S^{\lambda}$$

where λ^* is obtained from λ by removing the first row.

So rank$(H_{m-1}(M_{m,n}))$ = # pairs of SYT of sizes m and n, whose shapes differ by a row (Garst pair).

For each Garst pair we construct a cycle and a cocycle of top dimension.

P

1	3
2	
4	

Q

1	2	4	6
3	5		
7			
8			

P

1	3	∞	∞
2	∞		
4			
∞			

Q

1	2	4	6
3	5		
7			
8			

P

1	3	∞	∞
2	∞		
4			
∞			

Q

1	2	4	6
3	5		
7			
8			

P

2	3	∞	∞
4	∞		
∞			

Q

1	2	4	6
3	5		
7			

1

8

P

2	4	∞	∞
∞	∞		

Q

1	2	4	6
3	5		

3 1

7 8

P

2	4	∞	
∞	∞		

Q

1	2	4	
3	5		

∞ 3 1

6 7 8

P

2	∞	∞	
∞			

Q

1	2	4	
3			

$$\begin{matrix} 4 & \infty & 3 & 1 \\ 5 & 6 & 7 & 8 \end{matrix}$$

P

2	∞		
∞			

Q

1	2		
3			

$$\begin{matrix} \infty & 4 & \infty & 3 & 1 \\ 4 & 5 & 6 & 7 & 8 \end{matrix}$$

P

∞	∞		

Q

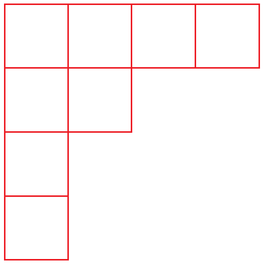
1	2		

2	∞	4	∞	3	1
3	4	5	6	7	8

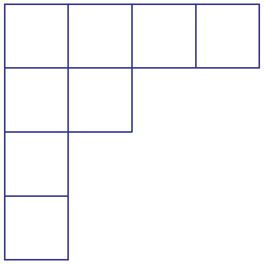
$\infty \quad 2 \quad \infty \quad 4 \quad \infty \quad 3 \quad 1$

$2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8$

P

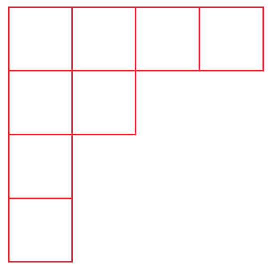


Q

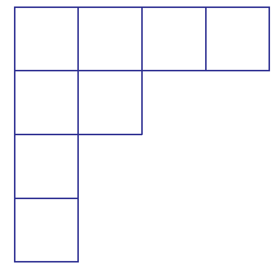


∞	∞	2	∞	4	∞	3	1
1	2	3	4	5	6	7	8

P



Q



$\infty\ \infty\ 2\ \infty\ 4\ \infty\ 3\ 1$

$1\ 2\ 3\ 4\ 5\ 6\ 7\ 8$

$$\begin{array}{cccccccc} \infty & \infty & 2 & \infty & 4 & \infty & 3 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array}$$

Cocycle in $M_{4,8}$:

$$\gamma(P, Q) = (2, 3) \wedge (4, 5) \wedge (3, 7) \wedge (1, 8)$$

Cycle

1	3	∞	∞
2	∞		
4			
∞			

1	2	4	6
3	5		
7			
8			

Cycle

1

2	3	∞	∞
4	∞		
∞			

8

1	2	4	6
3	5		
7			

Cycle

1	3		
2	4	∞	∞
∞	∞		

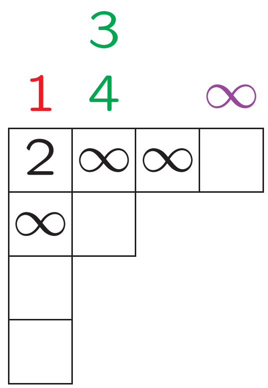
8	7		
1	2	4	6
3	5		

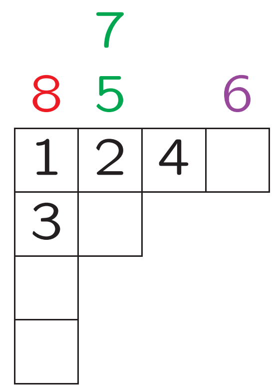
Cycle

1	3		∞
2	4	∞	
∞	∞		

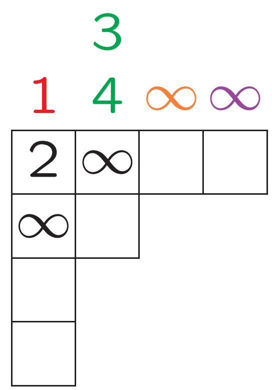
8	7		6
1	2	4	
3	5		

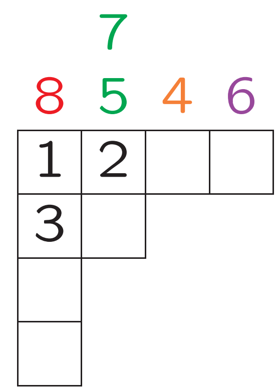
Cycle



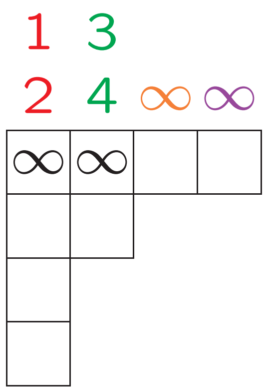


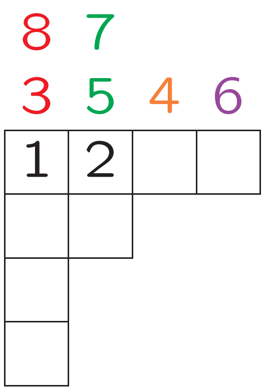
Cycle



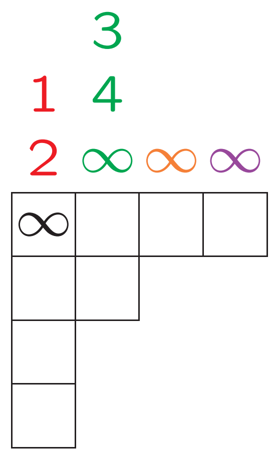


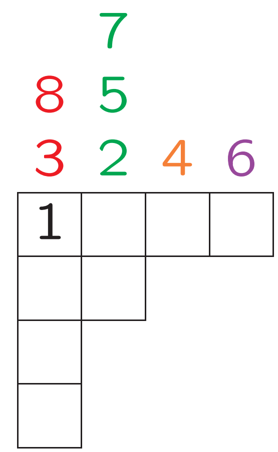
Cycle



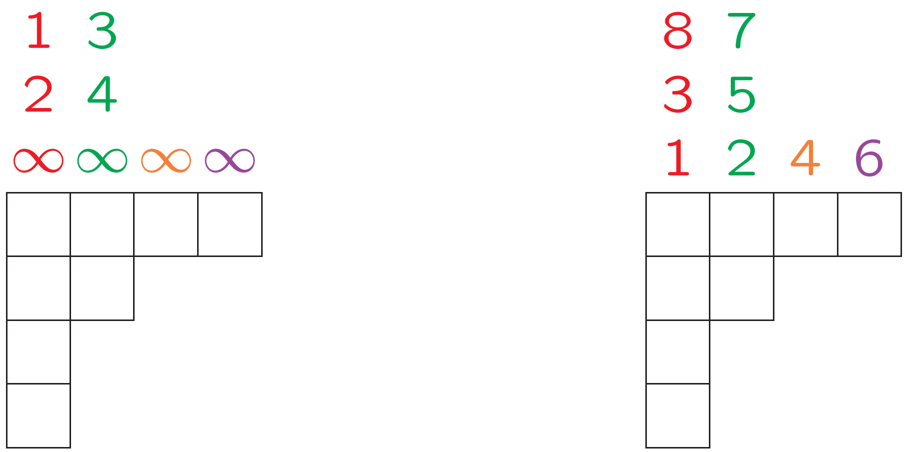


Cycle

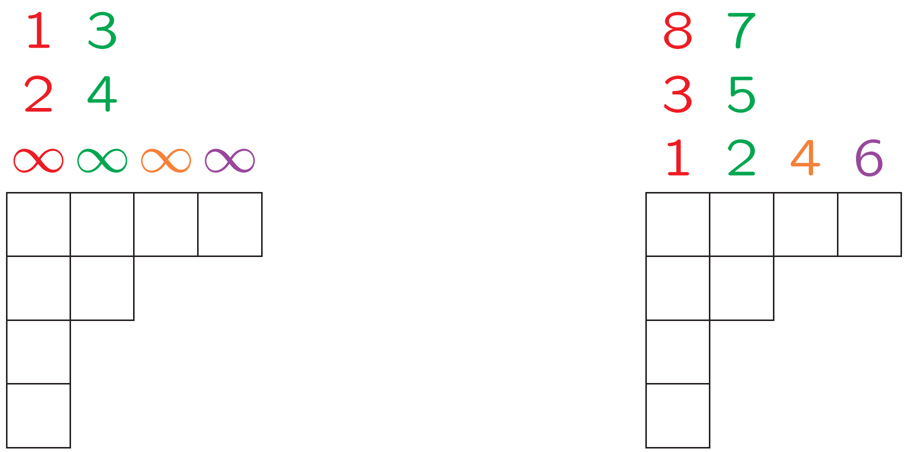




Cycle

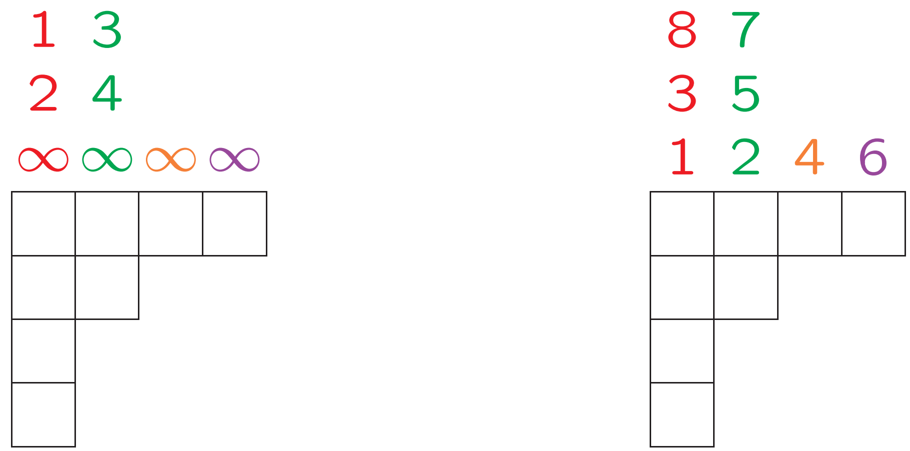


Cycle



$$M_{\{1,2\},\{1,3,8\}},\ M_{\{3,4\},\{2,5,7\}},\ M_{\emptyset,\{4\}},\ M_{\emptyset,\{6\}}$$

Cycle



$M_{\{1,2\},\{1,3,8\}}$, $M_{\{3,4\},\{2,5,7\}}$, $M_{\emptyset,\{4\}}$, $M_{\emptyset,\{6\}}$

Complexes are pseudomanifolds. So top homology is cyclic.

