# Bounded Degree Graph Complexes

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Let $G$ be a graph on node set $\{1, 2, \ldots, n\}$.

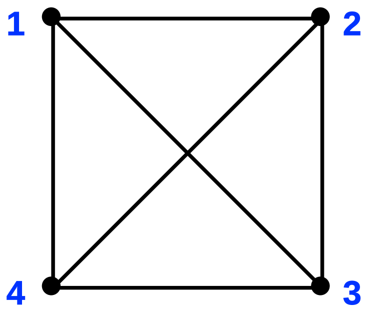
$\Delta(G, r) :=$ simplicial complex of subgraphs of $G$ for which each node has degree $\leq r$.

vertices := edges
faces := subgraphs

$\Delta(K_n, 1)$ is called the **matching complex**.

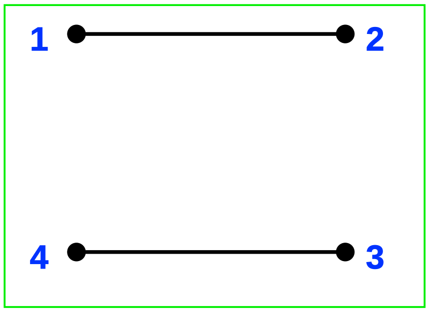
$\Delta(K_{m,n}, 1)$ is called the **chessboard complex**.

