

Freeness in the codimension 1 homology of the chessboard complex

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Joint work with **John Shareshian**

Chessboard Complex: $M_{m,n}$

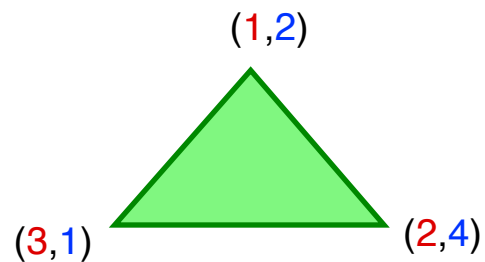
vertex set $:= \{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$

face $:= \{(i_1, j_1), \dots, (i_k, j_k)\} \quad i_s \neq i_t, \quad j_s \neq j_t \quad \forall s, t$

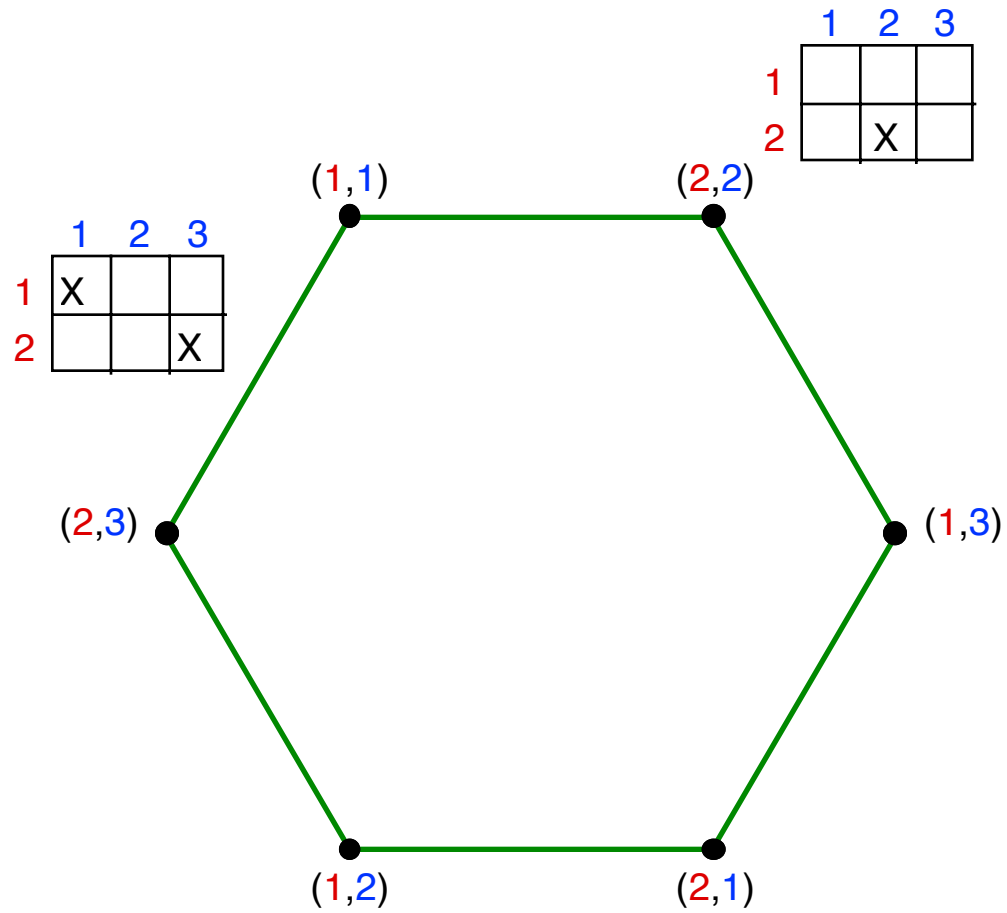
facet of $M_{3,4}$:

	1	2	3	4
1		×		
2				×
3	×			

$\leftrightarrow \{(1, 2), (2, 4), (3, 1)\}$



$M_{2,3}$



Garst (1979): Tits coset complexes

Vrécica & Živaljević (1992): Computational geometry (Colored Tverberg Problem) - connectivity bounds

Reiner & Roberts (1997): Commutative algebra (minimal free resolutions) - bounded degree bipartite graph complexes

Babson & Reiner (2001): Type selected subcomplexes of Coxeter-like complexes

Assume $m \leq n$. $\dim M_{m,n} = m - 1$. m $\begin{matrix} & & & n \\ & & & \square & \square & \square & \square & \square & \square \\ & & & \square & \square & \square & \square & \square & \square \\ m & & & \square & \square & \square & \square & \square & \square \end{matrix}$

Björner, Lovász, Vrc̆ica, Živaljević (1994): Let

$$\nu(m, n) = \min\left\{m, \left\lfloor \frac{m + n + 2}{3} \right\rfloor\right\} - 1.$$

Then $M_{m,n}$ is $(\nu(m, n) - 1)$ -connected. Consequently

$$H_t(M_{m,n}) = 0 \quad \forall t < \nu(m, n).$$

BLVZ Conjecture: $H_{\nu(m,n)}(M_{m,n}) \neq 0$.

