Representations of the symmetric group on the homology of matching complexes

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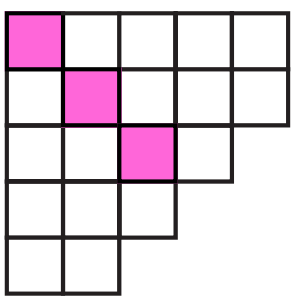
Littlewood:

$$\prod_{i\le j}(1-x_ix_j)\prod_i(1-x_i)^{-1} = \sum_{\lambda=\lambda'}(-1)^{(|\lambda|-d(\lambda))/2}\, s_\lambda$$

λ is a partition
λ' is the conjugate of λ
s_λ is a Schur function

$d(\lambda)$ is the size of the diagonal of λ.

λ =



d(λ) = 3

Macdonald: Weyl denominator formula for root system B_n

Burge: Combinatorial proof - sign reversing involution

Stanley: Representation theoretic interpretation?

- Jozefiak and Weyman: minimal free resolution of quotient of a polynomial ring

- Sigg: homology of free two-step nilpotent Lie algebra

Extract degree n terms

$$\sum_p (-1)^p \, e_p[h_2] \, h_{n-2p} = \sum_p (-1)^p \sum_{\substack{\lambda : \; \lambda \vdash n \\ \lambda = \lambda' \\ d(\lambda) = n-2p}} s_\lambda$$

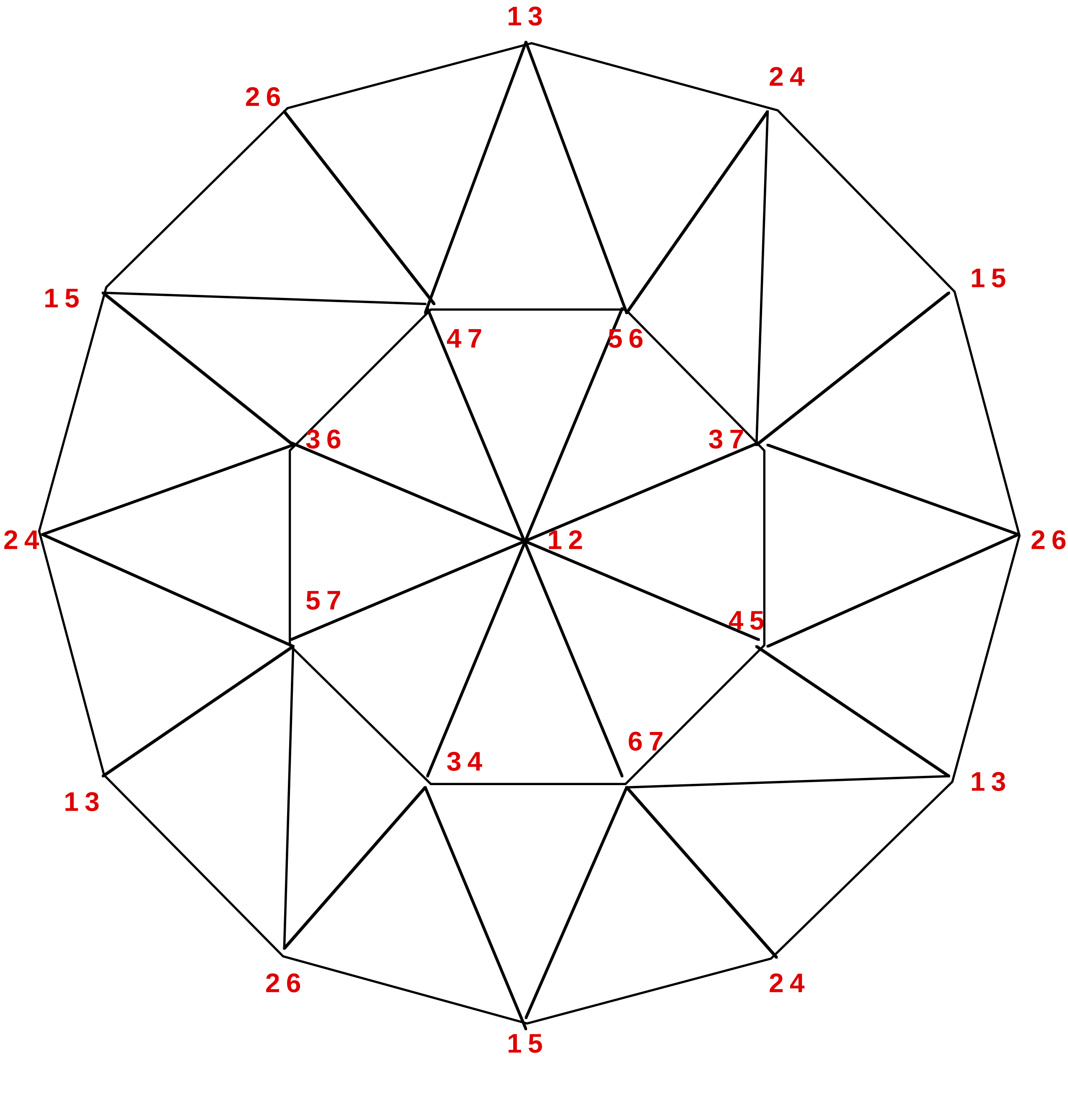
Hopf trace formula for the **matching complex**

The matching complex M_n is the simplicial complex of matchings on

$$[n] := \{1, 2, \ldots, n\}$$

vertices	:=	2 element subsets of $[n]$
faces	:=	sets of pairwise disjoint 2 element subsets of $[n]$