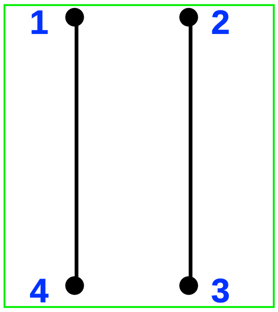
G =

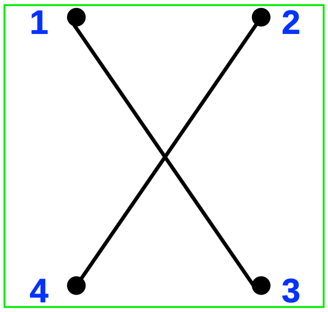


r = 1

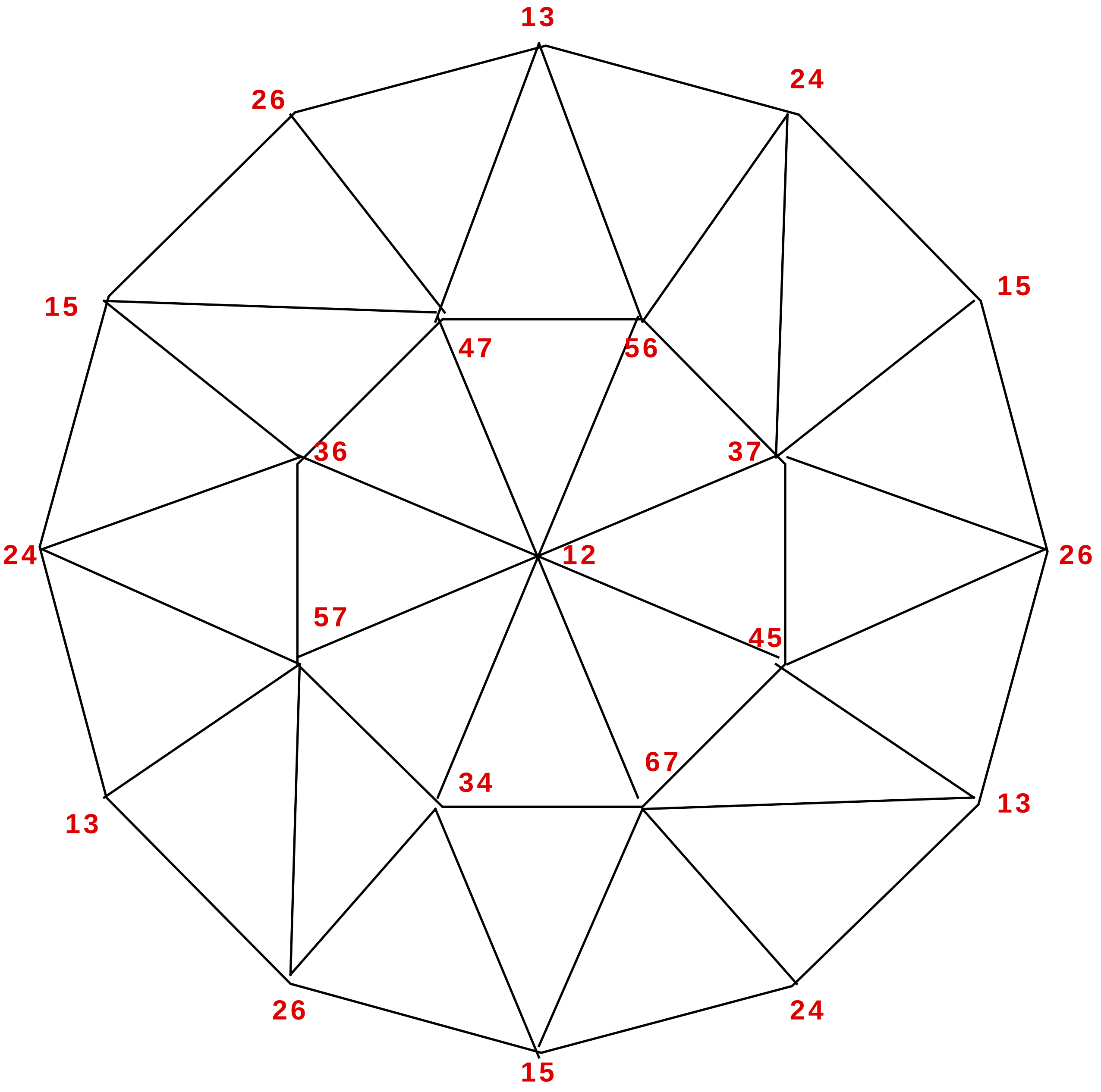
**Maximal bounded degree subgraphs :**





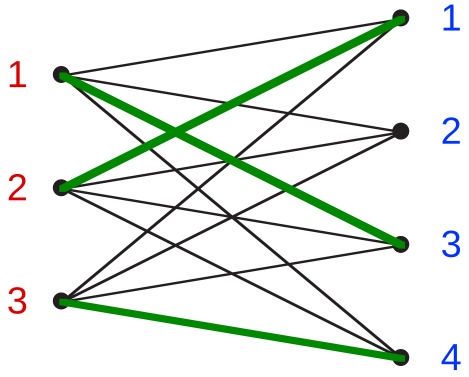


$\Delta(G,1) =$

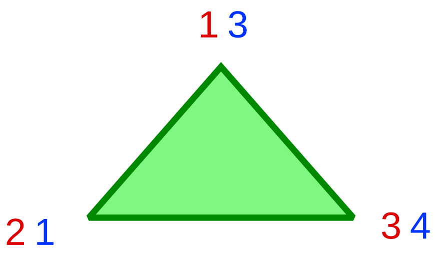


$M_7 := \Delta(K_7,1)$

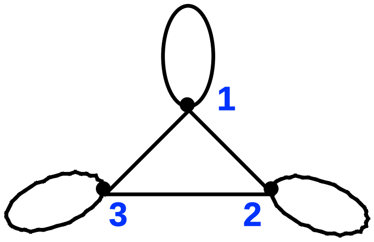
# Chessboard Complex $\Delta(K_{3,4},1)$



| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | | | X | |
| 2 | X | | | |
| 3 | | | | X |

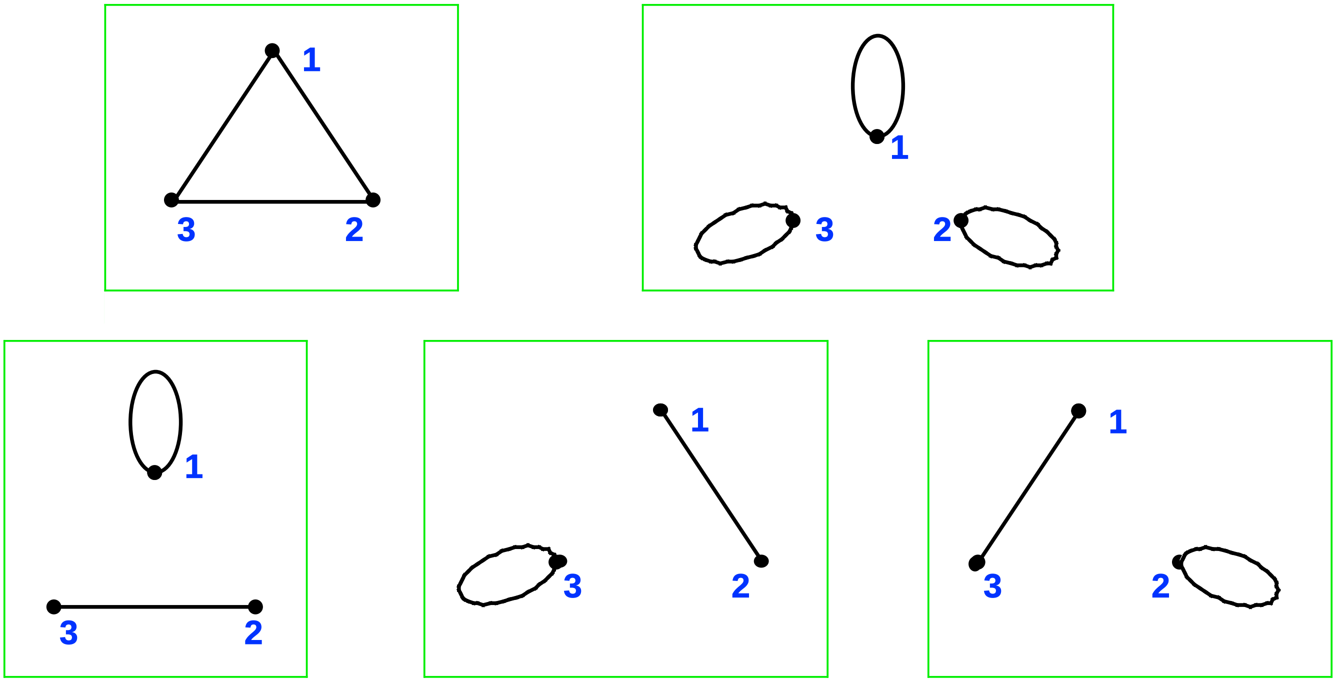


G =

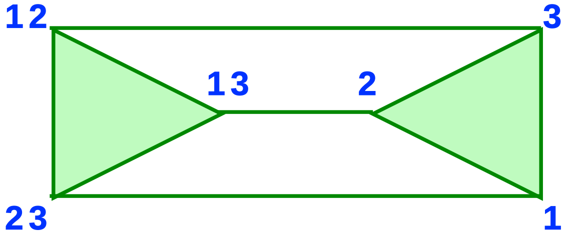


r = 2

Maximal bounded degree subgraphs:



$\Delta(G, 2) =$



Tits coset complexes: Garst(1979)

Commutative algebra (minimal free resolutions): Józefiak & Weyman(1988), Reiner & Roberts(1997)

Quillen's poset of non-trivial $p$-subgroups of a finite group: Bouc(1990)

Computational Geometry (Colored Tverberg Problem): Vrécica & Živaljević(1992)

Homology of nilpotent Lie algebras: Sigg(1996)

Vassiliev knot invariants (not $k$-connected graph complexes): Babson, Björner, Linusson, Shareshian & Welker (1999)

Combinatorial simplicial complexes: Björner, Lovász, Vrécica & Živaljević(1992), Ziegler(1992), Karaguezian(1994), Friedman & Hanlon(1998), Björner & Eriksson (1999), Karaguezian, Reiner & MW(1999), MW(1999), Dong(1999), Dong & MW(2000), Shareshian & MW(2000)

# Tits Coset Complexes

Let $G$ be a group and $G^1, \ldots, G^m$ a family of subgroups. Form a simplicial complex

$$\Delta(G; G^1, \ldots, G^m)$$

whose vertices are the cosets of the subgroups and whose facets are of the form $\{gG^1, \ldots, gG^m\}$ where $g \in G$.

Examples: Coxeter complexes, Tits buildings, chessboard complexes

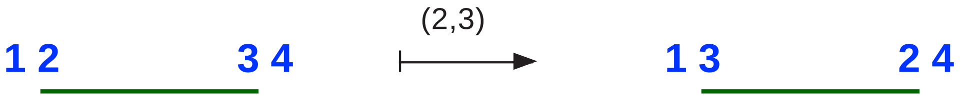
$\Delta(K_{m,n}, 1) = \Delta(G; G^1, \ldots, G^m)$ where

$G = \mathfrak{S}_n$ and $G^i = \{\sigma \in \mathfrak{S}_n \mid \sigma(i) = i\}$ for $i = 1, \ldots, m$.

Garst (1979): $\Delta(K_{m,n}, 1)$ is Cohen-Macaulay if and only if $2m - 1 \leq n$.

# Homology of the Matching Complex

The symmetric group $\mathfrak{S}_n$ acts on $M_n := \Delta(K_n, 1)$ by permuting node labels.



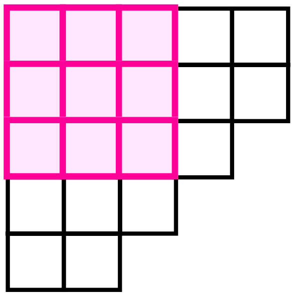
This induces a representation of $\mathfrak{S}_n$ on $\tilde{H}_p(M_n; \mathbb{C})$.

**Theorem (Bouc, 1990).** As $\mathfrak{S}_n$-modules,

$$\tilde{H}_{p-1}(M_n; \mathbb{C}) \cong \bigoplus_{\substack{\lambda:\ \lambda \vdash n \\ \lambda = \lambda' \\ d(\lambda) = n-2p}} S^\lambda$$

where $S^\lambda$ is the irreducible $\mathfrak{S}_n$-module indexed by $\lambda$ and $d(\lambda)$ is the size of the Durfee square of $\lambda$.

$\lambda =$



$d(\lambda) = 3$

**Hopf trace formula** for the matching complex:

$$\sum_p (-1)^p \mathrm{ch}\, C_{p-1}(M_n) = \sum_p (-1)^p \mathrm{ch}\, \tilde{H}_{p-1}(M_n)$$

where ch is the Frobenius characteristic.

By Bouc's theorem, this is equivalent to the **identity of Littlewood**

$$\prod_{i \leq j} (1 - x_i x_j) \prod_i (1 + x_i)^{-1} = \sum_{\lambda = \lambda'} (-1)^{(|\lambda| + d(\lambda))/2} s_\lambda$$

where $s_\lambda$ is the **Schur function** indexed by $\lambda$.

# Proofs of Bouc's Theorem

- Bouc(1990) - homology of matching complex; Quillen's study of homotopy properties of poset of non-trivial $p$-subgroups of a finite group

- Józefiak & Weyman(1988) - homology of complex of $GL_n(\mathbb{C})$-modules arising from study of minimal free resolutions of quotient of a polynomial ring

- Karaguezian(1994) - homology of matching complex

- Sigg(1996) - homology of free two-step nilpotent Lie algebra

- Reiner & Roberts(1997) - homology of bounded degree graph complexes; commutative algebra

# Laplacian Proof of Bouc's Theorem

Dong & MW(1999) (ideas borrowed from Sigg's work on the homology of the free two-step nilpotent Lie algebra and from Friedman & Hanlon's work on the homology of the chess-board complex)

The **combinatorial Laplacian** $\Lambda_p : C_p(M_n) \to C_p(M_n)$ is defined by

$$\Lambda_p = \delta_{p-1}\partial_p + \partial_{p+1}\delta_p$$

where $\partial$ is the boundary map and $\delta$ is the coboundary map.