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Garst (1979): True for $n \geq 2m - 1$ (top homology).

Friedman & Hanlon (1998): $H_{\nu(m,n)}(M_{m,n})$ infinite iff $n \geq 2m - 5$ or $(m, n) = (6, 6), (7, 7), (8, 9)$

Chessboard complex - homology groups $H_{\nu(m,n)}(M_{m,n})$

$m \setminus n$	2	3	4	5	6	7
1	\mathbb{Z}	\mathbb{Z}^2	\mathbb{Z}^3	\mathbb{Z}^4	\mathbb{Z}^5	\mathbb{Z}^6
2	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}^5	\mathbb{Z}^{11}	\mathbb{Z}^{19}	\mathbb{Z}^{29}
3		\mathbb{Z}^4	\mathbb{Z}^2	\mathbb{Z}^{14}	\mathbb{Z}^{47}	\mathbb{Z}^{104}
4			\mathbb{Z}^{15}	\mathbb{Z}^{20}	\mathbb{Z}^5	\mathbb{Z}^{225}
5				\mathbb{Z}_3	\mathbb{Z}^{152}	\mathbb{Z}^{98}
6					$\mathbb{Z}^{25} \oplus \mathbb{Z}_3^{10}$	\mathbb{Z}_3
7						$\mathbb{Z}^{588} \oplus \mathbb{Z}_3^{66}$

Shareshian & MW (2000): BLVZ conjecture true for all m, n .

Finite $H_{\nu(m,n)}(M_{m,n})$ always has 3-torsion.

Moreover, if $m + n \equiv 1 \pmod{3}$ and $n \leq 2m - 5$ then

$$H_{\nu(m,n)}(M_{m,n}) \cong \mathbb{Z}_3$$

Q: When does **infinite** $H_{\nu(m,n)}(M_{m,n})$ have 3-torsion?

Conjecture 1: Shareshian & MW

- If $n > 2m - 5$ then $H_{\nu(m,n)}(M_{m,n})$ torsion-free
- if $n \leq 2m - 5$ then $H_{\nu(m,n)}(M_{m,n})$ has 3-torsion.

Conjecture 1: Shareshian & MW

- If $n > 2m - 5$ then $H_{\nu(m,n)}(M_{m,n})$ torsion-free
- if $n \leq 2m - 5$ then $H_{\nu(m,n)}(M_{m,n})$ has 3-torsion.

$(m, n) \neq (6, 6), (7, 7), (8, 9)$ then $H_{\nu(m,n)}(M_{m,n})$ is finite. So Shareshian-MW Theorem applies.

$(m, n) = (6, 6), (7, 7)$ computer found 3-torsion.

$(m, n) = (8, 9)$ open

Conjecture 1: Shareshian & MW

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- if $n \leq 2m - 5$ then $H_{\nu(m,n)}(M_{m,n})$ has 3-torsion.

$2m - n \leq 1 \Rightarrow \nu(m, n) = m - 1$. Top homology always free.

$2m - n = 2, 3, 4 \Rightarrow \nu(m, n) = m - 2$. Codimension 1 homology

Conjecture 1: Shareshian & MW

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$2m - n = 2, 3, 4 \Rightarrow \nu(m, n) = m - 2$. Codimension 1 homology

Shareshian & MW: If $2m - n = 2$ then

$$H_{\nu(m,n)}(M_{m,n}) = \mathbb{Z}^{c_m-1},$$

where $c_m = \frac{1}{m+1} \binom{2m}{m}$

Conjecture 2. Shareshian & MW

- If $i \geq m - 2$ then $H_i(M_{m,n})$ is torsion-free
- If $\nu(m, n) \leq i \leq m - 3$ then $H_i(M_{m,n})$ has 3-torsion.

Conjecture 2. Shareshian & MW

- If $i \geq m - 2$ then $H_i(M_{m,n})$ is torsion-free
- If $\nu(m, n) \leq i \leq m - 3$ then $H_i(M_{m,n})$ has 3-torsion

$i \geq m - 1$: top homology trivially torsion-free

$i = m - 2, n = m$: $M_{m,n}$ collapses to $(m - 2)$ -dim. complex.

