On the property M conjecture for the Heisenberg Lie algebra

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A Lie algebra L has *Property M* if

$$H_*(L \otimes \mathbb{C}[t]/(t^k)) \cong H_*(L)^{\otimes k} \quad \forall k > 0$$

Strong Macdonald Conjecture (Hanlon 1988):
Semisimple $\Longrightarrow$ property M

This implies Macdonald's root system conjectures.

Hanlon also conjectured that the following Lie algebras have property M

- Lie algebra T_n of strictly upper triangular $n \times n$ matrices
- Heisenberg Lie algebra H_{2n+1}

Results

Cherednik (1995): Macdonald root system conjecture

Fishel, Grojnowski, Teleman (2001): Strong Macdonald Conjecture

Kumar (1999): False for T_n, $n \geq 4$

Still open for the Heisenberg Lie algebra H_{2n+1}. Even for $H_3 = T_3$.

Basis for T_3:

$$e = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad f = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad x = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Bracket for T_3:

$$[e, f] = ef - fe = x$$

$$[f, e] = -x$$

All other brackets of basis elements $= 0$

Lie algebra Homology

$$\partial_r : \wedge^r L \to \wedge^{r-1} L$$

$\partial_r(u_1 \wedge \cdots \wedge u_r) =$

$$\sum_{i<j} (-1)^{i+j+1}\, u_1 \wedge \cdots \wedge \widehat{u}_i \wedge \cdots \wedge \widehat{u}_j \wedge \cdots \wedge u_r \wedge [u_i, u_j]$$

Homology of Koszul complex $(\wedge^r L, \partial_r)$:

$$H_r(L) := \ker \partial_r / \operatorname{im} \partial_{r+1}$$

Example: $L = T_3$

$\bigwedge^r T_3 = 0$ for $r \geq 4$ since $\dim T_3 = 3$.

$\bigwedge^3 T_3 = \langle e \wedge f \wedge x \rangle$

$$\begin{aligned} \partial(e \wedge f \wedge x) &= x \wedge [e, f] - f \wedge [e, x] + e \wedge [f, x] \\ &= x \wedge x \\ &= 0 \end{aligned}$$

So $\ker \partial_3 = \langle e \wedge f \wedge x \rangle$ & $\operatorname{im} \partial_4 = 0 \Longrightarrow$

$$H_3(T_3) = \langle e \wedge f \wedge x \rangle$$

.

$$
\begin{aligned}
H_r(T_3) &= 0 && \text{for } r \geq 4 \\
H_3(T_3) &= \langle e \wedge f \wedge x \rangle \\
H_2(T_3) &= \langle e \wedge x, f \wedge x \rangle \\
H_1(T_3) &= \langle e, f \rangle \\
H_0(T_3) &= \langle 1 \rangle
\end{aligned}
$$

So

$$\sum_r \dim H_r(T_3) y^r = 1 + 2y + 2y^2 + y^3$$

Property M for T_3 is

$$\sum_{r \geq 0} \dim H_r(T_3 \otimes \mathbb{C}[t]/(t^k)) y^r = (1+2y+2y^2+y^3)^k$$

Extended Lie algebra

Lie algebra L, associative algebra A
$L \otimes A$ is the Lie algebra with bracket

$$[u \otimes a, v \otimes b] := [u, v] \otimes ab$$

Basis for $L_k := T_3 \otimes \mathbb{C}[t]/(t^k)$:

$$\{e_0, \dots, e_{k-1}, f_0, \dots, f_{k-1}, x_0, \dots, x_{k-1}\},$$

where $e_i = e \otimes t^i$, $f_i = f \otimes t^i$, $x_i = x \otimes t^i$.

The only nonzero brackets on basis elements:

$$[e_i, f_j] = -[f_j, e_i] = x_{i+j} \qquad \text{where } i + j < k.$$

$$\wedge L_k = \bigoplus_{m,n,p} \wedge^m(E) \otimes \wedge^n(F) \otimes \wedge^p(X),$$

where $E = \langle e_0, \dots, e_{k-1}\rangle, F = \langle f_0, \dots, f_{k-1}\rangle$,
$X = \langle x_0, \dots, x_{k-1}\rangle$.

Koszul complex $(\wedge L_k, \partial)$ is graded by (e, f, x)-degree.

Homology graded by (e, f, x)-degree

$$H_*(L_k) = \bigoplus_{m,n,p} H_{m,n,p}(L_k)$$

$$\begin{aligned}
H_3(L_1) &= \langle e_0 \wedge f_0 \wedge x_0 \rangle = H_{111}(L_1) \\
H_2(L_1) &= \langle e_0 \wedge x_0, f_0 \wedge x_0 \rangle = H_{101}(L_1) \oplus H_{011}(L_1) \\
H_1(L_1) &= \langle e_0, f_0 \rangle = H_{100}(L_1) \oplus H_{010}(L_1) \\
H_0(L_1) &= \langle 1 \rangle = H_{000}(L_1)
\end{aligned}$$

So

$$\sum_{m,n,p} \dim H_{m,n,p}(L_1) u^m v^n w^p = 1+u+v+uw+vw+uvw$$

Refinement of Property M for T_3 is

$$\sum_{m,n,p} \dim H_{m,n,p}(L_k) u^m v^n w^p = (1+u+v+uw+vw+uvw)^k$$

Set $w = 0$

Theorem (Hanlon and MW)

$$\sum_{m,n} \dim H_{m,n,0}(L_k) u^m v^n = (1+u+v)^k$$

Equivalently,

$$\dim H_{m,n,0}(L_k) = \binom{k}{m, n, k-m-n}$$

Adin & Athanasiadis (1996): m=1 case.