Analogue of Hodge Theory (Kostant):

$$\tilde{H}_p(M_n;\mathbb{C}) \cong_{\mathfrak{S}_n} \ker \Lambda_p$$

**Lemma.** For all $\gamma \in C_p(M_n)$,

$$\Lambda_p(\gamma) = T \cdot \gamma$$

where

$$T = \sum_{1 \leq i < j \leq n} (i,j) \in \mathbb{C}\mathfrak{S}_n.$$

Another Littlewood formula and Pieri's rule enable us to easily decompose $C_p(M_n)$ into irreducibles:

$$C_{p-1}(M_n) \cong \bigoplus_{\lambda \in A_n} b^p_\lambda \, S^\lambda$$

where

$$A_n = \{\lambda \vdash n \mid \lambda = (\alpha_1, \dots, \alpha_d \mid \beta_1, \dots, \beta_d),\ \alpha_i \geq \beta_i\}$$

For $\lambda$ self-conjugate, $b^p_\lambda = 1$ if $d(\lambda) = n - 2p$ and is 0 otherwise.

How does $T$ act on irreducibles?

**Macdonald:** For all $\gamma \in S^\lambda$,

$$T \cdot \gamma = c_\lambda \gamma$$

where

$$c_\lambda = \sum_{i=1}^{d} \left( \binom{\alpha_i + 1}{2} - \binom{\beta_i + 1}{2} \right).$$

For $\lambda \in A_n$, we see $c_\lambda = 0$ iff $\lambda$ is self-conjugate. So

$$\ker \Lambda_p = \bigoplus_{\substack{\lambda :\ \lambda \vdash n \\ \lambda = \lambda'}} b_\lambda^p \, S^\lambda = \bigoplus_{\substack{\lambda :\ \lambda \vdash n \\ \lambda = \lambda' \\ d(\lambda) = n - 2p}} S^\lambda$$

**Open Problem.** Find a natural basis for $\tilde{H}_p(M_n;\mathbb{C})$ (or for $\ker \Lambda_p$) indexed by standard tableaux of self-conjugate shape.

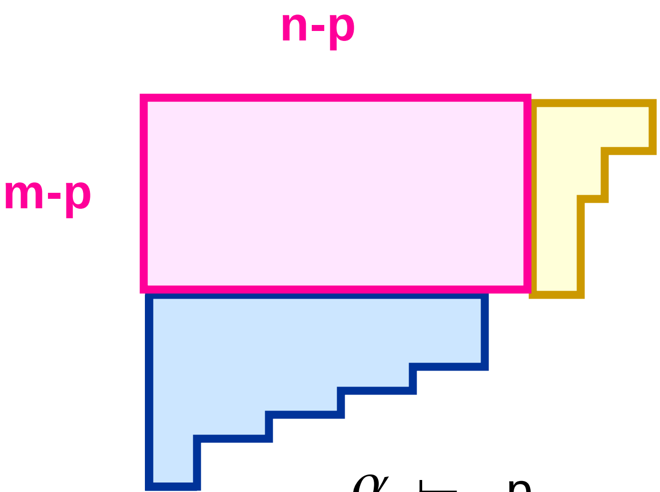
# Homology of the Chessboard Complex

The group $\mathfrak{S}_m \times \mathfrak{S}_n$ acts on $M_{m,n} := \Delta(K_{m,n}, 1)$ which induces a representation of $\mathfrak{S}_m \times \mathfrak{S}_n$ on $\tilde{H}_p(M_{m,n}; \mathbb{C})$.

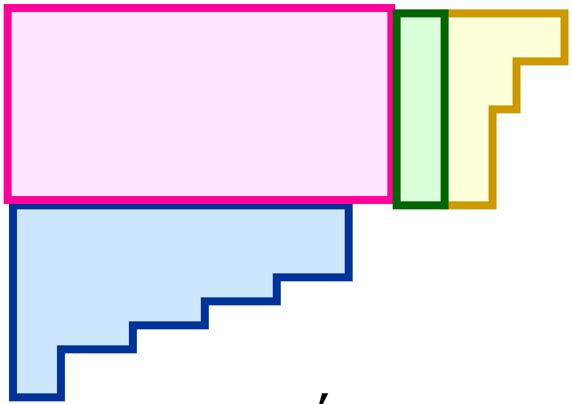
Friedman & Hanlon(1998): As $\mathfrak{S}_m \times \mathfrak{S}_n$-modules,

$$\tilde{H}_{p-1}(M_{m,n}; \mathbb{C}) \cong \bigoplus_{\lambda,\mu} S^{\lambda} \otimes S^{\mu}$$

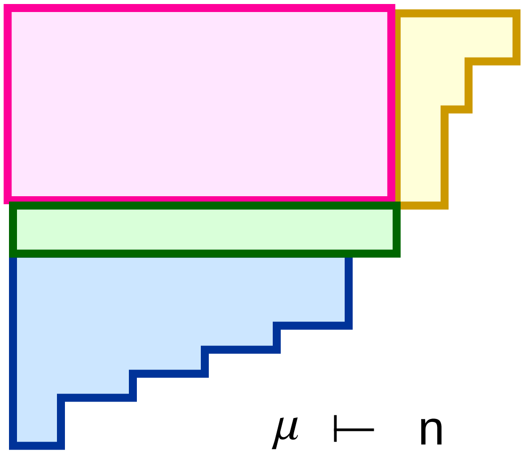
summed over all pairs of partitions $\lambda \vdash m$, $\mu \vdash n$ which can be obtained from a partition $\alpha \vdash p$ in the following way



$\alpha \vdash p$



$\lambda' \vdash m$



$\mu \vdash n$

# Torsion in the Matching Complex

**Björner, Lovász, Vrécica, Živaljević (1994):** Let

$\nu_n = \lfloor \frac{n+1}{3} \rfloor - 1$ & $\nu_{m,n} = \min\{m, \lfloor \frac{m+n+2}{3} \rfloor\} - 1$.