$n = m + 1$: $M_{m,n}$ is a pseudomanifold

$n = 2m - 2$: Shareshian-MW

$m + 2 \leq n < 2m - 2$: open

Conjecture 2. Shareshian & MW

- If $i \geq m - 2$ then $H_i(M_{m,n})$ is torsion-free
- If $\nu(m, n) \leq i \leq m - 3$ then $H_i(M_{m,n})$ has 3-torsion

Shareshian-MW (2000) $i = \nu(m, n)$ except $(m, n) = (8, 9)$

Jonsson (2005) proved this for all i when $n \geq m + 2$ and almost all i this when $n = m, m + 1$. Still open for $i = m - 3$ when $n = m, m + 1$

Shareshian & MW: If $n = 2m - 2$ then

$$H_{\nu(m,n)}(M_{m,n}) = \mathbb{Z}^{c_m-1},$$

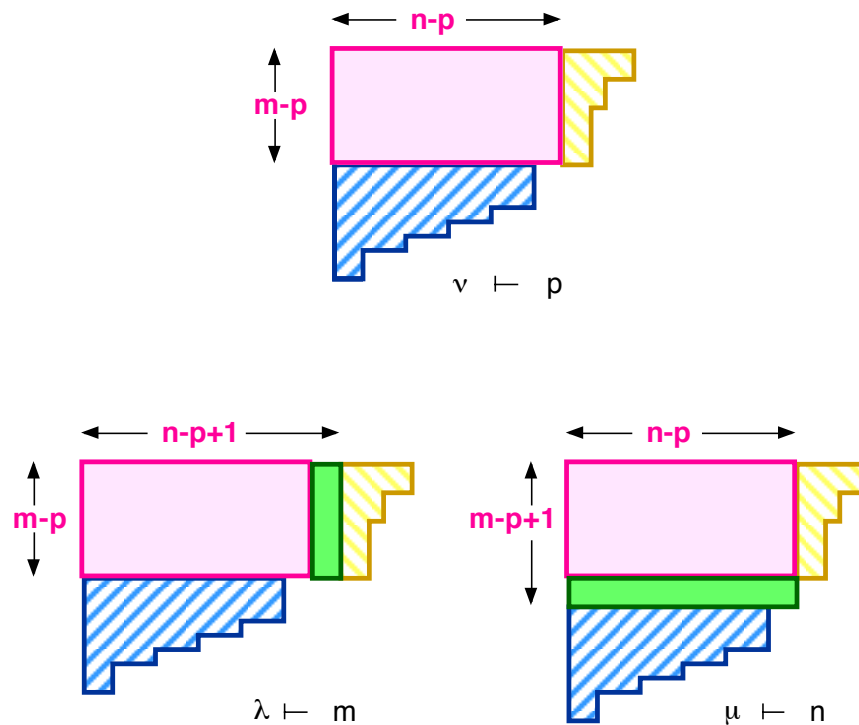
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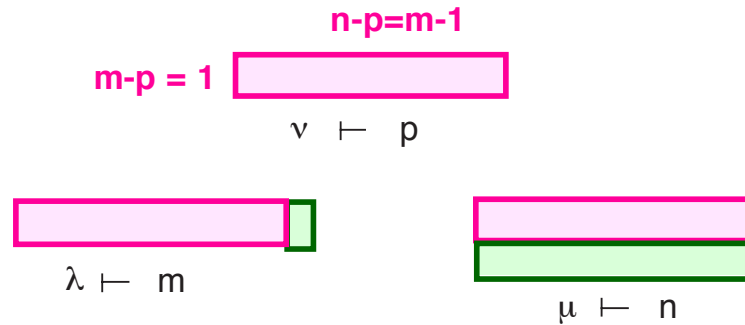
Proof technique:

Friedman & Hanlon(1998): As $(\mathfrak{S}_m \times \mathfrak{S}_n)$ -modules,

$$H_{p-1}(M_{m,n}; \mathbb{C}) \cong \bigoplus_{(\lambda, \mu) \in \mathcal{R}(m,n,p)} S^{\lambda'} \otimes S^{\mu}$$

where $\mathcal{R}(m, n, p)$ is the set of all pairs of partitions $(\lambda \vdash m, \mu \vdash n)$ which can be obtained from a partition $\nu \vdash p$ as follows





$$H_{m-2}(M_{m,2m-2}; \mathbb{C}) \cong_{\mathfrak{S}_m \times \mathfrak{S}_{2m-2}} S^{1^m} \otimes S^{(m-1, m-1)}$$

$$H_{m-2}(M_{m,2m-2}; \mathbb{C}) \cong_{\mathfrak{S}_{2m-2}} S^{(m-1, m-1)}$$

$$\dim H_{m-2}(M_{m,2m-2}; \mathbb{C}) = \#\text{SYT of shape } (m-1, m-1) = c_{m-1}$$

To show $H_{m-2}(M_{m,2m-2}; \mathbb{Z})$ is free we show $\cong S_{\mathbb{Z}}^{(m-1, m-1)}$

Presentation of abelian group $S_{\mathbb{Z}}^{\lambda}$:

Generators: tableaux of shape λ with distinct entries $1, 2, \dots, n$.

