Bounded Degree Graph Complexes

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Let G be a graph on node set $\{1, 2, \ldots, n\}$.

$\Delta(G, r) :=$ simplicial complex of subgraphs of G for which each node has degree $\leq r$.

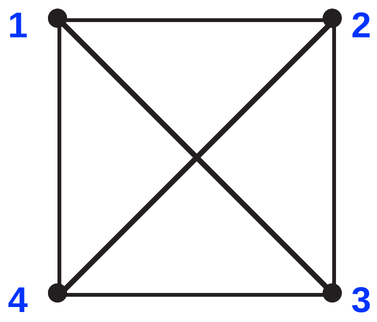
vertices := edges
faces := subgraphs

Matching complex $M_n := \Delta(K_n, 1)$
Chessboard complex $M_{m,n} := \Delta(K_{m,n}, 1)$

Matching Complex

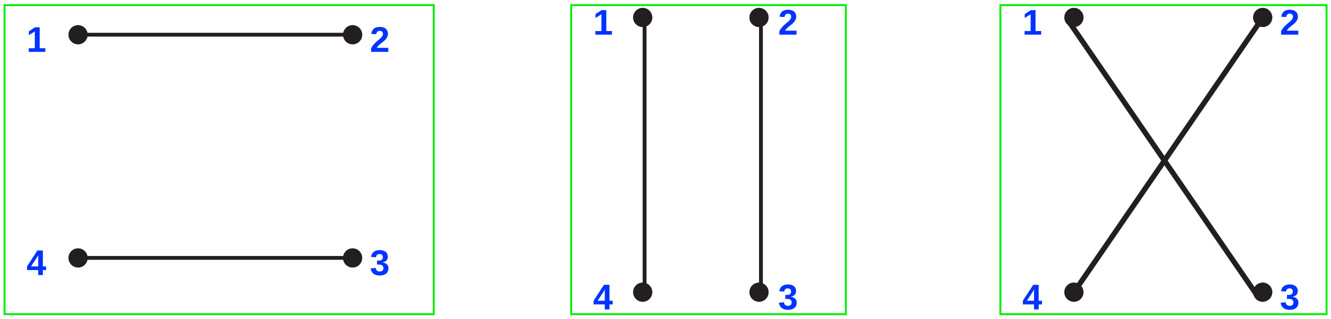
$M_4 = \Delta(K_4, 1)$

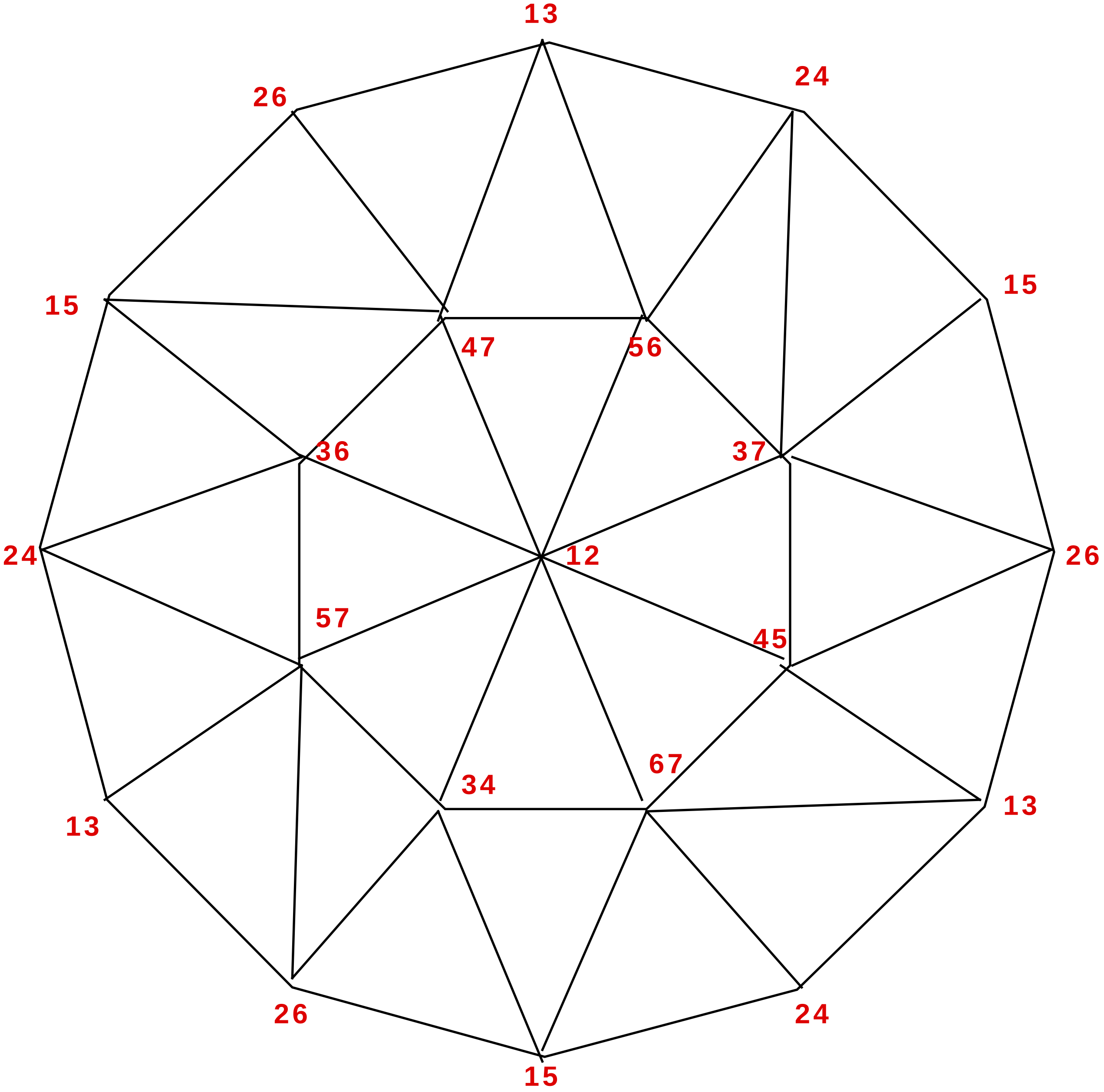
K_4 =



r = 1

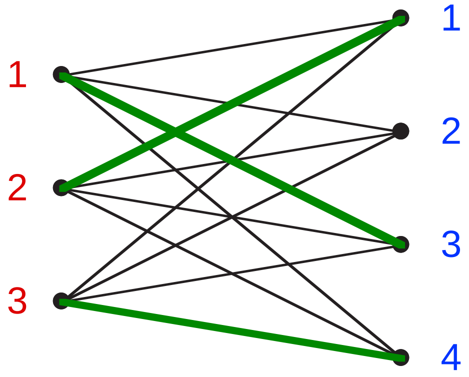
Maximal bounded degree subgraphs:



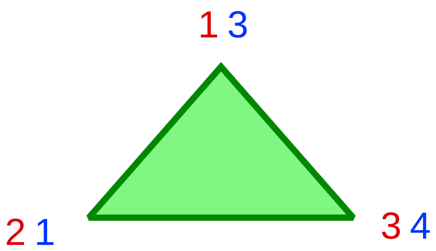


$M_7 := \Delta(K_7, 1)$

Chessboard Complex $M_{3,4} = \Delta(K_{3,4}, 1)$



	1	2	3	4
1			X	
2	X			
3				X



Tits coset complexes: Garst(1979)

Commutative algebra (minimal free resolutions): Józefiak & Weyman(1988), Reiner & Roberts(1997)

Quillen complexes: Bouc(1990)

Computational Geometry (Colored Tverberg Problem): Vrécica & Živaljević(1992)

Homology of nilpotent Lie algebras: Sigg(1996)

Vassiliev knot invariants (not k-connected graph complexes): Babson, Björner, Linusson, Shareshian & Welker (1999)