Then

- $M_n$ is $(\nu_n - 1)$-connected
- $M_{m,n}$ is $(\nu_{m,n} - 1)$-connected

**BLVZ Conjecture:**

- $\tilde{H}_{\nu_n}(M_n)$ does not vanish.
- $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ does not vanish.

**Bouc (1990):**

If $n \equiv 1 \bmod 3$, $n \geq 7$ then $\tilde{H}_{\nu_n}(M_n) \cong \mathbb{Z}_3$

If $n \equiv 0 \bmod 3$, $n \geq 12$ then $\tilde{H}_{\nu_n}(M_n)$ is a nontrivial finite 3-group with exponent at most 9.

Shareshian & MW (1999): $\tilde{H}_{\nu_n}(M_n)$ is nonvanishing for all $n$. Moreover, for all $n \geq 12$ (except possibly $n = 14$)

$$\tilde{H}_{\nu_n}(M_n) \cong \mathbb{Z}_3^{r_n}$$

where $r_n \geq 1$.

Main tool: Bouc's long exact sequence

$$\cdots \xrightarrow{\delta} \bigoplus_{a,h} \tilde{H}_{t-1}(M_{n-3}) \xrightarrow{\phi} \tilde{H}_t(M_n) \xrightarrow{\psi}$$

$$\bigoplus_{i,j} \tilde{H}_{t-2}(M_{n-4}) \xrightarrow{\delta} \bigoplus_{a,h} \tilde{H}_{t-2}(M_{n-3}) \xrightarrow{\phi} \cdots$$

where $a$ ranges over the set $\{1, 2\}$ and $h, i, j$ range over the set $\{3, \ldots, n\}$ with $i \neq j$.

Nonvanishing is an easy consequence.

To show that $\tilde{H}_{\nu_n}(M_n)$ has only 3-torsion:

For $\mathbf{n \equiv 0, 1}$ mod $\mathbf{3}$, Bouc's long exact sequence gives

$$\bigoplus_{a,h} \tilde{H}_{\nu_{n-3}}(M_{n-3}) \xrightarrow{\phi} \tilde{H}_{\nu_n}(M_n) \to 0$$

If $\tilde{H}_{\nu_{n-3}}(M_{n-3})$ has only 3-torsion then $\tilde{H}_{\nu_n}(M_n)$ has only 3-torsion.

**Base step of induction:** $n = 7$ and $n = 12$. Bouc shows

$$\tilde{H}_{\nu_7}(M_7) = \mathbb{Z}_3$$

Computer calculation of Heckenbach, Welker, Dumas, Saunders:

$$\tilde{H}_{\nu_{12}}(M_{12}) = \mathbb{Z}_3^{56}$$

Let $\mathbf{n \equiv 2 \bmod 3}$. We used Bouc's long exact sequence to construct a generating set of cycles for $\tilde{H}_{\nu_n}(M_n)$ which have the form

$$\alpha * \rho$$

where

$$\alpha \in \tilde{H}_1(M_A) \quad \text{and} \quad \rho \in \tilde{H}_{\nu_{n-5}}(M_{[n]-A})$$

for some $A \subseteq [n]$ such that $|A| = 5$.

Since $n-5 \equiv 0 \bmod 3$, we can apply the previous case if $n-5 \geq 12$. This gives $3\rho = 0$ which implies

$$3(\alpha * \rho) = \alpha * 3\rho = 0$$

.

## Open Problems:

- Eliminate use of the computer
- Determine $r_n$
- Show that there is only 3-torsion in all homology groups of $M_n$. (Babson, Björner, Linusson, Shareshian, Welker)

# Torsion in the Chessboard Complex

**Note:** If $2m-1 \leq n$ then $\nu_{m,n} = m-1$. So $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is free.

**Shareshian & MW (2000):** $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is nonvanishing for all $m$ and $n$. Moreover

- $n+m \equiv 1 \bmod 3$
  If $5 \leq m \leq n \leq 2m-5$ then
  $$\tilde{H}_{\nu_{m,n}}(M_{m,n}) \equiv \mathbb{Z}_3$$

- $n+m \equiv 0 \bmod 3$
  If $9 \leq m \leq n \leq 2m-9$ then $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is a 3-group with exponent at most 9.

- $n+m \equiv 2 \bmod 3$
  If $13 \leq m \leq n \leq 2m-13$ then $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is a 3-group with exponent at most 9.

