

# Posets of graphs, partitions and trees

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Poset  $\longrightarrow$  topological space

Order complex  $\Delta(P)$  of a poset  $P$  is the simplicial complex whose faces are the chains of  $P - \{\hat{0}, \hat{1}\}$ .

$$\mu_P(\hat{0}, \hat{1}) = \tilde{\chi}(\Delta(P))$$

- a poset of **graphs** - deletion of edges
- a poset of **partitions** - refinement
- a poset of **trees** - contraction of edges

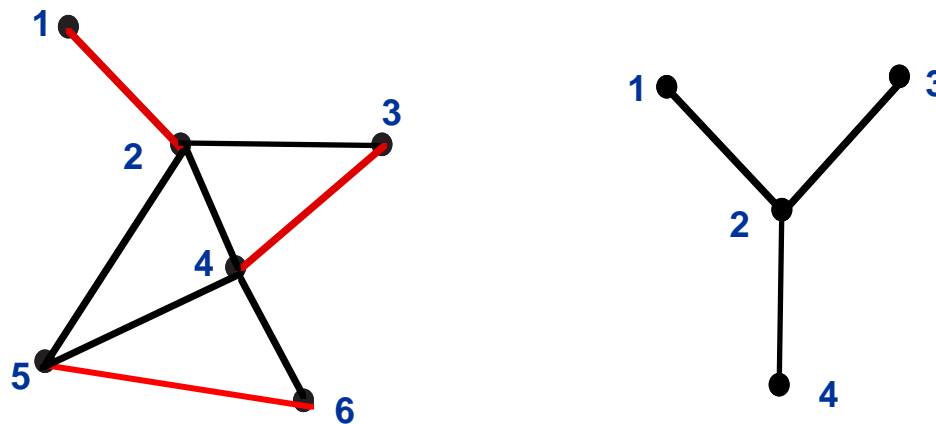
## Applications and Connections

- hyperplane & subspace arrangements
- complexity theory
- group theory
- Lie algebras
- knot theory
- commutative algebra
- algebraic geometry
- homotopy theory