$$H_1(M_{\{1,2\}\{1,3,8\}}) = \langle\alpha\rangle$$
$$H_1(M_{\{3,4\}\{2,5,7\}}) = \langle\beta\rangle$$

Define

$$\rho(P,Q) = \alpha \wedge \beta \in H_3(M_{4,8})$$

Theorem: $\{\rho(P,Q) : (P,Q)$ a Garst pair$\}$ is a basis for $H_{m-1}(M_{m,n})$.

Idea of Proof: We find an ordering of the pairs of standard Young tableaux

$$(P_1, Q_1), \ldots, (P_k, Q_k)$$

so that the matrix

$$(\langle \rho(P_i, Q_i), \gamma(P_j, Q_j) \rangle)_{i,j=1,\ldots,k}$$

is unitriangular.

So $H_{m-1}(M_{m,n})$ is generated by elements of the form

$$[A \mid B] \wedge \tau,$$

where

- $A \subseteq [m]$, $B \subseteq [n]$ and $|A| = |B| - 1$
- $[A \mid B]$ is a fundamental cycle of the pseudomanifold $M_{A,B}$
- $\tau \in H_{m-1-|A|}(M_{[m]-A,[n]-B})$.

This factorization of generating cycles in the top homology enables us to push through the induction step of the torsion result.

The base step $H_{\nu(5,5)}(M_{5,5}) \stackrel{\sim}{=} \mathbb{Z}_3$ was obtained using the software package of Dumas, Heckenbech, Saunders, Welker.