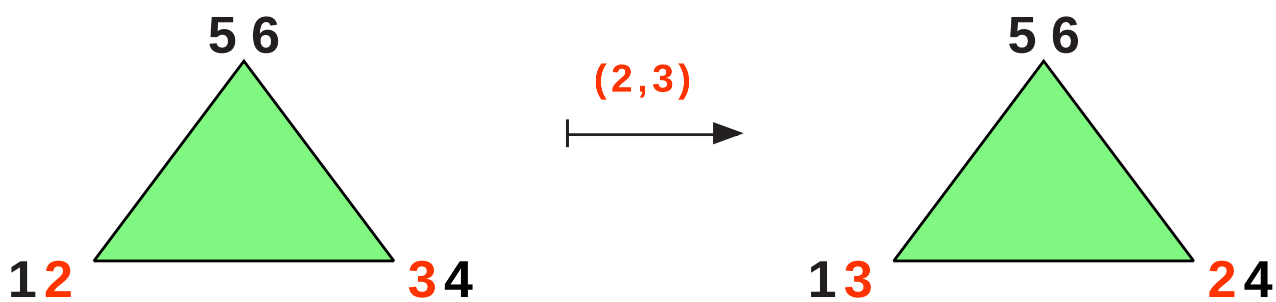
$M_4 =$



Piece of M_7

Homology of the Matching Complex

The symmetric group $\mathfrak{S}_n$ acts on M_n by permuting node labels.



This induces a representation of $\mathfrak{S}_n$ on $C_p(M_n;\mathbb{C})$ and $\tilde{H}_p(M_n;\mathbb{C})$.

$$C_{p-1}(M_n) \cong (S^{1^p}[S^2] \times S^{n-2p}) \uparrow_{\mathfrak{S}_p[\mathfrak{S}_2]\times\mathfrak{S}_{n-2p}}^{\mathfrak{S}_n}$$

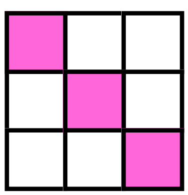
where S^λ is the irreducible $\mathfrak{S}_n$-module indexed by λ

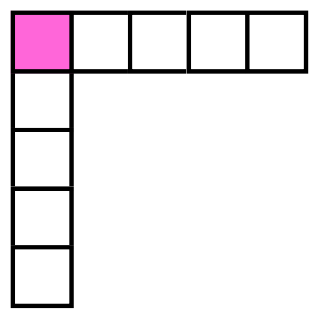
Theorem (Bouc, 1990). As $\mathfrak{S}_n$-modules,

$$\tilde{H}_{p-1}(M_n;\mathbb{C}) \cong \bigoplus_{\substack{\lambda:\ \lambda\vdash n \\ \lambda=\lambda' \\ d(\lambda)=n-2p}} S^{\lambda}$$

Example: $\tilde{H}_{p-1}(M_9;\mathbb{C})$

Self-conjugate shapes with 9 cells:





p = $\frac{9 - 3}{2}$ = 3

p = $\frac{9 - 1}{2}$ = 4

$\tilde{H}_2(M_9;\mathbb{C}) = S^{333}$

$\tilde{\beta}_2(M_9) = \frac{9!}{5\cdot4\cdot3\cdot4\cdot3\cdot2\cdot3\cdot2}$

$= 42$

$\tilde{H}_3(M_9;\mathbb{C}) = S^{51111}$

$\tilde{\beta}_3(M_9) = \frac{9!}{9\cdot4\cdot3\cdot2\cdot4\cdot3\cdot2\cdot}$

$= 70$

Bouc's Theorem

- Bouc(1990) - Quillen complexes

- Józefiak & Weyman(1988) - homology of complex of $GL_n(\mathbb{C})$-modules arising from study of minimal free resolutions of quotient of a polynomial ring

- Karaguezian(1994) - homology of bounded degree graph complexes

- Sigg(1996) - homology of free two-step nilpotent Lie algebra

- Reiner & Roberts(1997) - minimal free resolutions - homology of bounded degree graph complexes

Laplacian Proof of Bouc's Theorem

Dong & MW(1999) (ideas borrowed from Sigg's work on the homology of the free two-step nilpotent Lie algebra and from Friedman & Hanlon's work on the homology of the chess-board complex)

The **combinatorial Laplacian** $\Lambda_p : C_p(M_n, \mathbb{C}) \to C_p(M_n, \mathbb{C})$ is defined by

$$\Lambda_p = \delta_{p-1}\partial_p + \partial_{p+1}\delta_p$$

where ∂ is the boundary map and δ is the coboundary map.

Analogue of Hodge Theory (Kostant):

$$\tilde{H}_p(M_n; \mathbb{C}) \cong_{\mathfrak{S}_n} \ker \Lambda_p$$

Lemma For all $\gamma \in C_p(M_n;\mathbb{C})$,

$$\Lambda_p(\gamma) = T \cdot \gamma$$

where

$$T = \sum_{1 \le i < j \le n} (i,j) \ \in \ \mathbb{C}\mathfrak{S}_n.$$

How does T act on irreducibles?

Macdonald: For all $\gamma \in S^\lambda$,

$$T \cdot \gamma = c_\lambda \gamma$$

where

$$c_\lambda = \sum_{i=1}^{d} \left(\binom{\alpha_i + 1}{2} - \binom{\beta_i + 1}{2} \right)$$

for

$$\lambda = (\alpha_1, \ldots, \alpha_d \mid \beta_1, \ldots, \beta_d)$$

Now decompose $C_p(M_n;\mathbb{C})$ into irreducibles by using another Littlewood formula,

$$\prod_{i\le j}(1+x_ix_j) = \sum_\lambda s_\lambda$$

summed over partitions λ of the form

$$(\alpha_1+1,\ldots,\alpha_d+1 \mid \alpha_1,\ldots,\alpha_d).$$

By Pieri's rule

$$C_{p-1}(M_n;\mathbb{C}) \cong \bigoplus_{\lambda\in A_n} b_\lambda^p\, S^\lambda$$

where

$$A_n = \{\lambda \vdash n \mid \lambda = (\alpha_1,\ldots,\alpha_d \mid \beta_1,\ldots,\beta_d),\ \alpha_i \ge \beta_i\}$$

For λ self-conjugate, $b_\lambda^p = 1$ if $d(\lambda) = n-2p$ and is 0 otherwise.