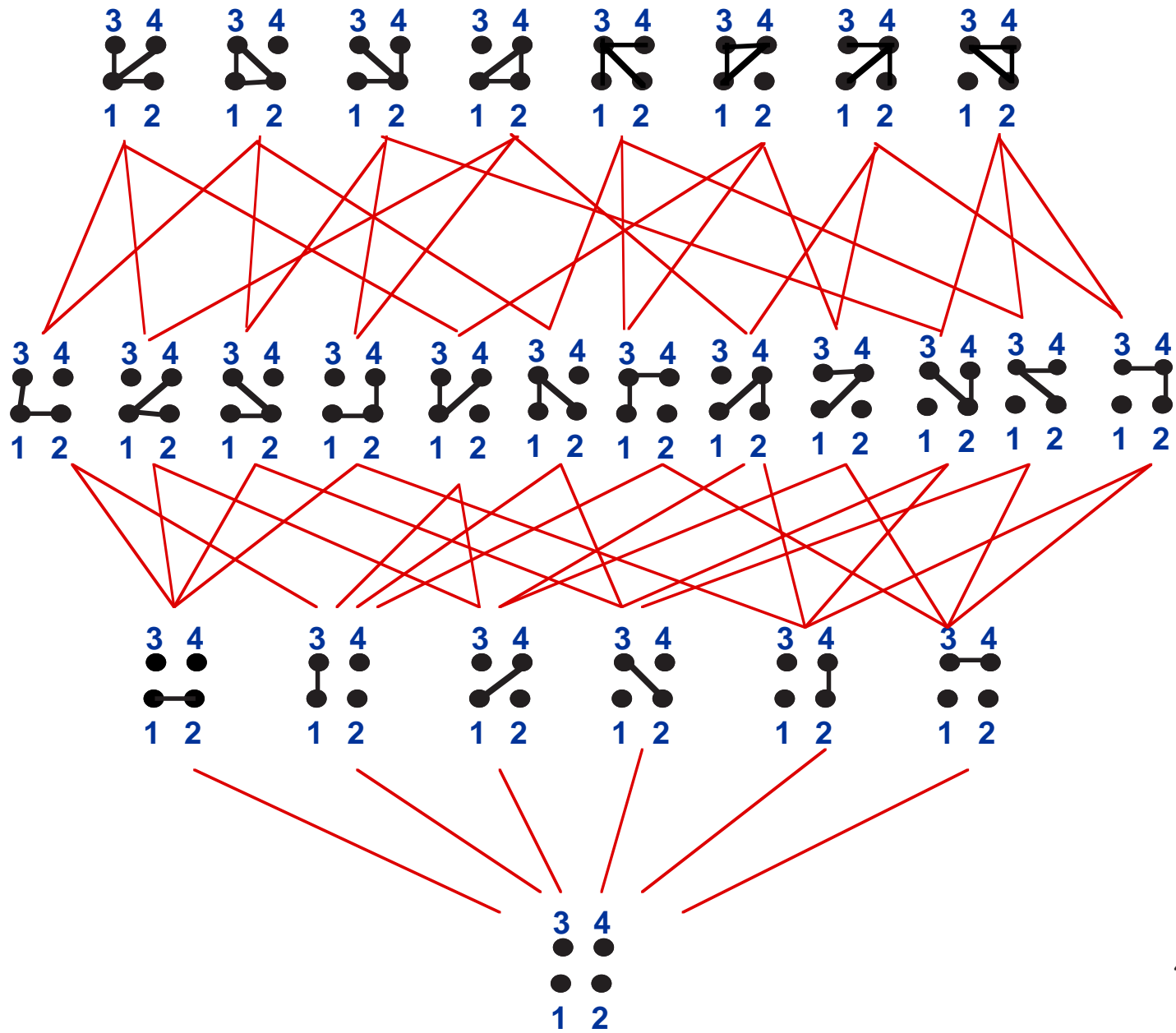
## The no-perfect matching poset

Linusson, Shareshian & Welker

$\text{NPM}_{2n}$  = poset of graphs on node set  $[2n]$  with **no** perfect matching.



Order relation is deletion of edges.



## Betti numbers

$n = 2$     0, 0, **1**

$n = 3$     0, 0, 0, 0, 0, **9**, 0, 0, 0, 0

$n = 4$     0, 0, 0, 0, 0, 0, 0, 0, **225**, 0, 0, 0, 0, 0, 0, 0, ...

$n = 5$     0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, **11025**, 0, 0, 0, 0, ...

## Betti numbers

$$\begin{aligned}n = 2 & \quad 0, 0, 1^2 \\n = 3 & \quad 0, 0, 0, 0, 0, (1 \cdot 3)^2, 0, 0, 0, 0 \\n = 4 & \quad 0, 0, 0, 0, 0, 0, 0, 0, (1 \cdot 3 \cdot 5)^2, 0, 0, 0, 0, 0, 0, \dots \\n = 5 & \quad 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, (1 \cdot 3 \cdot 5 \cdot 7)^2, 0, 0, \dots\end{aligned}$$

## Betti numbers

$n = 2$	0, 0, $1^2$
$n = 3$	0, 0, 0, 0, 0, $(1 \cdot 3)^2$ , 0, 0, 0, 0
$n = 4$	0, 0, 0, 0, 0, 0, 0, 0, $(1 \cdot 3 \cdot 5)^2$ , 0, 0, 0, 0, 0, 0, ...
$n = 5$	0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, $(1 \cdot 3 \cdot 5 \cdot 7)^2$ , 0, 0, ...