Shareshian & MW: $H_{\nu(m,n)}(M_{m,n})$ is nonvanishing for all m and n. Has 3-torsion whenever finite. Moreover

- $m + n \equiv 1 \bmod 3$
  If $m \leq n \leq 2m - 5$ then
  $$H_{\nu(m,n)}(M_{m,n}) \cong \mathbb{Z}_3$$

- $m + n \equiv 0 \bmod 3$
  If $m \leq n \leq 2m - 9$ then $H_{\nu(m,n)}(M_{m,n})$ is a 3-group with exponent at most 9.

- $m + n \equiv 2 \bmod 3$
  If $m \leq n \leq 2m - 13$ then $H_{\nu(m,n)}(M_{m,n})$ is a 3-group with exponent at most 9.

Conjecture: Shareshian & MW

- If $n > 2m - 5$ then $H_{\nu(m,n)}(M_{m,n})$ torsion-free
- if $n \leq 2m - 5$ then $H_{\nu(m,n)}(M_{m,n})$ has 3-torsion.

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$(m, n) \neq (6, 6), (7, 7), (8, 9)$ then $H_{\nu(m,n)}(M_{m,n})$ is finite. So Shareshian-MW Theorem applies.

$(m, n) = (6, 6), (7, 7)$ computer found 3-torsion.

$(m, n) = (8, 9)$ open

Conjecture: Shareshian & MW

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- if $n \leq 2m-5$ then $H_{\nu(m,n)}(M_{m,n})$ has 3-torsion.

$2m-n \leq 1 \Rightarrow \nu(m,n) = m-1$. Top homology always free.
$2m-n = 2,3,4 \Rightarrow \nu(m,n) = m-2$. Codimension 1 homology

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Shareshian & MW: If $2m - n = 2$ then

$$H_{\nu(m,n)}(M_{m,n}) = \mathbb{Z}^{c_{m-1}},$$

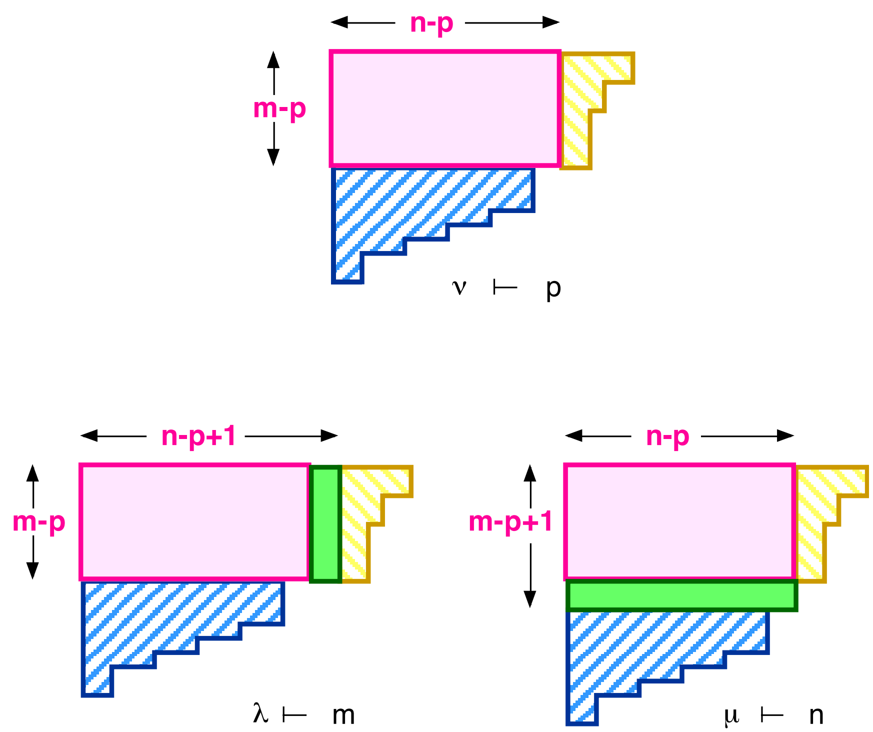
where $c_m = \frac{1}{m+1}\binom{2m}{m}$

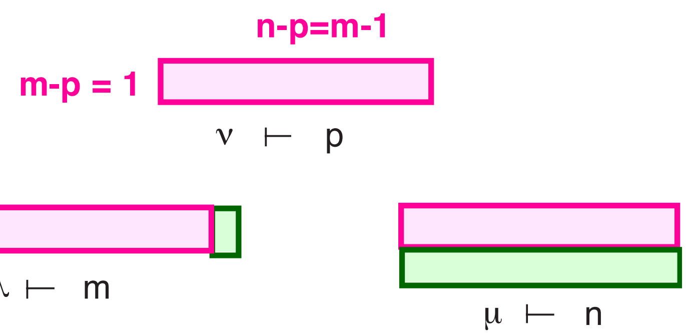
$2m - n = 3, 4$ open.