Column relations: $\sigma T = \text{sgn}(\sigma) T \quad \forall \sigma \in C(T)$

$$\begin{array}{|c|c|} \hline 3 & 2 \\ \hline 1 & \\ \hline \end{array} = - \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array}$$

Garnir relations:

Let $w_{i,j}(T) :=$ word obtained by reading top i entries of column $j + 1$ followed by entries of column j from row i down.

7	1	5	10
3	4	2	
9	8		
11	6		

$$w_{2,2} = 52486$$

Define

$$g_{i,j}(T) = \sum_{\sigma} \text{sgn}(\sigma) \sigma T$$

summed over all $\sigma \in \mathfrak{S}_n$ such that σT is obtained from T by replacing the red entries of column $j + 1$ with a subword of $w_{i,j}$ and replacing the red entries of column j with the complementary subword.

7	1	5	10
3	4	2	
9	8		
11	6		

$$w_{2,2} = 52486$$

$$g_{2,2} = \begin{array}{c} \begin{array}{|c|c|c|c|} \hline 7 & 1 & 5 & 10 \\ \hline 3 & 4 & 2 & \\ \hline 9 & 8 & & \\ \hline 11 & 6 & & \\ \hline \end{array} - \begin{array}{|c|c|c|c|} \hline 7 & 1 & 5 & 10 \\ \hline 3 & 2 & 4 & \\ \hline 9 & 8 & & \\ \hline 11 & 6 & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline 7 & 1 & 5 & 10 \\ \hline 3 & 2 & 8 & \\ \hline 9 & 4 & & \\ \hline 11 & 6 & & \\ \hline \end{array} - \begin{array}{|c|c|c|c|} \hline 7 & 1 & 5 & 10 \\ \hline 3 & 2 & 6 & \\ \hline 9 & 4 & & \\ \hline 11 & 8 & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline 7 & 1 & 2 & 10 \\ \hline 3 & 5 & 4 & \\ \hline 9 & 8 & & \\ \hline 11 & 6 & & \\ \hline \end{array} \pm \dots$$

The Garnir relations are the relations $g_{i,j}(T) = 0 \quad \forall i, j$

Let $T^\lambda = \{\text{tableaux of shape } \lambda\}$

$S_{\mathbb{Z}}^\lambda = \langle T^\lambda \mid \text{Garnir relations \& column relations} \rangle$

$S_{\mathbb{Z}}^{(m-1, m-1)}$ relations

$$\begin{array}{|c|c|c|} \hline \cdots & a_j & \cdots \\ \hline \cdots & b_j & \cdots \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline \cdots & b_j & \cdots \\ \hline \cdots & a_j & \cdots \\ \hline \end{array} = 0$$

$$\begin{array}{|c|c|c|c|} \hline \cdots & a_{j-1} & a_j & \cdots \\ \hline \cdots & b_{j-1} & \cdot & \cdots \\ \hline \end{array} - \begin{array}{|c|c|c|c|} \hline \cdots & a_j & a_{j-1} & \cdots \\ \hline \cdots & b_{j-1} & \cdot & \cdots \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline \cdots & a_j & b_{j-1} & \cdots \\ \hline \cdots & a_{j-1} & \cdot & \cdots \\ \hline \end{array} = 0$$

$$\begin{array}{|c|c|c|c|} \hline \cdots & \cdot & a_j & \cdots \\ \hline \cdots & b_{j-1} & b_j & \cdots \\ \hline \end{array} - \begin{array}{|c|c|c|c|} \hline \cdots & \cdot & a_j & \cdots \\ \hline \cdots & b_j & b_{j-1} & \cdots \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline \cdots & \cdot & b_j & \cdots \\ \hline \cdots & a_j & b_{j-1} & \cdots \\ \hline \end{array} = 0$$

Chessboard Complex: $M_{m,n}$

vertex set $:= \{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$

face $:= \{(i_1, j_1), \dots, (i_k, j_k)\} \quad i_s \neq i_t, \quad j_s \neq j_t \quad \forall s, t$

facet of $M_{3,4}$:

	1	2	3	4	
1		×			↔
2				×	
3	×				

$\leftrightarrow \{(1, 2), (2, 4), (3, 1)\}$

Oriented simplex $(1, 2) \wedge (2, 4) \wedge (3, 1) \in C_2(M_{3,4})$

$$(1, 2) \wedge (2, 4) \wedge (3, 1) = -(2, 4) \wedge (1, 2) \wedge (3, 1)$$

Let $\phi : S_{\mathbb{Z}}^{(m-1)^2} \rightarrow H_{m-2}(M_{m,2m-2})$ be the homomorphism defined on generators by

$$\phi \left(\begin{array}{|c|c|c|c|} \hline a_1 & a_2 & \cdots & a_{m-1} \\ \hline b_1 & b_2 & \cdots & b_{m-1} \\ \hline \end{array} \right)$$

$$= [1 \mid a_1, b_1] \wedge [2 \mid a_2, b_2] \wedge \dots \wedge [m-1 \mid a_{m-1}, b_{m-1}],$$

where $[j \mid a_j, b_j]$ denotes the fundamental cycle $(j, a_j) - (j, b_j)$ of $H_0(M_{\{j\}, \{a_j, b_j\}})$