The proof is much harder than that for the matching complex

**Difficulties**:

- For $m+n \equiv 0,1 \bmod 3$

$$\bigoplus \tilde{H}_{\nu_{m-2,n-1}}(M_{m-2,n-1}) + \bigoplus \tilde{H}_{\nu_{m-1,n-2}}(M_{m-1,n-2})$$

$$\xrightarrow{\phi} \tilde{H}_{\nu_{m,n}}(M_{m,n}) \to 0$$

  But $m-2$ and $n-1$ could be sufficiently far apart to make homology free.

- Computer provides only

$$\tilde{H}_{\nu_{5,5}}(M_{5,5}) \cong \mathbb{Z}_3.$$

  $\tilde{H}_{\nu_{6,6}}(M_{6,6})$ and $\tilde{H}_{\nu_{7,7}}(M_{7,7})$ have free parts by the Friedman-Hanlon result.

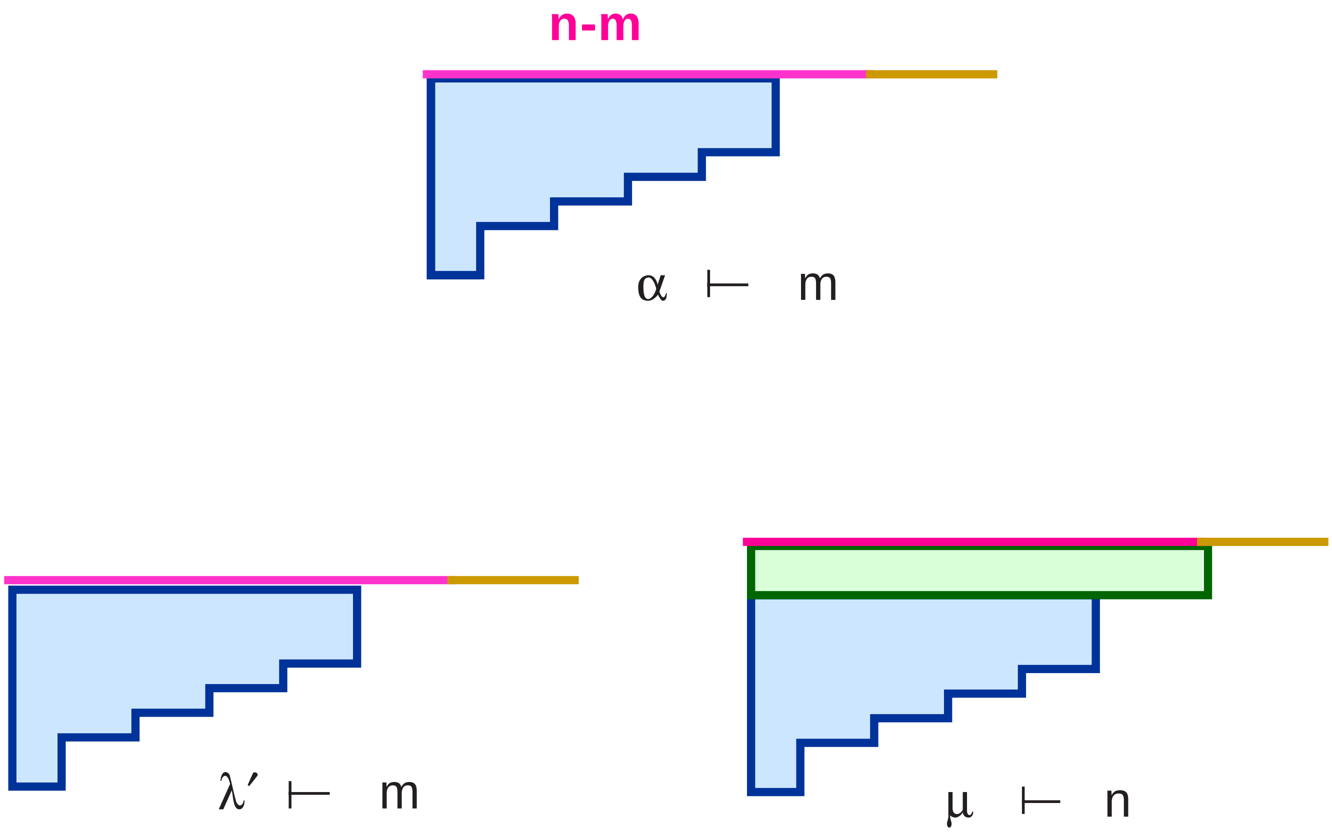
  $\tilde{H}_{\nu_{7,8}}(M_{7,8})$ is too big for the computer.

# Basis for Top Homology and Cohomology

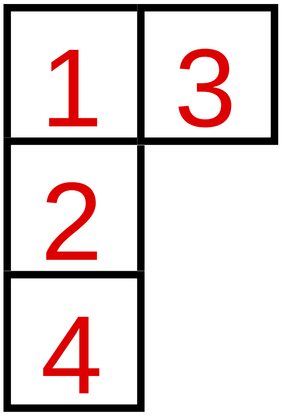
Recall Friedman-Hanlon

$$\tilde{H}_{p-1}(M_{m,n};\mathbb{C}) \cong \bigoplus_{\lambda,\mu} S^{\lambda} \otimes S^{\mu}$$