Linusson, Shareshian & Welker (2000):  $\Delta(\text{NPM}_{2n})$  has the homotopy type of a wedge of

$$(2n - 3)!!^2 := (1 \cdot 3 \cdots (2n - 1))^2$$

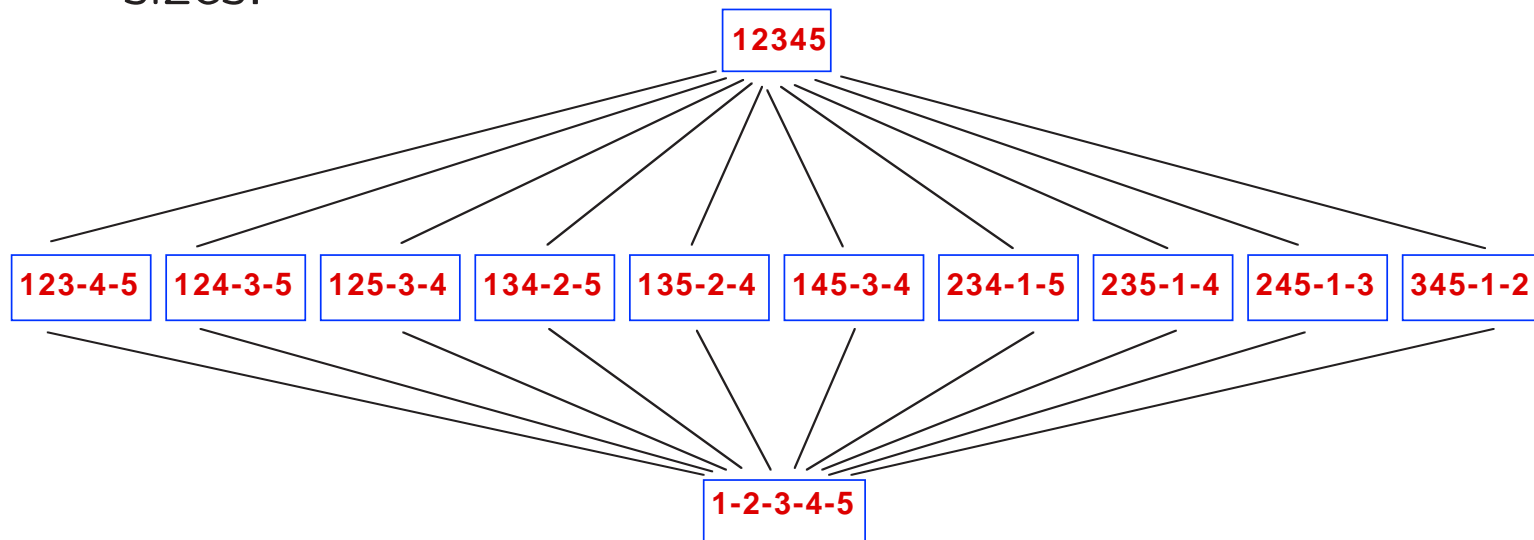
spheres of dimension  $(3n - 4)$ .

## The odd block size partition poset

Calderbank, Hanlon & Robinson

$\Pi_n$  = lattice of partitions of  $[n]$  ordered by refinement.

$\Pi_n^{\text{odd}}$  = subposet of  $\Pi_n$  consisting of partitions with odd block sizes.



$\Delta(\Pi_5^{\text{odd}})$  has homotopy type of a wedge of 9 0-spheres.

Björner, Stanley (1983):  $\Delta(\Pi_{2n-1}^{\text{odd}})$  has the homotopy type of a wedge of  $(2n-3)!!^2$   $(n-3)$ -spheres.

Björner-Stanley + Linusson-Shareshian-Welker implies

$$\tilde{H}_{n-3}(\Pi_{2n-1}^{\text{odd}}) \cong \tilde{H}_{3n-4}(\text{NPM}_{2n})$$