Clear that column relations map to 0.

$$\phi \left(\begin{array}{|c|c|c|} \hline \cdots & a_j & \cdots \\ \hline \cdots & b_j & \cdots \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline \cdots & b_j & \cdots \\ \hline \cdots & a_j & \cdots \\ \hline \end{array} \right) = (\cdots [j \mid a_j, b_j] \cdots) + (\cdots [j \mid b_j, a_j] \cdots)$$

Check that Garnir relations map to boundary relations, i.e.,

$$\phi \left(\begin{array}{|c|c|c|c|} \hline \cdots & a_{j-1} & a_j & \cdots \\ \hline \cdots & b_{j-1} & \cdot & \cdots \\ \hline \end{array} - \begin{array}{|c|c|c|c|} \hline \cdots & a_j & a_{j-1} & \cdots \\ \hline \cdots & b_{j-1} & \cdot & \cdots \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline \cdots & a_j & b_{j-1} & \cdots \\ \hline \cdots & a_{j-1} & \cdot & \cdots \\ \hline \end{array} \right)$$

is a boundary

Claim: ϕ is surjective.

Theorem: $H_{m-2}(M_{m,2m-2})$ is generated by elements of the form

$$[1 \mid a_1, b_1] \wedge [2 \mid a_2, b_2] \wedge \dots \wedge [m-1 \mid a_{m-1}, b_{m-1}],$$

where $[j \mid a_j, b_j] := (j, a_j) - (j, b_j)$.

$$S_{\mathbb{Z}}^{(m-1)^2} \xrightarrow{\phi} H_{m-2}(M_{m,2m-2}) \xrightarrow{\pi} H_{m-2}(M_{m,2m-2}) / H_{m-2}(M_{m,2m-2})_{\text{tor}}$$

surjection between free abelian groups of the same rank c_{m-1}

So $H_{m-2}(M_{m,2m-2})$ is free of rank c_{m-1} .

Theorem: $H_{m-2}(M_{m,2m-2})$ is generated by elements of the form

$$[1 \mid a_1, b_1] \wedge [2 \mid a_2, b_2] \wedge \dots \wedge [m-1 \mid a_{m-1}, b_{m-1}],$$

where $[i \mid a_i, b_i] := (i, a_i) - (i, b_i)$.

Proof idea:

Tail end of a long exact sequence: $n = 2m - 2$

$$\bigoplus_{i \in [m] \setminus \{1\}} H_{m-3}(M_{[m] \setminus \{1, i\}, [n] \setminus \{1\}}) \oplus$$

$$\bigoplus_{j \in [n] \setminus \{1\}} H_{m-3}(M_{[m] \setminus \{1\}, [n] \setminus \{1, j\}})$$

$$\xrightarrow{\varphi} H_{m-2}(M_{m,n}) \rightarrow 0$$

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$$\varphi(\rho) = \begin{cases} [1, i \mid 1] \wedge \rho & \text{if } \rho \in H_{m-3}(M_{[m] \setminus \{1, i\}, [n] \setminus \{1\}}) \\ [1 \mid 1, j] \wedge \rho & \text{if } \rho \in H_{m-3}(M_{[m] \setminus \{1\}, [n] \setminus \{1, j\}}) \end{cases}$$

where $[1, i \mid 1] = (1, 1) - (i, 1) \in H_0(M_{\{1, i\}, \{1\}})$ and
 $[1 \mid 1, j] = (1, j) - (1, 1) \in H_0(M_{\{1\}, \{1, j\}})$

$$\bigoplus_{i \in [m] \setminus \{1\}} H_{m-3}(M_{[m] \setminus \{1, i\}, [n] \setminus \{1\}}) \oplus$$

$$\bigoplus_{j \in [n] \setminus \{1\}} H_{m-3}(M_{[m] \setminus \{1\}, [n] \setminus \{1, j\}})$$

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where $[1, i \mid 1] = (1, 1) - (i, 1) \in H_0(M_{\{1, i\}, \{1\}})$ and
 $[1 \mid 1, j] = (1, j) - (1, 1) \in H_0(M_{\{1\}, \{1, j\}})$

Top Homology

Friedman-Hanlon \Rightarrow Garst: As \mathfrak{S}_n -modules

$$H_{m-1}(M_{m,n}; \mathbb{C}) \cong \bigoplus_{\substack{\lambda \vdash n \\ \lambda_1 = n - m}} f^{\lambda^*} S^\lambda$$

where λ^* is obtained from λ by removing the first row.

So $\text{rank}(H_{m-1}(M_{m,n})) = \#$ pairs of SYT of sizes m and n , whose shapes differ by a row (**Garst pair**).