Friedman & Hanlon(1998): As $(\mathfrak{S}_m \times \mathfrak{S}_n)$-modules,

$$H_{p-1}(M_{m,n};\mathbb{C}) \cong \bigoplus_{(\lambda,\mu)\in\mathcal{R}(m,n,p)} S^{\lambda'} \otimes S^{\mu}$$

where $\mathcal{R}(m,n,p)$ is the set of all pairs of partitions $(\lambda \vdash m,\ \mu \vdash n)$ which can be obtained from a partition $\nu \vdash p$ as follows





$$H_{m-2}(M_{m,2m-2};\mathbb{C}) \cong_{\mathfrak{S}_m \times \mathfrak{S}_{2m-2}} S^{1^m} \otimes S^{(m-1,m-1)}$$

$$H_{m-2}(M_{m,2m-2};\mathbb{C}) \cong_{\mathfrak{S}_{2m-2}} S^{(m-1,m-1)}$$

dim $H_{m-2}(M_{m,2m-2};\mathbb{C}) = \#$SYT of shape $(m-1, m-1) = c_{m-1}$

To show $H_{m-2}(M_{m,2m-2};\mathbb{Z})$ is free we show $\cong S_{\mathbb{Z}}^{(m-1,m-1)}$

Presentation of Specht module $S^{\lambda}_{\mathbb{Z}}$:

Generators: tableaux of shape λ with distinct entries $1, 2, \ldots, n$.

Column relations: $\sigma\ T = \text{sgn}(\sigma)\ T \qquad \forall \sigma \in C(T)$

$$\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 & \\ \cline{1-1} \end{array} = - \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1} \end{array}$$

Garnir relations:

Let $w_{i,j}(T) :=$ word obtained by reading top i entries of column $j+1$ followed by entries of column j from row i down.

7	1	5	10
3	4	2	
9	8		
11	6		

$$w_{2,2} = 52486$$

Define

$$g_{i,j}(T) = \sum_{\sigma} \mathrm{sgn}(\sigma)\ \sigma T$$

summed over all $\sigma \in \mathfrak{S}_n$ such that σT is obtained from T by replacing the red entries of column $j+1$ with a subword of $w_{i,j}$ and replacing the red entries of column j with the complementary subword.

$$\begin{array}{cccc} 7 & 1 & 5 & 10 \\ 3 & 4 & 2 & \\ 9 & 8 & & \\ 11 & 6 & & \end{array} \qquad w_{2,2} = 52486$$

$$g_{2,2} = \begin{array}{cccc} 7 & 1 & 5 & 10 \\ 3 & 4 & 2 & \\ 9 & 8 & & \\ 11 & 6 & & \end{array} - \begin{array}{cccc} 7 & 1 & 5 & 10 \\ 3 & 2 & 4 & \\ 9 & 8 & & \\ 11 & 6 & & \end{array} + \begin{array}{cccc} 7 & 1 & 5 & 10 \\ 3 & 2 & 8 & \\ 9 & 4 & & \\ 11 & 6 & & \end{array} - \begin{array}{cccc} 7 & 1 & 5 & 10 \\ 3 & 2 & 6 & \\ 9 & 4 & & \\ 11 & 8 & & \end{array} + \begin{array}{cccc} 7 & 1 & 2 & 10 \\ 3 & 5 & 4 & \\ 9 & 8 & & \\ 11 & 6 & & \end{array} \pm \ldots$$

The Garnir relations are the relations $g_{i,j}(T) = 0 \quad \forall i, j$

Let $T^\lambda = \{\text{tableaux of shape } \lambda\}$

$S^\lambda_{\mathbb{Z}} = \langle T^\lambda \mid \text{Garnir relations \& column relations} \rangle$