**Question:** Is there a deeper relationship?

Symmetric group  $\mathfrak{S}_{2n-1}$  acts on poset  $\Pi_{2n-1}^{\text{odd}}$

$$(2, 3) [134 - 58 - 267] = [124 - 58 - 367]$$

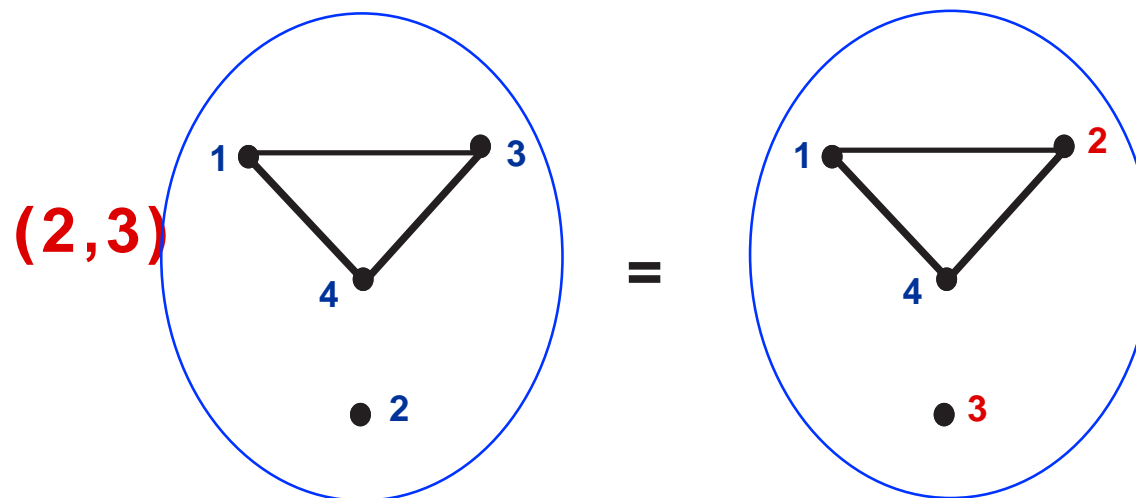
Induces a representation of  $\mathfrak{S}_{2n-1}$  on homology  $\tilde{H}_{n-3}(\Pi_{2n-1}^{\text{odd}})$ .

Calderbank-Hanlon-Robinson (1986)

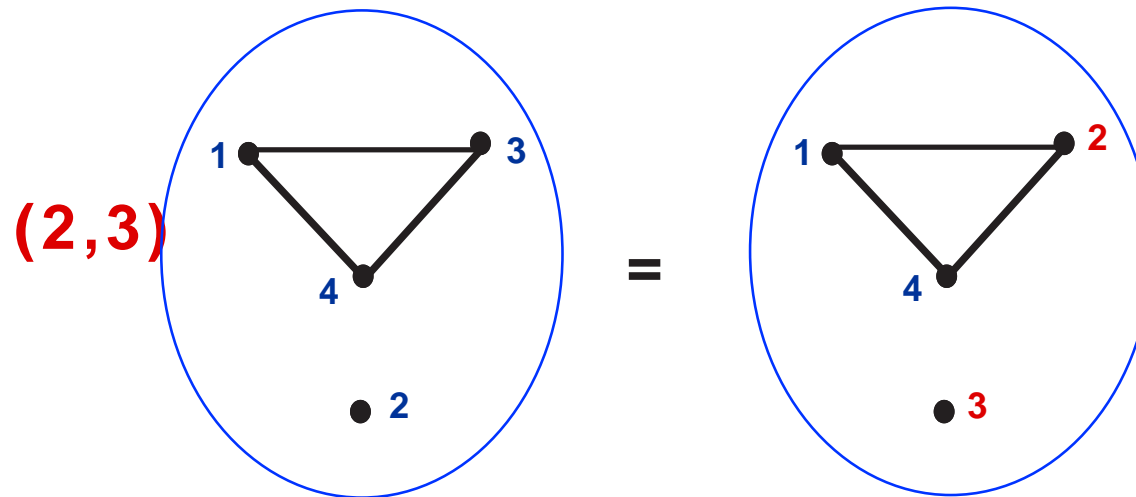
$$\text{ch}(\tilde{H}_{n-3}(\Pi_{2n-1}^{\text{odd}})) = (-1)^{n-1} \left[ \sum_{r \geq 1} h_{2r-1} \right]^{-1} \Big|_{\text{deg} 2n-1}$$

Hanlon-Wachs (1995): Lie  $k$ -algebras; basis for cohomology

Symmetric group  $\mathfrak{S}_{2n}$  acts on the poset  $\text{NPM}_{2n}$  by relabeling graph nodes.