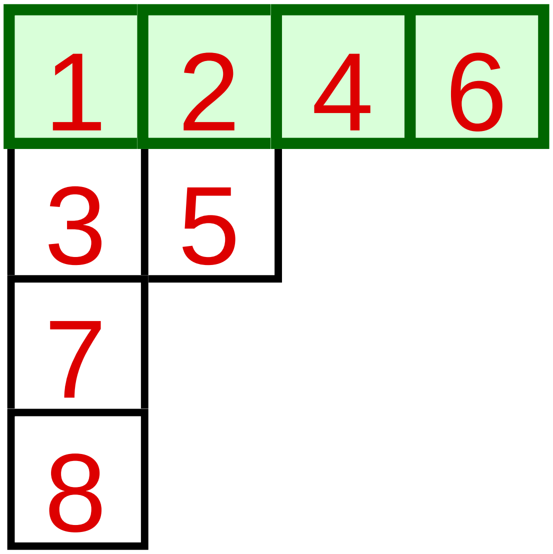
When $p = m \leq n$



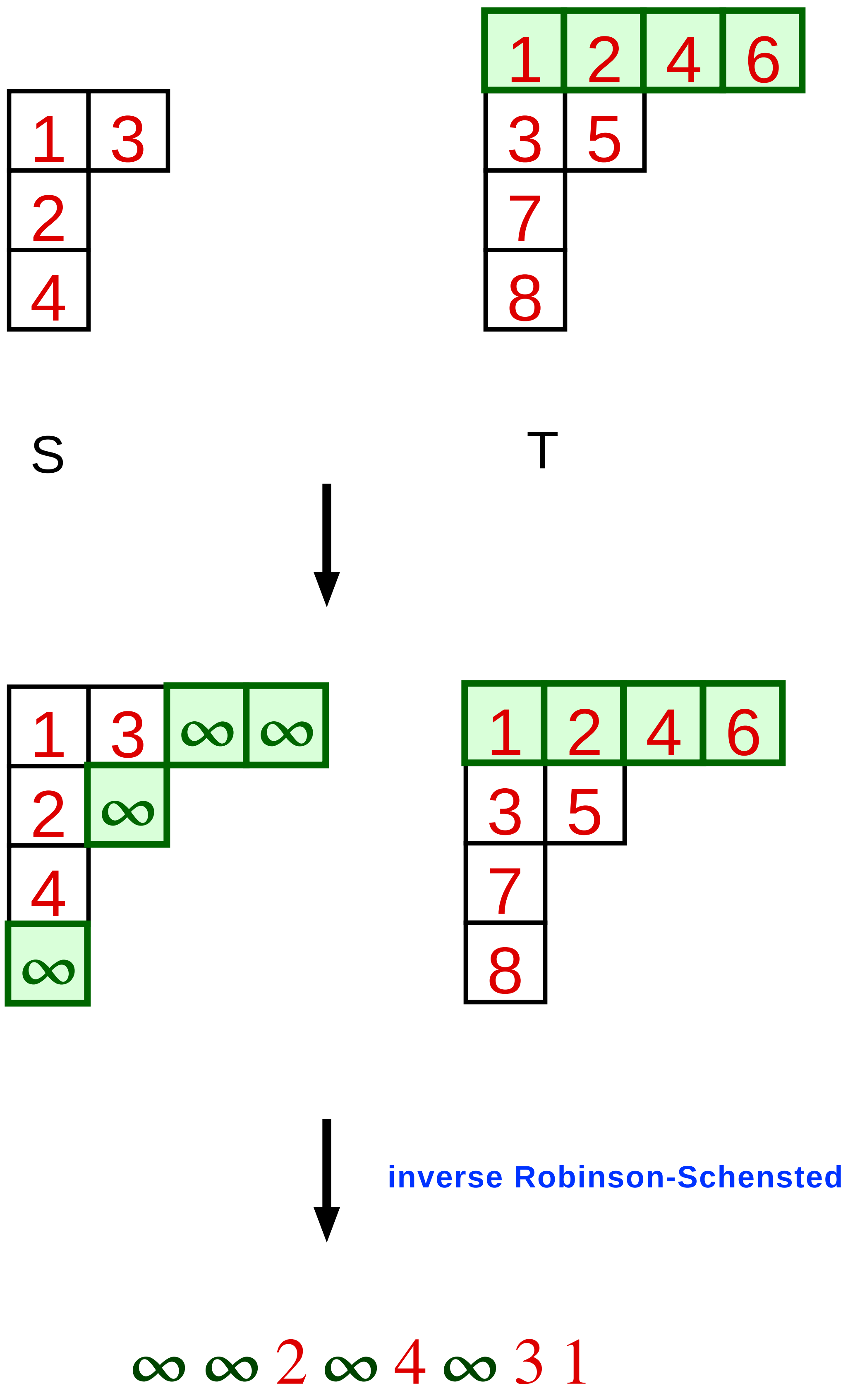
From a pair of standard young tableaux of shape $\lambda$ and $\mu$ we construct a cycle and a cocycle of dimension $m-1$ using the Robinson-Schensted correspondence.