For $\lambda \in A_n$, we see $c_\lambda = 0$ iff λ is self-conjugate. So

$$\ker \Lambda_{p-1} = \bigoplus_{\substack{\lambda:\ \lambda \vdash n \\ \lambda = \lambda'}} b_\lambda^p \, S^\lambda = \bigoplus_{\substack{\lambda:\ \lambda \vdash n \\ \lambda = \lambda' \\ d(\lambda) = n-2p}} S^\lambda$$

Problem. Find a natural basis for $\tilde{H}_p(M_n; \mathbb{C})$ (or for $\ker \Lambda_p$) indexed by standard tableaux of self-conjugate shape.

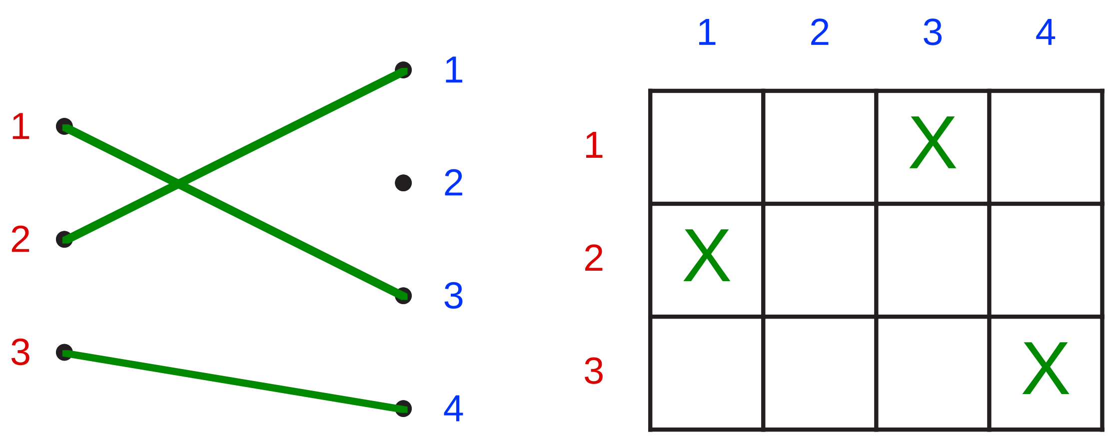
Chessboard Complex $M_{m,n}$

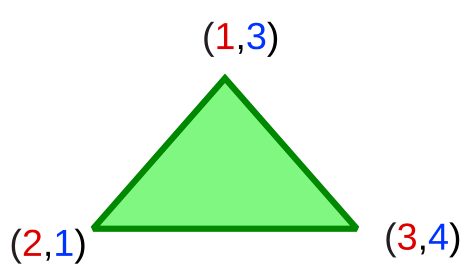
vertex set $:= [m] \times [n]$

face $:= \{(i_1, j_1), \ldots, (i_k, j_k)\}$

$i_s \neq i_t, \; j_s \neq j_t \; \forall s, t$

facet of $M_{3,4}$:





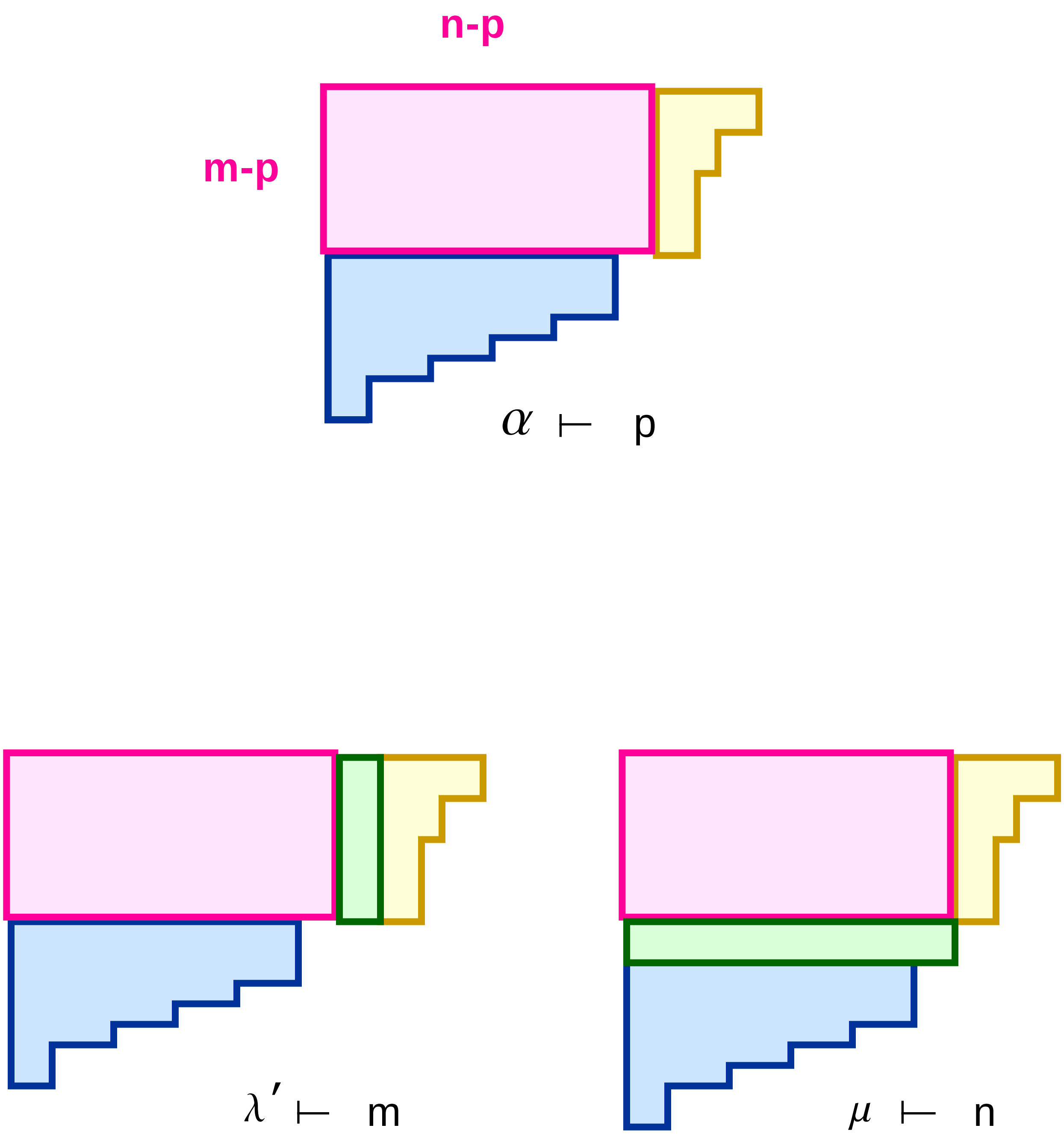
Homology of the Chessboard Complex

The group $\mathfrak{S}_m \times \mathfrak{S}_n$ acts on $M_{m,n}$ which induces a representation of $\mathfrak{S}_m \times \mathfrak{S}_n$ on $\tilde{H}_p(M_{m,n};\mathbb{C})$.

Friedman & Hanlon(1998): As $\mathfrak{S}_m \times \mathfrak{S}_n$-modules,

$$\tilde{H}_{p-1}(M_{m,n};\mathbb{C}) \cong \bigoplus_{\lambda,\mu} S^{\lambda} \otimes S^{\mu}$$

summed over all pairs of partitions $\lambda \vdash m$, $\mu \vdash n$ which can be obtained from a partition $\alpha \vdash p$ in the following way


n-p
m-p
$\alpha \vdash$ p
$\lambda' \vdash$ m
$\mu \vdash$ n

Garst(1979): Tits coset complexes - computed top homology

Vrécica & Živaljević(1992): Computational geometry (Colored Tverberg Problem) - connectivity bounds

Reiner & Roberts(1997): Commutative algebra (minimal free resolutions) -bounded degree bipartite graph complexes

Björner, Lovász, Vrécica, Živaljević (1994): Let

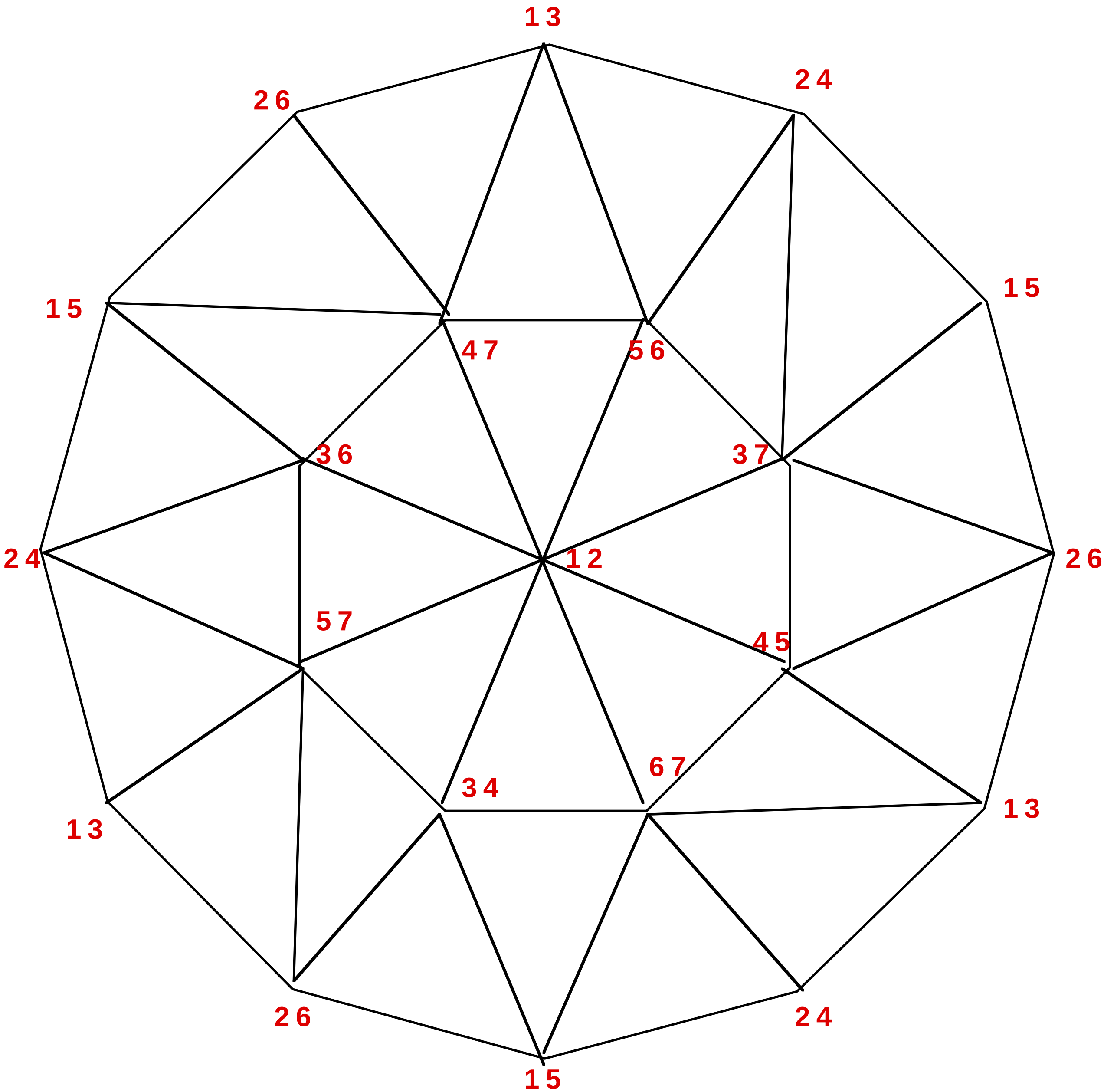
$\nu_n = \lfloor \frac{n+1}{3} \rfloor - 1$ & $\nu_{m,n} = \min\{m, \lfloor \frac{m+n+2}{3} \rfloor\} - 1$.