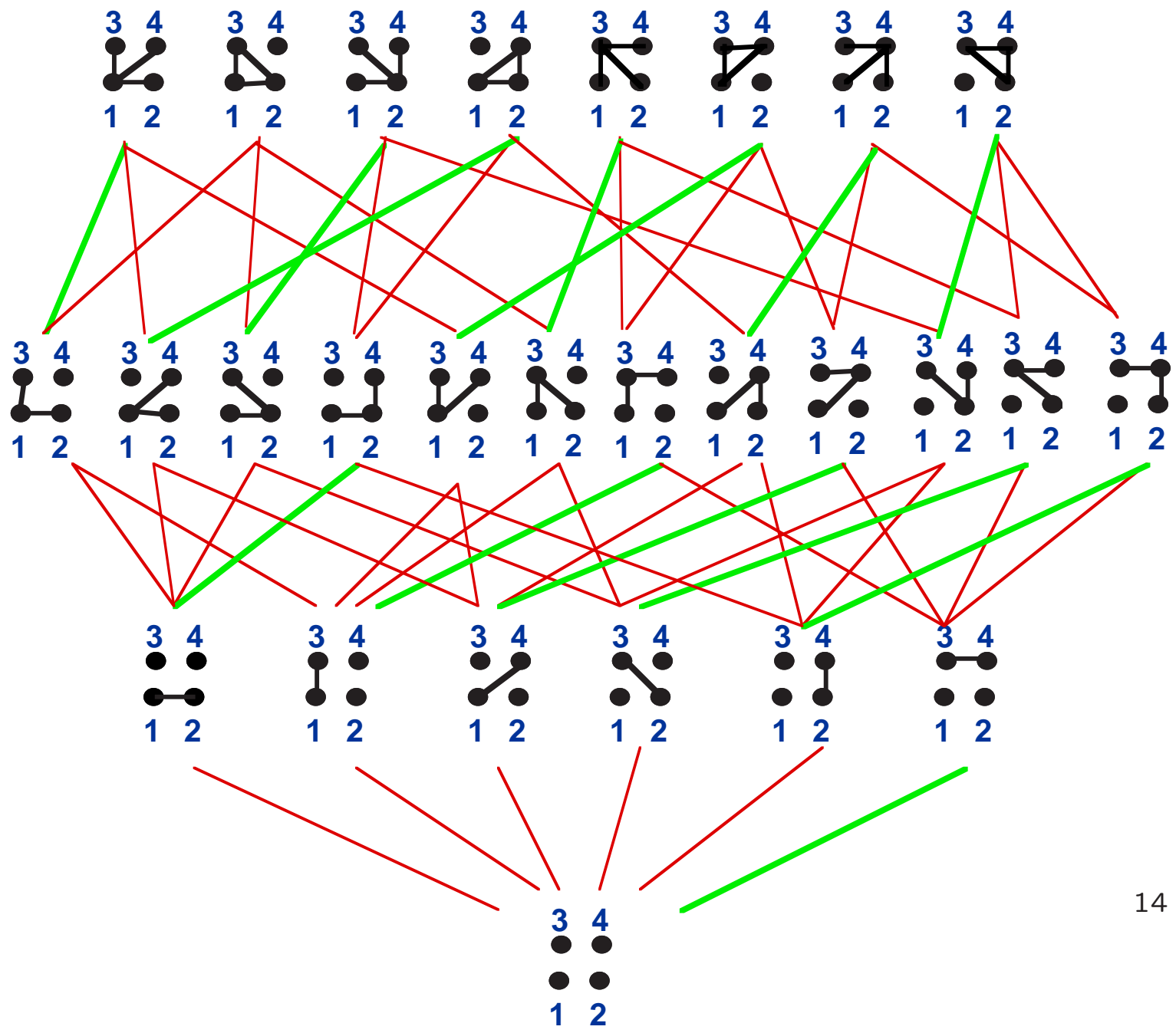
Symmetric group  $\mathfrak{S}_{2n}$  acts on the poset  $\text{NPM}_{2n}$  by relabeling graph nodes.