For each Garst pair we construct a **cycle** and a **cocycle** of top dimension.

P

1	3
2	
4	

Q

1	2	4	6
3	5		
7			
8			

P

1	3	∞	∞
2	∞		
4			
∞			

Q

1	2	4	6
3	5		
7			
8			

P

1	3	∞	∞
2	∞		
4			
∞			

Q

1	2	4	6
3	5		
7			
8			

P

2	3	∞	∞
4	∞		
∞			

Q

1	2	4	6
3	5		
7			

1

8

P

2	4	∞	∞
∞	∞		

Q

1	2	4	6
3	5		

3 1
7 8

P

2	4	∞	
∞	∞		

Q

1	2	4	
3	5		

∞ 3 1
6 7 8

P

2	∞	∞	
∞			

Q

1	2	4	
3			

4 ∞ 3 1
5 6 7 8

P

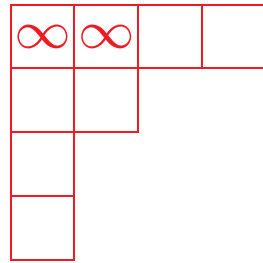
2	∞		
∞			

Q

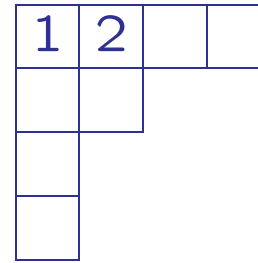
1	2		
3			

∞ 4 ∞ 3 1
4 5 6 7 8

P

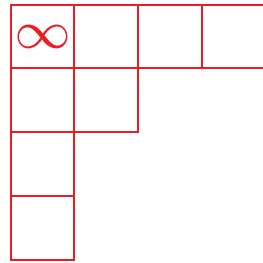


Q

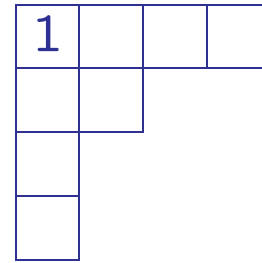


2 ∞ 4 ∞ 3 1
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P

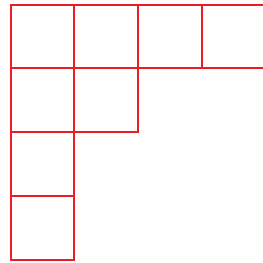


Q

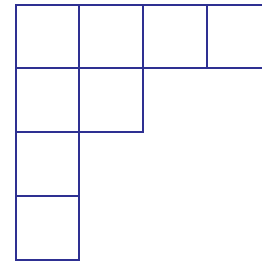


∞ 2 ∞ 4 ∞ 3 1
2 3 4 5 6 7 8

P

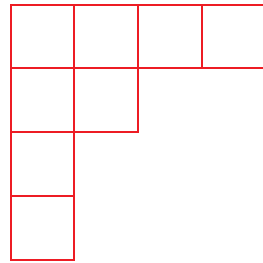


Q

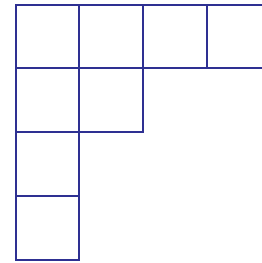


$\infty \infty 2 \infty 4 \infty 3 1$
1 2 3 4 5 6 7 8

P



Q



$\infty \infty 2 \infty 4 \infty 3 1$
 $1 2 3 4 5 6 7 8$



Cocycle in $M_{4,8}$:

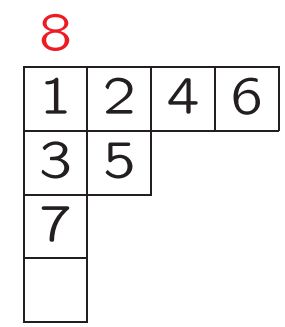
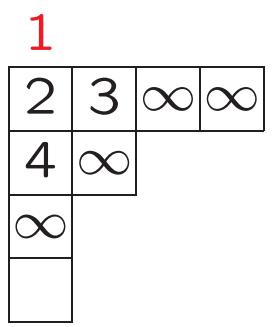
$$\gamma(P, Q) = (2, 3) \wedge (4, 5) \wedge (3, 7) \wedge (1, 8)$$

Cycle

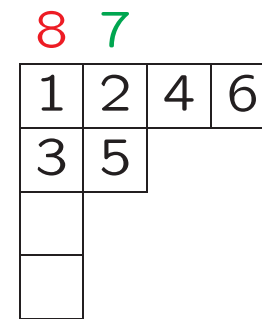
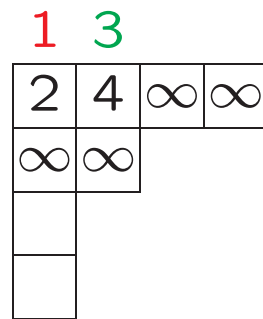
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2	∞		
4			
∞			