Combinatorial simplicial complexes: Björner, Lovász, Vrécica & Živaljević(1992), Ziegler(1992), Karaguezian(1994), Friedman & Hanlon(1998), Björner & Eriksson (1999), Karaguezian, Reiner & MW(1999), MW(1999), Dong(1999), Dong & MW(2000), Shareshian & MW(2000)

Tits Coset Complexes

Let G be a group and $G^1, \ldots, G^m$ a family of subgroups. Form a simplicial complex

$$\Delta(G; G^1, \ldots, G^m)$$

whose vertices are the cosets of the subgroups and whose facets are of the form $\{gG^1, \ldots, gG^m\}$ where $g \in G$.

Examples: Coxeter complexes, Tits buildings, chessboard complexes

$M_{m,n} = \Delta(G; G^1, \ldots, G^m)$ where

$G = \mathfrak{S}_n$ and $G^i = \{\sigma \in \mathfrak{S}_n \mid \sigma(i) = i\}$ for $i = 1, \ldots, m$.

Garst (1979): $M_{m,n}$ is Cohen-Macaulay if and only if $2m - 1 \leq n$.

Quillen Complexes

$S(G,p)$ = partially ordered set of nontrivial p-subgroups of a finite group G ordered by inclusion.

$\Delta(S(G,p))$ = simplicial complex whose faces are chains of the poset

Bouc considers a subposet of $S(\mathfrak{S}_n, 2)$:
T_n = poset of 2-subgroups of $\mathfrak{S}_n$ that contain a transposition.

Inclusion map from the subposet A_n of nontrivial abelian subgroups of $\mathfrak{S}_n$ that are generated by transpositions, to T_n induces a homotopy equivalence.

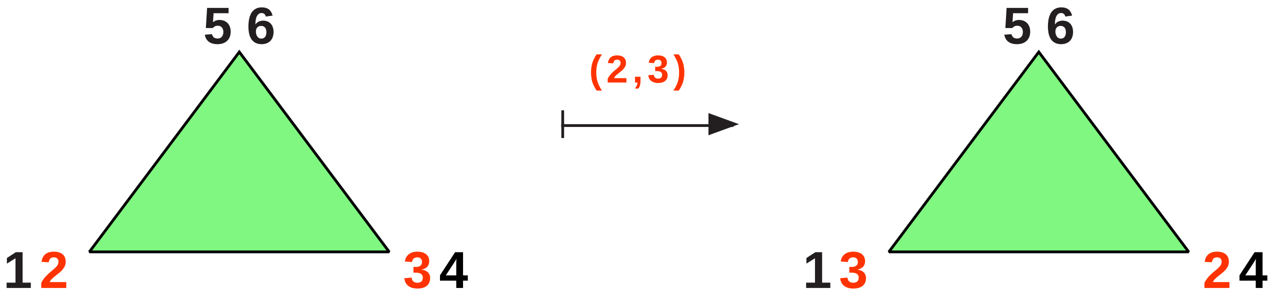
A_n is the same as the poset of nonempty faces of the matching complex.

$$\langle(1,4),(3,6),(5,8)\rangle \to \{\{1,4\},\{3,6\},\{5,8\}\}$$

So $\Delta(T_n) \simeq M_n$

Homology of the Matching Complex

The symmetric group $\mathfrak{S}_n$ acts on M_n by permuting node labels.



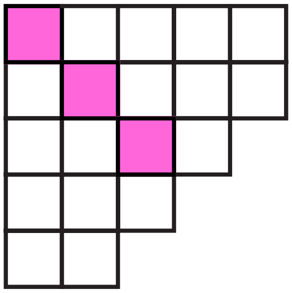
This induces a representation of $\mathfrak{S}_n$ on $\tilde{H}_p(M_n; \mathbb{C})$.