Shareshian & Wachs (2002): As  $\mathfrak{S}_{2n-1}$ -modules

$$\tilde{H}_{3n-4}(\text{NPM}_{2n}) \downarrow_{\mathfrak{S}_{2n-1}}^{\mathfrak{S}_{2n}} \cong \text{sgn} \otimes \tilde{H}_{n-3}(\Pi_{2n-1}^{\text{odd}})$$





Consequence of Linusson-Shareshian-Welker Morse matching:

$$\tilde{H}_{3n-4}(\text{NPM}_{2n}) \downarrow_{\mathfrak{S}_{2n-1}}^{\mathfrak{S}_{2n}} \cong \text{sgn} \otimes \tilde{H}_{\text{top}}(\text{FC}_{2n-1}),$$

where  $\text{FC}_{2n-1}$  is the poset of factor critical graphs on node set  $[2n - 1]$ .

A graph is called **factor critical** if removal of any node results in a graph with a perfect matching.

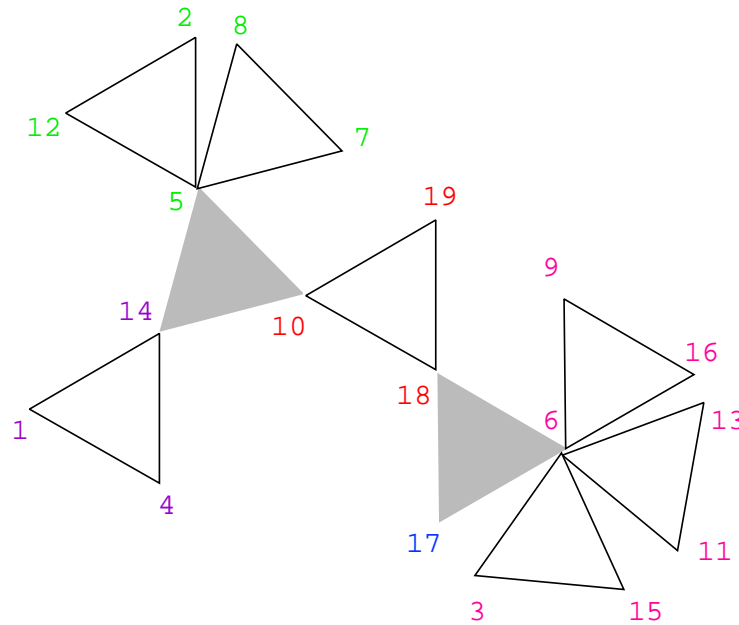
The critical triangle trees index a basis for  $\tilde{H}^{\text{top}}(\text{FC}_{2n-1})$ ,

$$\{\gamma_T \mid T \in \text{CTT}_{2n-1}\},$$

$\text{CTT}_{2n-1} :=$  set of critical triangle trees on node set  $[2n - 1]$ .

## Triangle splitting basis for $\tilde{H}_{n-3}(\Pi_{2n-1}^{\text{odd}})$

For  $T \in \text{CTT}_{2n-1}$ , **split**  $T$  by choosing some triangles of  $T$  and removing all the edges of these triangles. This disconnects the graph. The components induce a partition in  $\Pi_{2n-1}^{\text{odd}}$ .



$$1, 4, 14 - 12, 5, 2, 8, 7 - 10, 18, 19 - 17 - 3, 15, 11, 13, 16, 9, 6$$

Let  $\Pi_T$  be the subposet consisting of all the partitions obtained by splitting  $T$ .