S



T

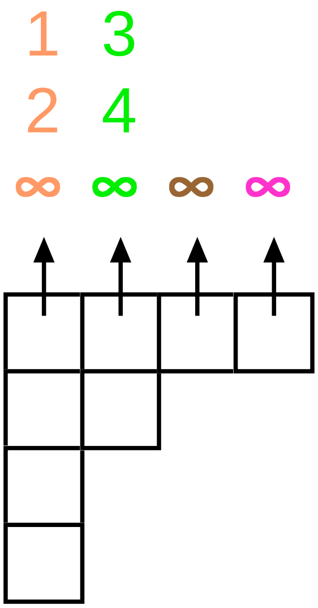

1 3
2
4
1 2 4 6
3 5
7
8
S
T
1 3 ∞ ∞
2 ∞
4
∞
1 2 4 6
3 5
7
8
inverse Robinson-Schensted
∞ ∞ 2 ∞ 4 ∞ 3 1

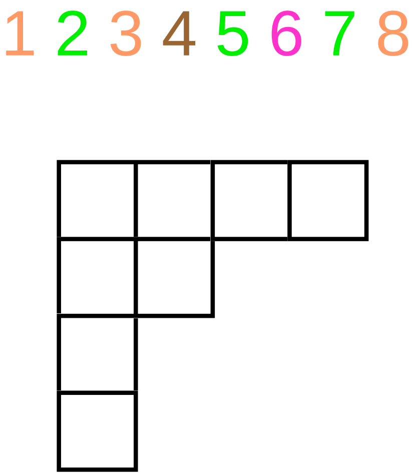
Cocycle:

$$\begin{array}{cccccccc} \infty & \infty & 2 & \infty & 4 & \infty & 3 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array}$$

$$\gamma(S,T) = (18, 23, 37, 45)$$

Cycle:





$M_{\{1,2\}\{1,3,8\}}$ and $M_{\{3,4\}\{2,5,7\}}$ are pseudomanifolds. So top homology is cyclic. Let

$$\begin{aligned}\tilde{H}_1(M_{\{1,2\}\{1,3,8\}}) &= \langle \alpha \rangle \\ \tilde{H}_1(M_{\{3,4\}\{2,5,7\}}) &= \langle \beta \rangle\end{aligned}$$

Define

$$\rho(S,T) = \alpha * \beta \in \tilde{H}_3(M_{4,8})$$

Shareshian & MW:

- $\{\rho(S,T) \mid (S,T)$ is a Friedman-Hanlon pair$\}$ is a basis for $\tilde{H}_{m-1}(M_{m,n})$
- $\{\gamma(S,T) \mid (S,T)$ is a Friedman-Hanlon pair$\}$ is a basis for the free part of $\tilde{H}^{m-1}(M_{m,n})$

Idea of Proof: We order the pairs of standard Young tableaux

$$(S_1, T_1), \ldots, (S_k, T_k)$$

so that the matrix

$$(\langle \rho(S_i, T_i), \gamma(S_j, T_j) \rangle)_{i,j=1,\ldots,t}$$

is triangular with 1's on the diagonal. From this we can establish the result.

# Generalizations and Variations

Hypergraph version: Björner, Lovász, Vrécica & Živaljević (1992), Björner & Eriksson (1999), Karaguezian, Reiner & MW (1999)

Directed graph and multigraph versions: Björner & Welker(1998), MW(1999)

General Bounded Degree: Reiner & Roberts (1997), Karaguezian, Reiner & MW (1999), Dong (1999), MW (1999)

## Littlewood Identities:

$$\prod_{i\leq j}(1-x_ix_j)\prod_i(1+x_i)^{-1}=\sum_{\lambda=\lambda'}(-1)^{(|\lambda|+d(\lambda))/2}s_\lambda$$

$$\prod_{i<j}(1-x_ix_j)=\sum_{\mu\in\mathcal{B}}(-1)^{|\mu|/2}s_\mu$$

where $\mathcal{B}$ is the set of all partitions of form $(\alpha_1,\ldots,\alpha_d\,|\,\alpha_1+1,\ldots,\alpha_d+1)$ for some $d$.

Combine to get

$$\prod_{i,j}(1-x_ix_j)\prod_i(1+x_i)^{-1}=\\ \sum_{\substack{\lambda=\lambda'\\ \mu\in\mathcal{B}}}(-1)^{(|\lambda\otimes\mu)|+d(\lambda))/2}\;s_{\lambda\otimes\mu}$$

## Representation Theoretic Interpretation?

MW (1999): Representation theoretic interpretation - Hopf Trace formula for the **directed matching complex**.

Let $DM_n$ be the matching complex of the complete digraph $([n], [n] \times [n])$.

MW: As $\mathfrak{S}_n$-modules,

$$\tilde{H}_{p-1}(DM_n; \mathbb{C}) \cong \bigoplus_{\substack{|\lambda| + |\mu| = n \\ \lambda = \lambda' \\ d(\lambda) = n - 2p \\ \mu \in \mathcal{B}}} S^{\lambda \otimes \mu}$$

# Generalization to Multigraphs

Let $M_n^m$ be the matching complex of the complete multigraph on node set $[n]$ with $m$ distinct edges between each (unordered) pair of vertices.

MW: As $\mathfrak{S}_n$-modules,

$$\tilde{H}_{p-1}(M_n^m;\mathbb{C}) \cong \bigoplus_{\substack{|\lambda|+|\mu|=n \\ \lambda=\lambda' \\ d(\lambda)=n-2p \\ \mu'\in\mathcal{B}}} (m-1)^{|\mu|/2}\, S^{\lambda\otimes\mu}$$

Proof Technique: We consider the natural simplicial map

$$M_n^m \longrightarrow M_n$$

and use a generalization of the Quillen Fiber Lemma due to Björner, MW & Welker (1999) to express the homology of $M_n^m$ in terms of that of $M_n$.