Theorem (Bouc, 1990). As $\mathfrak{S}_n$-modules,

$$\tilde{H}_{p-1}(M_n;\mathbb{C}) \cong \bigoplus_{\substack{\lambda:\ \lambda \vdash n \\ \lambda = \lambda' \\ d(\lambda) = n-2p}} S^\lambda$$

where S^λ is the irreducible $\mathfrak{S}_n$-module indexed by λ and $d(\lambda)$ is the size of the diagonal of λ.

$\lambda =$



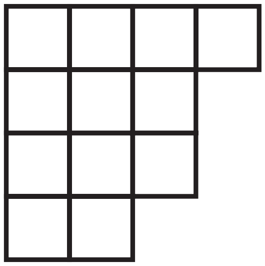
$d(\lambda) = 3$

Irreducible Representations of $\mathfrak{S}_n$

Let λ be a partition of n, i.e., a decreasing sequence of positive integers whose sum is n.

$\lambda = (4, 3, 3, 2)$ is a partition of 12

λ =



A **tableau** of shape λ is a filling of the cells of λ with integers $1, 2, \ldots, n$.

A tableau is **standard** if the entries of the rows and columns increase.

3	12	1	5
7	4	10	
11	6	2	
9	8		

tableau

1	3	5	11
2	4	10	
6	8	12	
7	9		

standard tableau

S^λ is the complex vector space generated by tableaux of shape λ subject to certain relations.

$\mathfrak{S}_n$ acts on S^λ by permuting tableau entries.

$$(2,3)\quad \begin{array}{|c|c|c|}\hline 1 & \mathbf{2} & \mathbf{3} \\ \hline 4 & 5 \\ \cline{1-2}\end{array} \quad = \quad \begin{array}{|c|c|c|}\hline 1 & \mathbf{3} & \mathbf{2} \\ \hline 4 & 5 \\ \cline{1-2}\end{array}$$

S^λ is an irreducible representation of $\mathfrak{S}_n$.

The standard tableaux form a basis for S^λ.

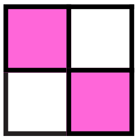
Basis for $S^{3,2}$:

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}$$

$\dim S^\lambda = \#$ standard tableaux of shape λ.

Example: $\tilde{H}_{p-1}(M_4;\mathbb{C})$

Self-conjugate shapes with 4 cells:



$$p = \frac{n-d}{2} = \frac{4-2}{2} = 1$$

$$\tilde{H}_0(M_4;\mathbb{C}) = S^{2,2} \qquad \tilde{H}_i(M_4;\mathbb{C}) = 0 \quad \forall i \neq 0$$

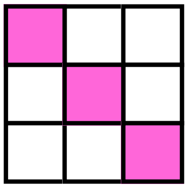
Standard tableaux of shape $2,2$:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$$

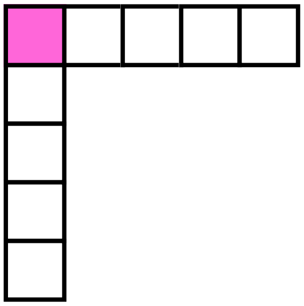
$$\tilde{\beta}_0(M_4) = 2, \qquad \tilde{\beta}_i(M_4) = 0 \quad \forall i \neq 0$$

Example: $\tilde{H}_{p-1}(M_9;\mathbb{C})$

Self-conjugate shapes with 9 cells:



$$p = \frac{9-3}{2} = 3$$



$$p = \frac{9-1}{2} = 4$$

$$\tilde{H}_2(M_9;\mathbb{C}) = S^{3,3,3}$$

$$\tilde{\beta}_2(M_9) = \frac{9!}{5\cdot4\cdot3\cdot4\cdot3\cdot2\cdot3\cdot2}$$

$$= 42$$

$$\tilde{H}_3(M_9;\mathbb{C}) = S^{51111}$$

$$\tilde{\beta}_3(M_9) = \frac{9!}{9\cdot4\cdot3\cdot2\cdot4\cdot3\cdot2\cdot}$$

$$= 70$$

Laplacian Proof of Bouc's Theorem

Dong & MW(1999) (ideas borrowed from Sigg's work on the homology of the free two-step nilpotent Lie algebra and from Friedman & Hanlon's work on the homology of the chessboard complex)

The **combinatorial Laplacian** $\Lambda_p : C_p(M_n, \mathbb{C}) \to C_p(M_n, \mathbb{C})$ is defined by

$$\Lambda_p = \delta_{p-1}\partial_p + \partial_{p+1}\delta_p$$

where ∂ is the boundary map and δ is the coboundary map.