Then

- M_n is $(\nu_n - 1)$-connected
- $M_{m,n}$ is $(\nu_{m,n} - 1)$-connected

BLVZ Conjecture:

- $\tilde{H}_{\nu_n}(M_n; \mathbb{Z}) \neq 0$
- $\tilde{H}_{\nu_{m,n}}(M_{m,n}; \mathbb{Z}) \neq 0$



Piece of M_7

Bouc (1990):

If $n \equiv 1 \bmod 3$, $n \geq 7$ then $\tilde{H}_{\nu_n}(M_n) \cong \mathbb{Z}_3$

If $n \equiv 0 \bmod 3$, $n \geq 12$ then $\tilde{H}_{\nu_n}(M_n)$ is a nontrivial finite 3-group with exponent at most 9.

Shareshian & MW (1999): $\tilde{H}_{\nu_n}(M_n)$ is nonvanishing for all n. Moreover, for all $n \geq 12$ (except possibly $n = 14$)

$$\tilde{H}_{\nu_n}(M_n) \cong \mathbb{Z}_3^{r_n}$$

where $r_n \geq 1$.

Proof idea: Bouc shows for $n = 0, 1 \bmod 3$, $\tilde{H}_{\nu_n}(M_n)$ is generated by cycles of the form

$$\alpha * \beta$$

where $\alpha \in \tilde{H}_0(M_{\{a,b,c\}})$, $\quad \beta \in \tilde{H}_{\nu_{n-3}}(M_{[n]\setminus\{a,b,c\}})$. Since $3(\alpha * \beta) = \alpha * 3\beta$, induction can be used.

Base step of induction for 1 mod 3:

$$\tilde{H}_{\nu_7}(M_7) = \mathbb{Z}_3$$

Our improvement:

Base step for $n \equiv 0 \bmod 3$: Computer calculation using homology software of Heckenbach, Welker, Dumas, Saunders:

$$\tilde{H}_{\nu_{12}}(M_{12}) = \mathbb{Z}_3^{56}$$

$n \equiv 2 \bmod 3$: We show that $\tilde{H}_{\nu_n}(M_n)$ is generated by cycles of the form

$$\alpha * \beta,$$

$$\alpha \in \tilde{H}_0(M_{\{a,b,c,d,e\}}), \ \ \beta \in \tilde{H}_{\nu_{n-5}}(M_{[n]\setminus\{a,b,c,d,e\}})$$

Since $3(\alpha * \beta) = \alpha * 3\beta$ and $n - 5 \equiv 0 \bmod 3$, we can apply the 0 mod 3 case when $n - 5 \geq 12$

Open Problems:

- Eliminate use of the computer
- Determine r_n
- Show that there is only 3-torsion in all homology groups of M_n (conjecture of Babson, Björner, Linusson, Shareshian, Welker)

Chessboard Complex - homology groups

$$\tilde{H}_{\nu_{m,n}}(M_{m,n})$$

$m \backslash n$	2	3	4	5	6	7
1	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^3$	$\mathbb{Z}^4$	$\mathbb{Z}^5$	$\mathbb{Z}^6$
2	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^5$	$\mathbb{Z}^{11}$	$\mathbb{Z}^{19}$	$\mathbb{Z}^{29}$
3		$\mathbb{Z}^4$	$\mathbb{Z}^2$	$\mathbb{Z}^{14}$	$\mathbb{Z}^{47}$	$\mathbb{Z}^{104}$
4			$\mathbb{Z}^{15}$	$\mathbb{Z}^{20}$	$\mathbb{Z}^5$	$\mathbb{Z}^{225}$
5				$\mathbb{Z}_3$	$\mathbb{Z}^{152}$	$\mathbb{Z}^{98}$
6					$\mathbb{Z}^{25} \oplus \mathbb{Z}_3^{10}$	$\mathbb{Z}_3$
7						$\mathbb{Z}^{588} \oplus \mathbb{Z}_3^{66}$

Torsion in the Chessboard Complex

Note: If $2m - 1 \leq n$ then $\nu_{m,n} = m - 1$. So $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is top homology which is free.

Shareshian & MW (2000): $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is non-vanishing for all m and n. Moreover

- $n + m \equiv 1 \bmod 3$
  If $5 \leq m \leq n \leq 2m - 5$ then
  $$\tilde{H}_{\nu_{m,n}}(M_{m,n}) \equiv \mathbb{Z}_3$$

- $n + m \equiv 0 \bmod 3$
  If $9 \leq m \leq n \leq 2m - 9$ then $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is a 3-group with exponent at most 9.

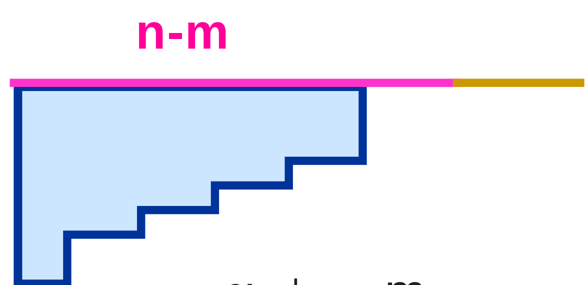
- $n + m \equiv 2 \bmod 3$
  If $13 \leq m \leq n \leq 2m - 13$ then $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is a 3-group with exponent at most 9.