$S_{\mathbb{Z}}^{(m-1,m-1)}$ relations

$$\begin{array}{|c|c|c|}\hline \cdots & a_j & \cdots \\ \hline \cdots & b_j & \cdots \\ \hline \end{array} + \begin{array}{|c|c|c|}\hline \cdots & b_j & \cdots \\ \hline \cdots & a_j & \cdots \\ \hline \end{array} = 0$$

$$\begin{array}{|c|c|c|c|}\hline \cdots & a_{j-1} & a_j & \cdots \\ \hline \cdots & b_{j-1} & \cdot & \cdots \\ \hline \end{array} - \begin{array}{|c|c|c|c|}\hline \cdots & a_j & a_{j-1} & \cdots \\ \hline \cdots & b_{j-1} & \cdot & \cdots \\ \hline \end{array} + \begin{array}{|c|c|c|c|}\hline \cdots & a_j & b_{j-1} & \cdots \\ \hline \cdots & a_{j-1} & \cdot & \cdots \\ \hline \end{array} = 0$$

$$\begin{array}{|c|c|c|c|}\hline \cdots & \cdot & a_j & \cdots \\ \hline \cdots & b_{j-1} & b_j & \cdots \\ \hline \end{array} - \begin{array}{|c|c|c|c|}\hline \cdots & \cdot & a_j & \cdots \\ \hline \cdots & b_j & b_{j-1} & \cdots \\ \hline \end{array} + \begin{array}{|c|c|c|c|}\hline \cdots & \cdot & b_j & \cdots \\ \hline \cdots & a_j & b_{j-1} & \cdots \\ \hline \end{array} = 0$$

Let $\phi : S_{\mathbb{Z}}^{(m-1)^2} \longrightarrow H_{m-2}(M_{m,2m-2})$ be the homomorphism defined on generators by

$$\phi\left(\begin{array}{|c|c|c|c|}\hline a_1 & a_2 & \cdots & a_{m-1} \\ \hline b_1 & b_2 & \cdots & b_{m-1} \\ \hline \end{array}\right)$$

$$= [1 \mid a_1, b_1] \wedge [2 \mid a_2, b_2] \wedge \ldots \wedge [m-1 \mid a_{m-1}, b_{m-1}],$$

where $[j \mid a_j, b_j]$ denotes the fundamental cycle $(j, a_j) - (j, b_j)$ of $H_0(M_{\{j\},\{a_j,b_j\}})$

Clear that column relations map to 0.

$$\phi\left(\begin{array}{|c|c|c|}\hline \cdots & a_j & \cdots \\ \hline \cdots & b_j & \cdots \\ \hline \end{array} + \begin{array}{|c|c|c|}\hline \cdots & b_j & \cdots \\ \hline \cdots & a_j & \cdots \\ \hline \end{array}\right) = (\cdots [j \mid a_j, b_j] \cdots) + (\cdots [j \mid b_j, a_j] \cdots)$$

Check that Garnir relations map to boundary relations, i.e.,

$$\phi\left(\begin{array}{|c|c|c|c|}\hline \cdots & a_{j-1} & a_j & \cdots \\ \hline \cdots & b_{j-1} & \cdot & \cdots \\ \hline \end{array} - \begin{array}{|c|c|c|c|}\hline \cdots & a_j & a_{j-1} & \cdots \\ \hline \cdots & b_{j-1} & \cdot & \cdots \\ \hline \end{array} + \begin{array}{|c|c|c|c|}\hline \cdots & a_j & b_{j-1} & \cdots \\ \hline \cdots & a_{j-1} & \cdot & \cdots \\ \hline \end{array}\right)$$

is a boundary

Claim: ϕ is surjective.

Theorem: $H_{m-2}(M_{m,2m-2})$ is generated by elements of the form

$$[1 \mid a_1, b_1] \wedge [2 \mid a_2, b_2] \wedge \ldots \wedge [m-1 \mid a_{m-1}, b_{m-1}],$$

where $[j \mid a_j, b_j] := (j, a_j) - (j, b_j)$.

$S_{\mathbb{Z}}^{(m-1)^2} \xrightarrow{\phi} H_{m-2}(M_{m,2m-2}) \xrightarrow{\pi} H_{m-2}(M_{m,2m-2})/H_{m-2}(M_{m,2m-2})_{\text{tor}}$

surjection between free abelian groups of the same rank c_{m-1}

So $H_{m-2}(M_{m,2m-2})$ is free of rank c_{m-1}.