1	2	4	6
3	5		
7			
8			

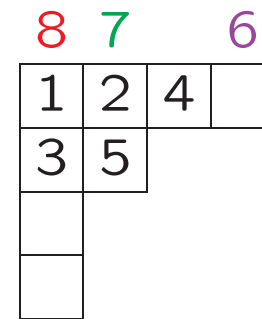
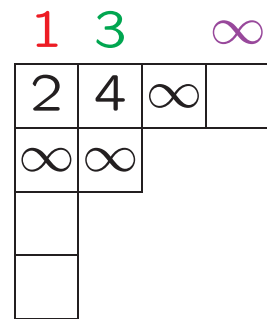
Cycle



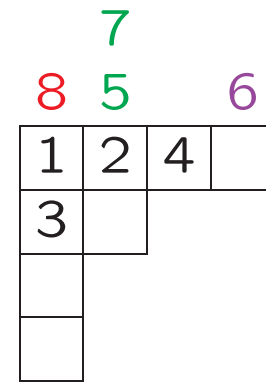
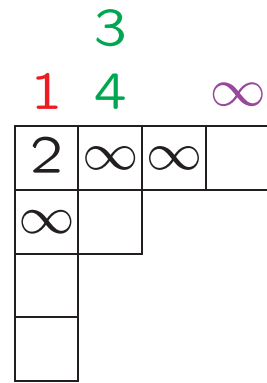
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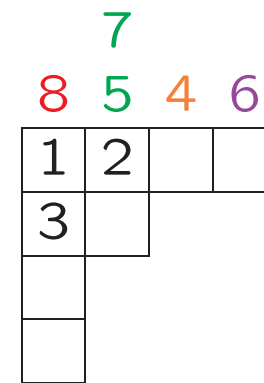
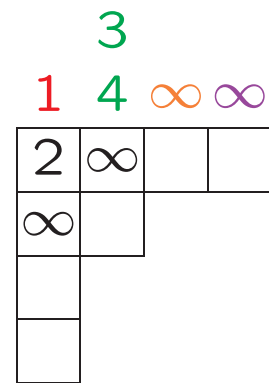
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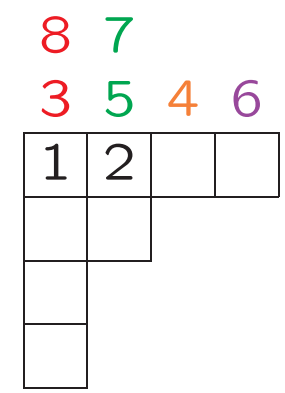
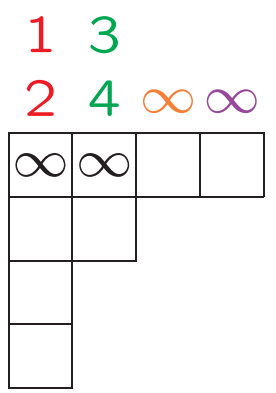
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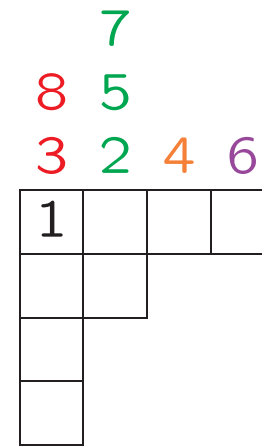
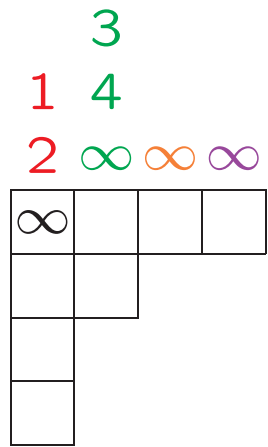
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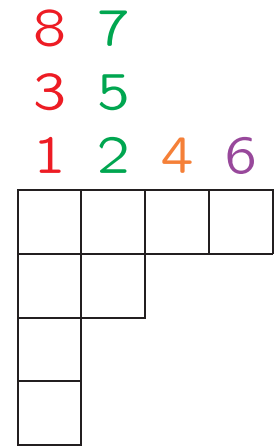
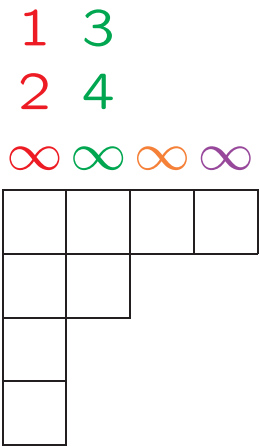
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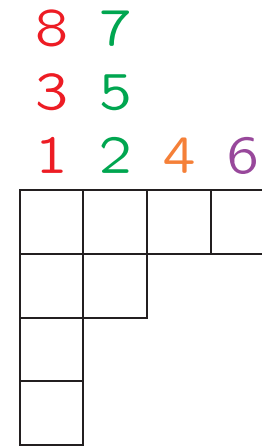
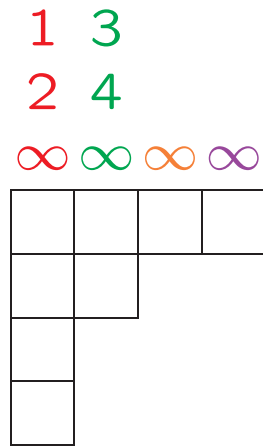
Cycle