Basis for Top Homology and Cohomology

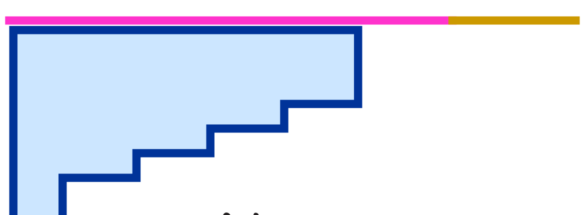
Recall Friedman-Hanlon

$$\tilde{H}_{p-1}(M_{m,n};\mathbb{C}) \cong \bigoplus_{\lambda,\mu} S^{\lambda} \otimes S^{\mu}$$

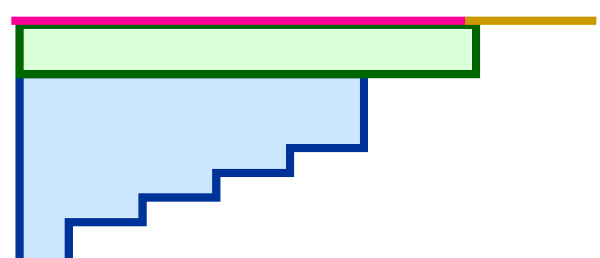
When $p = m \leq n$



$\alpha \vdash m$

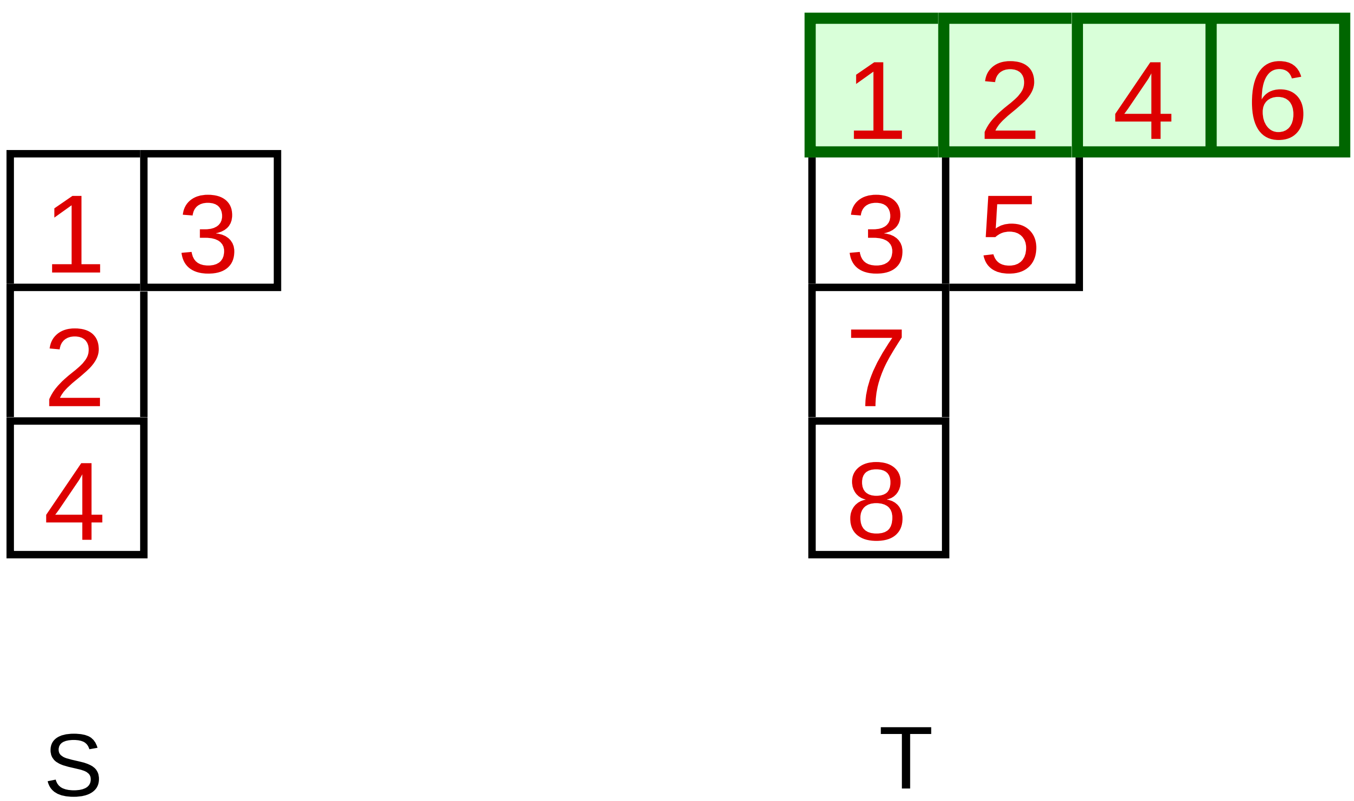


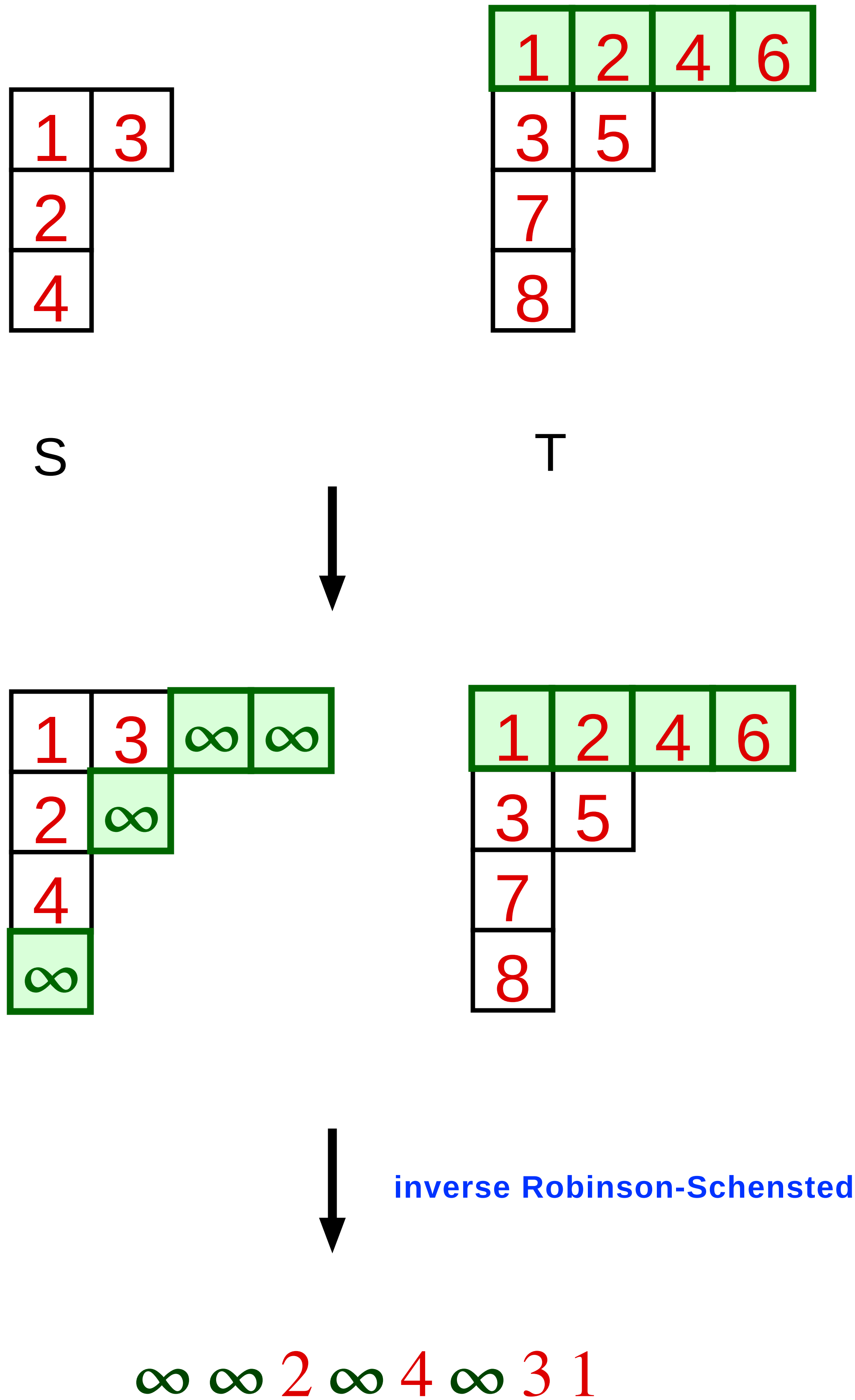
$\lambda' \vdash m$



$\mu \vdash n$

From a pair of young tableaux of shape λ and μ we construct a cycle and a cocycle of dimension $m-1$ using the Robinson-Schensted correspondence.



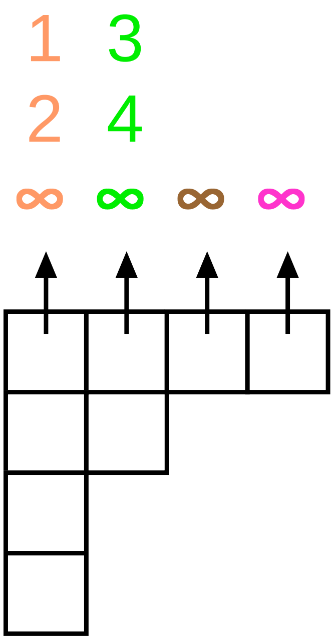

1 3
2
4
S
1 2 4 6
3 5
7
8
T
1 3 ∞ ∞
2 ∞
4
∞
1 2 4 6
3 5
7
8
inverse Robinson-Schensted
∞ ∞ 2 ∞ 4 ∞ 3 1

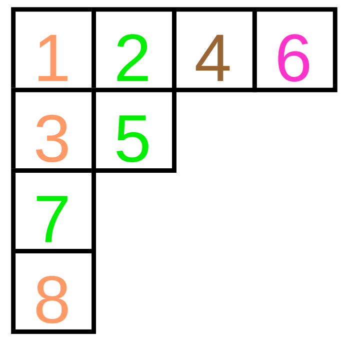
Cocycle:

∞	∞	2	∞	4	∞	3	1
1	2	3	4	5	6	7	8

$$\gamma(S,T) = (23, 45, 37, 18)$$

Cycle:





$M_{\{1,2\}\{1,3,8\}}$ and $M_{\{3,4\}\{2,5,7\}}$ are pseudomanifolds. So top homology is cyclic. Let

$$\tilde{H}_1(M_{\{1,2\}\{1,3,8\}}) = \langle \alpha \rangle$$
$$\tilde{H}_1(M_{\{3,4\}\{2,5,7\}}) = \langle \beta \rangle$$

Define

$$\rho(S,T) = \alpha * \beta \in \tilde{H}_3(M_{4,8})$$

Shareshian & MW:

- $\{\rho(S,T) \mid (S,T)$ is a Friedman-Hanlon pair$\}$ is a basis for $\tilde{H}_{m-1}(M_{m,n})$
- $\{\gamma(S,T) \mid (S,T)$ is a Friedman-Hanlon pair$\}$ is a basis for the free part of $\tilde{H}^{m-1}(M_{m,n})$

Idea of Proof: We order the pairs of standard tableaux

$$(S_1, T_1), \ldots, (S_k, T_k)$$

so that the matrix

$$(\langle \rho(S_i, T_i), \gamma(S_j, T_j) \rangle)_{i,j=1,\ldots,t}$$

is triangular with 1's on the diagonal. From this we can establish the result.