To prove this we switch to cohomology

$$H^{m,n,p}(L_k) \cong H_{m,n,p}(L_k)$$

Coboundary map δ

$$\begin{aligned}
\delta(e_2 \wedge e_3 \wedge x_6) = \quad & e_2 \wedge e_3 \wedge e_0 \wedge f_6 \\
+ \quad & e_2 \wedge e_3 \wedge e_1 \wedge f_5 \\
+ \quad & e_2 \wedge e_3 \wedge e_2 \wedge f_4 \\
+ \quad & e_2 \wedge e_3 \wedge e_3 \wedge f_3 \\
+ \quad & e_2 \wedge e_3 \wedge e_4 \wedge f_2 \\
+ \quad & e_2 \wedge e_3 \wedge e_5 \wedge f_1 \\
+ \quad & e_2 \wedge e_3 \wedge e_6 \wedge f_0
\end{aligned}$$

$$\delta(e_2 \wedge e_3 \wedge e_4 \wedge f_2) = 0$$

$$\ker \delta_{m,n,0} = \wedge^m(E) \otimes \wedge^n(F)$$

$$H^{m,n,0}(L_k) = \wedge^m(E) \otimes \wedge^n(F)/\mathrm{im}\, \delta_{m-1,n-1,1}$$

Let $E_I = e_{i_1} \wedge \cdots \wedge e_{i_m}$ if $I = \{i_1 < \ldots < i_m\}$. So $H^{m,n,0}(L_k)$ has generating set

$$\{e_I \wedge f_J \mid |I| = m,\ |J| = n\}$$

subject to coboundary relations.

<u>Theorem</u> (Hanlon and MW)

$$\{e_I \wedge f_J \mid |I| = m,\ |J| = n,\ \min J \geq m\}$$

is a basis for $H^{m,n,0}(L_k)$.

Follows that

$$\dim H^{m,n,0}(L_k) = \binom{k}{m}\binom{k-m}{n} = \binom{k}{m, n, k-m-n}$$

To prove spanning we use coboundary relations to "straighten" $e_I \wedge e_J$ when min $J < |I|$.

Difficulties:

- Which coboundary relations?
- What does it mean to be straighter?

Straighten $e_2 \wedge e_3 \wedge e_5 \wedge f_1$

$0 = \delta(e_2 \wedge e_3 \wedge x_6) =$		$e_2 \wedge e_3 \wedge e_0 \wedge f_6$	better
	$+$	$e_2 \wedge e_3 \wedge e_1 \wedge f_5$	better
	$+$	$e_2 \wedge e_3 \wedge e_2 \wedge f_4$	0
	$+$	$e_2 \wedge e_3 \wedge e_3 \wedge f_3$	0
	$+$	$e_2 \wedge e_3 \wedge e_4 \wedge f_2$	better
	$+$	$e_2 \wedge e_3 \wedge e_5 \wedge f_1$	
	$+$	$e_2 \wedge e_3 \wedge e_6 \wedge f_0$	worse

Ordered alphabet

$$\mathcal{A} = \{\bar{0} < 0 < \bar{1} < 1 < \ldots < \overline{k-1} < k-1\}$$

Associate word in $\mathcal{A}^*$ with each wedge product

$$\mu(e_{i_1} \wedge \ldots \wedge e_{i_m} \wedge f_j) = \begin{cases} i_1 i_2 \cdots \overline{i_{j+1}} \cdots i_m & \text{if } j < m \\ i_1 i_2 \cdots i_m & \text{if } j \geq m \end{cases}$$

Straighter = lexicographically smaller word

$$\begin{aligned} 0 = \delta(e_2 \wedge e_5 \wedge x_4) &= e_2 \wedge e_5 \wedge e_0 \wedge f_4 && \mathbf{025} \\ &+ e_2 \wedge e_5 \wedge e_1 \wedge f_3 && \mathbf{125} \\ &+ e_2 \wedge e_5 \wedge e_2 \wedge f_2 && \\ &+ e_2 \wedge e_5 \wedge e_3 \wedge f_1 && \mathbf{2\bar{3}5} \\ &+ e_2 \wedge e_5 \wedge e_4 \wedge f_0 && \mathbf{\bar{2}45} \end{aligned}$$

By Poincaré duality we get a complementary basis for $H_{m,n,k}(L_k)$ and

$$\dim H_{m,n,k}(L_k) = \binom{k}{k-m, k-n, m+n-k}$$

<u>Conjecture</u>

$$\{e_I \wedge x_J \mid |I| = m,\ |J| = p,\ \{0, \ldots, p-1\} \subseteq I\}$$

is a basis for $H_{m,0,p}(L_k)$.

Follows that

$$\dim H_{m,0,p}(L_k) = \binom{k}{p}\binom{k-p}{m-p} = \binom{k}{p, m-p, k-m}$$

Again by Poincaré duality we get a complementary basis for $H^{m,k,p}(L_k)$ and

$$\dim H_{m,k,p}(L_k) = \binom{k}{k-m, k-p, m+p-k}$$