Analogue of Hodge Theory (Kostant):

$$\tilde{H}_p(M_n; \mathbb{C}) \cong_{\mathfrak{S}_n} \ker \Lambda_p$$

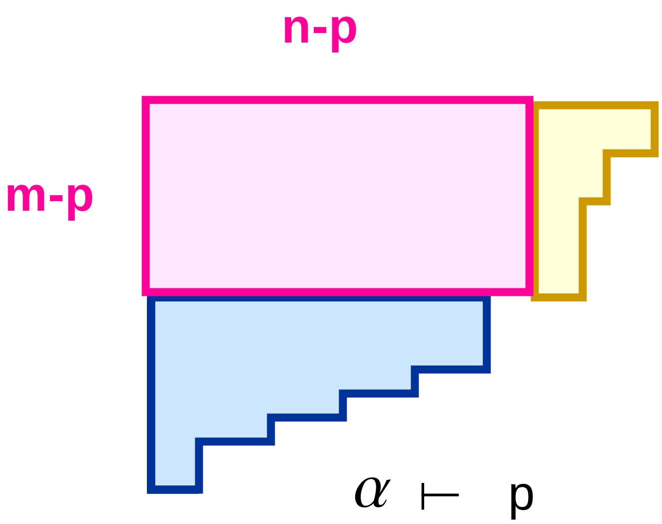
Homology of the Chessboard Complex

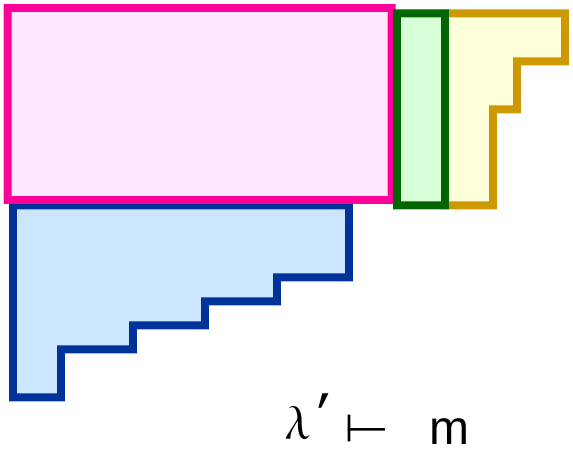
The group $\mathfrak{S}_m \times \mathfrak{S}_n$ acts on $M_{m,n}$ which induces a representation of $\mathfrak{S}_m \times \mathfrak{S}_n$ on $\tilde{H}_p(M_{m,n}; \mathbb{C})$.

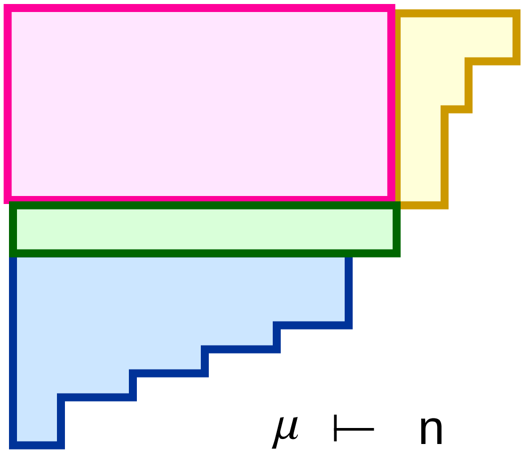
Friedman & Hanlon(1998): As $\mathfrak{S}_m \times \mathfrak{S}_n$-modules,

$$\tilde{H}_{p-1}(M_{m,n}; \mathbb{C}) \cong \bigoplus_{\lambda,\mu} S^{\lambda} \otimes S^{\mu}$$