Cycle

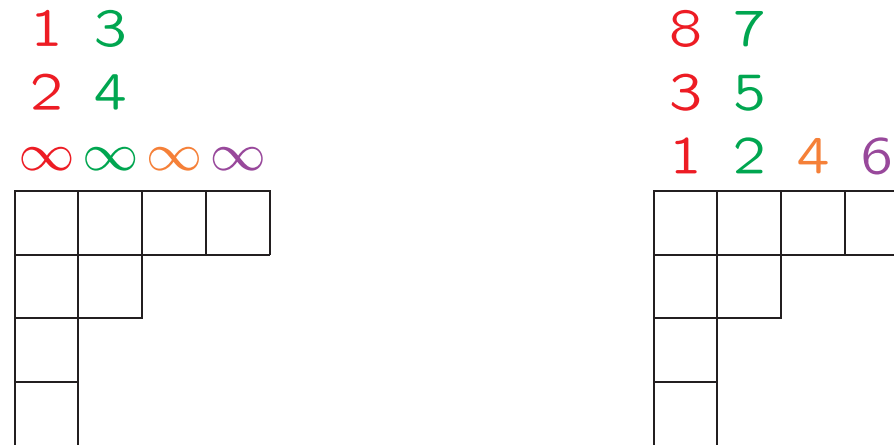


Cycle



$$M_{\{1,2\},\{1,3,8\}}, M_{\{3,4\},\{2,5,7\}}, M_{\emptyset,\{4\}}, M_{\emptyset,\{6\}}$$

Cycle



$$M_{\{1,2\},\{1,3,8\}}, M_{\{3,4\},\{2,5,7\}}, M_{\emptyset,\{4\}}, M_{\emptyset,\{6\}}$$

Complexes are pseudomanifolds. So top homology is cyclic.

$$H_1(M_{\{1,2\},\{1,3,8\}}) = \langle \alpha \rangle$$

$$H_1(M_{\{3,4\},\{2,5,7\}}) = \langle \beta \rangle$$

Define

$$\rho(P, Q) = \alpha \wedge \beta \in H_3(M_{4,8})$$

Theorem: $\{\rho(P, Q) : (P, Q) \text{ a Garst pair}\}$ is a basis for $H_{m-1}(M_{m,n})$.

Idea of Proof: We find an ordering of the pairs of standard Young tableaux

$$(P_1, Q_1), \dots, (P_k, Q_k)$$

so that the matrix

$$(\langle \rho(P_i, Q_i), \gamma(P_j, Q_j) \rangle)_{i,j=1,\dots,k}$$

is unitriangular.

So $H_{m-1}(M_{m,n})$ is generated by elements of the form

$$[A \mid B] \wedge \tau,$$

where

- $A \subseteq [m]$, $B \subseteq [n]$ and $|A| = |B| - 1$
- $[A \mid B]$ is a fundamental cycle of the pseudomanifold $M_{A,B}$
- $\tau \in H_{m-1-|A|}(M_{[m]-A,[n]-B})$.

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- $\tau \in H_{m-1-|A|}(M_{[m]-A,[n]-B})$.

Follows that $H_{m-2}(M_{m,2m-2})$ generated by cycles of the forms $[1 \mid 1, j] \wedge \rho$ and

$$[1, i \mid 1] \wedge [A \mid B] \wedge \tau = \pm [A \mid B] \wedge [1, i \mid 1] \wedge \tau = [A \mid B] \wedge \gamma$$

where $\gamma \in H_{m-2-|A|}(M_{[m]-A,[2m-2]-B})$. Can show $|A| = 1$.

Shareshian & MW (2000): $H_{\nu(m,n)}(M_{m,n})$ is nonvanishing for all m and n . Moreover

- $m + n \equiv 1 \pmod{3}$

If $m \leq n \leq 2m - 5$ then

$$H_{\nu(m,n)}(M_{m,n}) \cong \mathbb{Z}_3$$

- $m + n \equiv 0 \pmod{3}$

If $m \leq n \leq 2m - 9$ then $H_{\nu(m,n)}(M_{m,n})$ is a 3-group with exponent at most 9.

- $m + n \equiv 2 \pmod{3}$

If $m \leq n \leq 2m - 13$ then $H_{\nu(m,n)}(M_{m,n})$ is a 3-group with exponent at most 9.