$\Pi_T$  is isomorphic to the lattice of subsets of  $[n - 1]$ . So  $\Delta(\Pi_T)$  is homeomorphic to an  $(n - 3)$ -sphere.

$\rho_T :=$  fundamental cycle of  $\Delta(\Pi_T)$

Theorem:

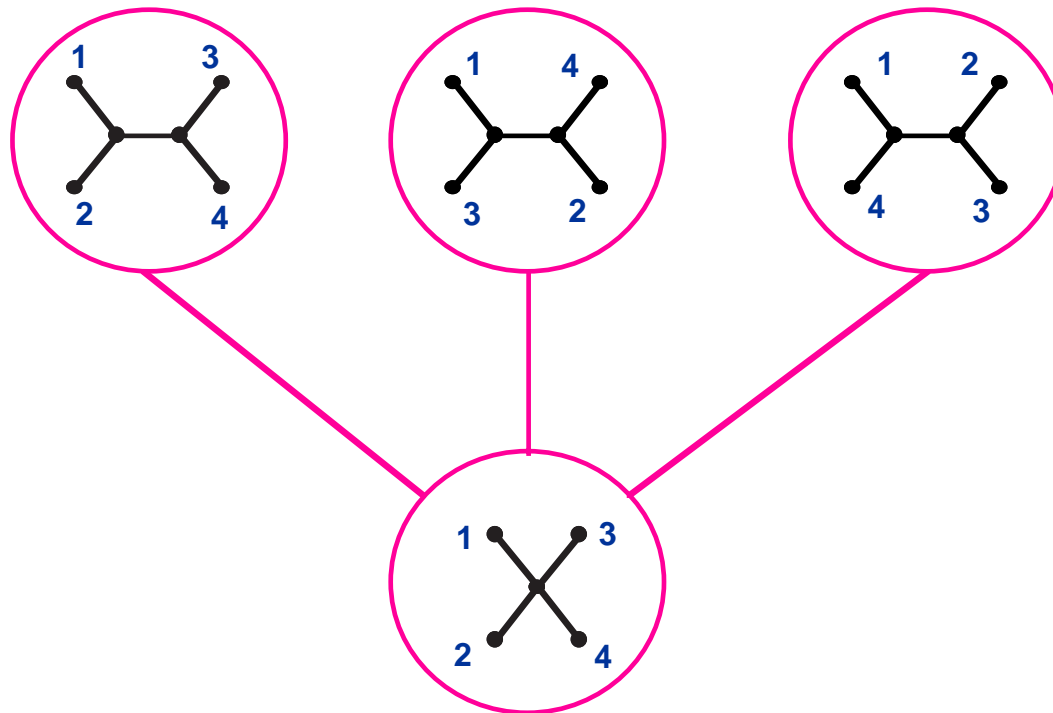
- $\{\rho_T \mid T \in \text{CTT}_{2n-1}\}$  is a basis for  $\tilde{H}_{n-3}(\Pi_{2n-1}^{\text{odd}})$
- The bijection  $\rho_T \mapsto \gamma_T$  determines an  $\mathfrak{S}_{2n-1}$ -module isomorphism

$$\tilde{H}_{n-3}(\Pi_{2n-1}^{\text{odd}}) \rightarrow \tilde{H}^{\text{top}}(\text{FC}_{2n-1})$$

## Tree poset Vogtmann (1990)

$\mathcal{T}_n$  = poset of trees on leaf set  $[n]$  where all internal nodes have degree  $\geq 3$ .

$T < T'$  if  $T$  can be obtained from  $T'$  by contracting internal edges.



$\Delta(\mathcal{T}_n)$  has the homotopy type of a wedge of  $(n-2)!$   $(n-4)$ -spheres.