Shareshian and MW (2001): $\tilde{H}_*(M_{m,n})$ is free for all $n \geq 2m-2$.

$$\tilde{H}_{m-2}(M_{m,2m-2}) \cong \mathbb{Z}^{c_{m-1}}$$

where c_m is the mth Catalan number.

Proof outline: Friedman and Hanlon =>

$$\tilde{H}_{m-2}(M_{m,2m-2};\mathbb{C}) \cong S^{1^m} \otimes S^{(m-1)^2}$$

YT_λ := set of Young tableaux of shape λ
SYT_λ := set of standard Young tableaux of shape λ.

For each $T \in YT_{(m-1)^2}$ construct

$$\rho_T \in \tilde{H}_{m-2}(M_{m,2m-2})$$

as follows

$T =$

a_1	a_2	$\cdots$	$a_{m\text{-}1}$
b_1	b_2	$\cdots$	$b_{m\text{-}1}$

$$\rho_T = \alpha_{\{1\},\{a_1,b_1\}} * \cdots * \alpha_{\{m-1\},\{a_{m-1},b_{m-1}\}}$$

where $\alpha_{\{i\},\{a_i,b_i\}}$ is the fundamental cycle

$$[(i, a_i) - (i, b_i)]$$

of $\tilde{H}_0(M_{\{i\},\{a_i,b_i\}})$

- Show map defined on polytabloids by

$$e_T \mapsto \rho_T$$

is a well-defined $\mathbb{Z}$-homomorphism

$$\varphi : S^{(m-1)^2} \to \tilde{H}_{m-2}(M_{m,2m-2})$$

by using the Garnir relations.

- Show

$$\{\rho_T \mid T \in \mathsf{YT}_{(m-1)^2}\}$$

spans $\tilde{H}_{m-2}(M_{m,2m-2})$ by using the decomposition of bottom homology cycles into smaller cycles.

So φ is onto.

By Friedman-Hanlon, rank of free part of $\tilde{H}_{m-2}(M_{m,2m-2})$ equals rank of $S^{(m-1)^2}$. So φ is an isomorphism which implies that $\tilde{H}_{m-2}(M_{m,2m-2})$ is free of rank $|SYT_{(m-1)^2}| = c_{m-1}$.

Monotone graph property is a property of a labeled graph that is closed under removal of edges and relabeling of vertices.

Examples:

- not connected
- not k-connected
- degree $\leq b$

Set of graphs on vertex set $[n]$ that have a monotone graph property P forms an $\mathfrak{S}_n$-simplicial complex $\Delta(P)$.

$$\begin{aligned} \text{vertices} &:= \text{2 element subsets of } [n] \\ \text{faces} &:= \text{edge sets of graphs on } [n] \\ &\quad\ \text{with Property } P \end{aligned}$$

Second example: $k = 2$, connections with Vasiliev knot invariants

Third example: $b = 1$, $\Delta(P) = M_n$

Generalizations and Variations

Hypergraph version: Björner, Lovász, Vrécica & Živaljević(1992), Björner & Eriksson(1999), Ksontini(2000), Shareshian(2000), Shareshian & MW(2001)

Directed graph and multigraph versions: Björner & Welker(1998), MW(1999)

General Bounded Degree: Reiner & Roberts(1997), Karaguezian, Reiner & MW(1999), Dong(1999), MW(1999)

Littlewood Identities:

$$\prod_{i\le j}(1-x_ix_j)\prod_i(1-x_i)^{-1} = \sum_{\lambda=\lambda'}(-1)^{(|\lambda|-d(\lambda))/2}s_\lambda$$

$$\prod_{i<j}(1-x_ix_j) = \sum_{\mu\in\mathcal{B}}(-1)^{|\mu|/2}s_\mu$$

where $\mathcal{B}$ is the set of all partitions of form $(\alpha_1,\ldots,\alpha_d \mid \alpha_1+1,\ldots,\alpha_d+1)$ for some d.

Combine to get

$$\prod_{i,j}(1-x_ix_j)\prod_i(1-x_i)^{-1} = \sum_{\substack{\lambda=\lambda' \\ \mu\in\mathcal{B}}}(-1)^{(|\lambda|+|\mu)|-d(\lambda))/2}\, s_\lambda s_\mu$$

MW (1999): Hopf Trace formula for the directed matching complex, DM_n

$$\tilde{H}_{p-1}(DM_n;\mathbb{C}) \cong \bigoplus_{\substack{|\lambda|+|\mu|=n \\ \lambda=\lambda' \\ d(\lambda)=n-2p \\ \mu\in\mathcal{B}}} S^{\lambda\otimes\mu}$$

Generalization to Multigraphs

Let M_n^m be the matching complex of the complete multigraph on node set $[n]$ with m distinct edges between each (unordered) pair of vertices.

MW: As $\mathfrak{S}_n$-modules,

$$\tilde{H}_{p-1}(M_n^r;\mathbb{C}) \cong \bigoplus_{\substack{|\lambda|+|\mu|=n \\ \lambda=\lambda' \\ d(\lambda)=n-2p \\ \mu' \in \mathcal{B}}} (r-1)^{|\mu|/2} \, S^{\lambda \otimes \mu}$$

To prove this we express the homology of M_n^r in terms of the homology of M_n. This relationship leads to a generalization of the Quillen Fiber Lemma due to Björner, MW & Welker (1999).