summed over all pairs of partitions $\lambda \vdash m$, $\mu \vdash n$ which can be obtained from a partition $\alpha \vdash p$ in the following way


n-p
m-p
$\alpha \vdash p$


$\lambda' \vdash m$


$\mu \vdash n$

Torsion in the Matching Complex

Björner, Lovász, Vrécica, Živaljević (1994): Let

$\nu_n = \lfloor \frac{n+1}{3} \rfloor - 1$ & $\nu_{m,n} = \min\{m, \lfloor \frac{m+n+2}{3} \rfloor\} - 1$.

Then

- M_n is $(\nu_n - 1)$-connected
- $M_{m,n}$ is $(\nu_{m,n} - 1)$-connected

BLVZ Conjecture:

- $\tilde{H}_{\nu_n}(M_n) \neq 0$
- $\tilde{H}_{\nu_{m,n}}(M_{m,n}) \neq 0$

Bouc (1990):

If $n \equiv 1 \bmod 3$, $n \geq 7$ then $\tilde{H}_{\nu_n}(M_n) \cong \mathbb{Z}_3$

If $n \equiv 0 \bmod 3$, $n \geq 12$ then $\tilde{H}_{\nu_n}(M_n)$ is a nontrivial finite 3-group with exponent at most 9.

Shareshian & MW (1999): $\tilde{H}_{\nu_n}(M_n)$ is nonvanishing for all n. Moreover, for all $n \geq 12$ (except possibly $n = 14$)

$$\tilde{H}_{\nu_n}(M_n) \cong \mathbb{Z}_3^{r_n}$$

where $r_n \geq 1$.

Main tool: Bouc's long exact sequence

$$\cdots \xrightarrow{\delta} \bigoplus_{a,h} \tilde{H}_{t-1}(M_{n-3}) \xrightarrow{\phi} \tilde{H}_t(M_n) \xrightarrow{\psi}$$

$$\bigoplus_{i,j} \tilde{H}_{t-2}(M_{n-4}) \xrightarrow{\delta} \bigoplus_{a,h} \tilde{H}_{t-2}(M_{n-3}) \xrightarrow{\phi} \cdots$$

where a ranges over the set $\{1, 2\}$ and h, i, j range over the set $\{3, \ldots, n\}$ with $i \neq j$.

Nonvanishing is a consequence.

To show that $\tilde{H}_{\nu_n}(M_n)$ has only 3-torsion:

For $\mathbf{n \equiv 0, 1}$ mod $\mathbf{3}$, Bouc's long exact sequence gives

$$\bigoplus_{a,h} \tilde{H}_{\nu_{n-3}}(M_{n-3}) \xrightarrow{\phi} \tilde{H}_{\nu_n}(M_n) \rightarrow 0$$

If $\tilde{H}_{\nu_{n-3}}(M_{n-3})$ has only 3-torsion then $\tilde{H}_{\nu_n}(M_n)$ has only 3-torsion.

Base step of induction: $n = 7$ and $n = 12$. Bouc shows

$$\tilde{H}_{\nu_7}(M_7) = \mathbb{Z}_3$$

Computer calculation of Heckenbach, Welker, Dumas, Saunders:

$$\tilde{H}_{\nu_{12}}(M_{12}) = \mathbb{Z}_3^{56}$$

Let $\mathbf{n \equiv 2}$ mod $\mathbf{3}$. We used Bouc's long exact sequence to construct a generating set of cycles for $\tilde{H}_{\nu_n}(M_n)$ which have the form

$$\alpha \, * \, \rho$$

where

$$\alpha \in \tilde{H}_1(M_A) \quad \text{and} \quad \rho \in \tilde{H}_{\nu_{n-5}}(M_{[n]-A})$$

for some $A \subseteq [n]$ such that $|A| = 5$.

Since $n - 5 \equiv 0$ mod 3, we can apply the previous case if $n - 5 \geq 12$. This gives $3\rho = 0$ which implies

$$3(\alpha * \rho) = \alpha * 3\rho = 0$$

.