So  $\Delta(\mathcal{T}_{n+1}) \simeq \Delta(\Pi_n)$

Robinson and Whitehouse (1996)

As  $\mathfrak{S}_n$ -modules

$$\tilde{H}_{n-3}(\mathcal{T}_{n+1}) \downarrow_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}} \cong \tilde{H}_{n-3}(\Pi_n)$$

As  $\mathfrak{S}_{n+1}$ -modules

$$\tilde{H}_{n-3}(\mathcal{T}_{n+1}) \cong \tilde{H}_{n-3}(\Pi_n) \uparrow_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}} - \tilde{H}_{n-2}(\Pi_{n+1})$$

## Other occurrences of Whitehouse module:

- Cyclic Lie operad - Kontsevich (1993)
- Not 2-connected graph complex - Vassiliev (1997), Babson, Björner, Linusson, Shareshian, Welker (1997), Turchin (1997)
- Kernel of Varchenko operator - Hanlon & Stanley (1998)
- Bounded block size partition poset - Sundaram (1998)
- Nonmodular partition poset - Sundaram (1999)

## Analogue of Whitehouse Module - Hanlon (1996)

$\mathcal{T}_{2n}^{\text{even}}$  = subposet of  $\mathcal{T}_{2n}$  consisting of trees in which all internal nodes have even degree.

- As  $\mathfrak{S}_{2n-1}$ -modules

$$\tilde{H}_{n-3}(\mathcal{T}_{2n}^{\text{even}}) \downarrow_{\mathfrak{S}_{2n-1}}^{\mathfrak{S}_{2n}} \cong \tilde{H}_{n-3}(\Pi_{2n-1}^{\text{odd}})$$

- As  $\mathfrak{S}_{2n}$ -modules

$$\tilde{H}_{n-3}(\mathcal{T}_{2n}^{\text{even}}) \cong \tilde{H}_{n-3}(\Pi_{2n-1}^{\text{odd}}) \uparrow_{\mathfrak{S}_{2n-1}}^{\mathfrak{S}_{2n}} - \tilde{H}_{n-2}(\Pi_{2n}^{\text{odd}})$$

Another Occurrence: Cyclic Lie 3-Operad - Wachs (1998)

Shareshian & Wachs:

$$\tilde{H}_{3n-4}(\text{NPM}_{2n}) \downarrow_{\mathfrak{S}_{2n-1}}^{\mathfrak{S}_{2n}} \cong \tilde{H}_{n-3}(\mathcal{T}_{2n}^{\text{even}}) \downarrow_{\mathfrak{S}_{2n-1}}^{\mathfrak{S}_{2n}}$$

Conjecture: As  $\mathfrak{S}_{2n}$ -modules

$$\tilde{H}_{3n-4}(\text{NPM}_{2n}) \cong \tilde{H}_{n-3}(\mathcal{T}_{2n}^{\text{even}})$$

## Generalizations

$$\Pi_{kn+1}^{1 \bmod k}$$

$$\mathcal{T}_{kn+2}^{2 \bmod k}$$

Björner:  $\Pi_{kn+1}^{1 \bmod k}$  is shellable

Trappmann & Ziegler, Wachs:  $\mathcal{T}_{kn+2}^{2 \bmod k}$  is shellable

Hanlon:  $\tilde{H}_{n-2}(\mathcal{T}_{kn+2}^{2 \bmod k}) \downarrow_{S_{kn+1}} \cong \tilde{H}_{n-2}(\Pi_{kn+1}^{1 \bmod k})$

Hanlon and Wachs: Brush basis for  $\tilde{H}_{n-2}(\Pi_{kn+1}^{1 \bmod k})$  and Lie  $k$ -algebra.

Shareshian & Wachs: Triangle splitting basis generalizes to a splitting basis for  $\tilde{H}_{n-2}(\Pi_{kn+1}^{1 \bmod k})$

**Q:** Is there a generalization of  $\text{NPM}_{2n}$  whose homology is

$$\cong_{\mathfrak{S}_{kn+1}} \tilde{H}_{n-2}(\Pi_{kn+1}^{1 \bmod k})$$

or more generally

$$\cong_{\mathfrak{S}_{kn+2}} \tilde{H}_{n-2}(\mathcal{T}_{kn+2}^{2 \bmod k})?$$