Open Problems:

- Eliminate use of the computer
- Determine r_n
- Show that there is only 3-torsion in all homology groups of M_n. (Babson, Björner, Linusson, Shareshian, Welker)

Torsion in the Chessboard Complex

Note: If $2m - 1 \leq n$ then $\nu_{m,n} = m - 1$. So $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is free.

Shareshian & MW (2000): $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is non-vanishing for all m and n. Moreover

- $n + m \equiv 1 \bmod 3$
  If $5 \leq m \leq n \leq 2m - 5$ then
  $$\tilde{H}_{\nu_{m,n}}(M_{m,n}) \equiv \mathbb{Z}_3$$
- $n + m \equiv 0 \bmod 3$
  If $9 \leq m \leq n \leq 2m - 9$ then $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is a 3-group with exponent at most 9.
- $n + m \equiv 2 \bmod 3$
  If $13 \leq m \leq n \leq 2m - 13$ then $\tilde{H}_{\nu_{m,n}}(M_{m,n})$ is a 3-group with exponent at most 9.

The proof is much harder than that for the matching complex

Difficulties:

- For $m+n \equiv 0,1 \bmod 3$

$$\bigoplus \tilde{H}_{\nu_{m-2,n-1}}(M_{m-2,n-1}) + \bigoplus \tilde{H}_{\nu_{m-1,n-2}}(M_{m-1,n-2})$$

$$\xrightarrow{\phi} \tilde{H}_{\nu_{m,n}}(M_{m,n}) \to 0$$

  But $m-2$ and $n-1$ could be sufficiently far apart to make homology free.

- Computer provides only

$$\tilde{H}_{\nu_{5,5}}(M_{5,5}) \cong \mathbb{Z}_3.$$

  $\tilde{H}_{\nu_{6,6}}(M_{6,6})$ and $\tilde{H}_{\nu_{7,7}}(M_{7,7})$ have free parts by the Friedman-Hanlon result.

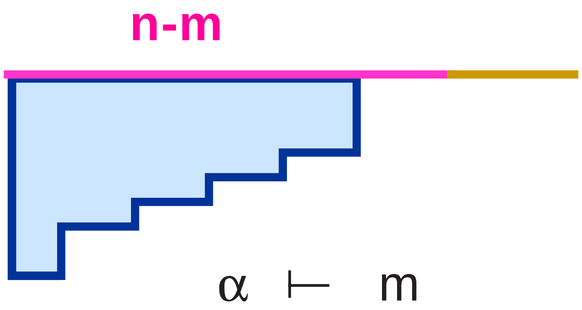
  $\tilde{H}_{\nu_{7,8}}(M_{7,8})$ is too big for the computer.

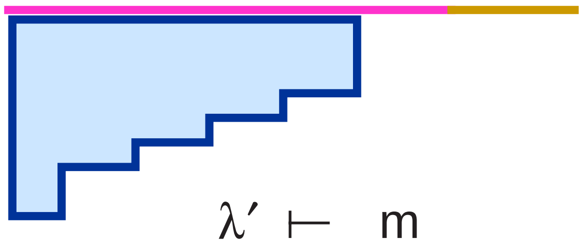
Basis for Top Homology and Cohomology

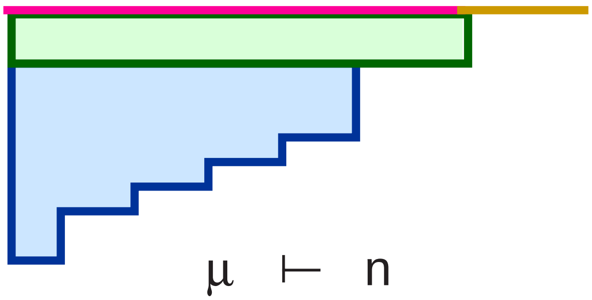
Recall Friedman-Hanlon

$$\tilde{H}_{p-1}(M_{m,n};\mathbb{C}) \cong \bigoplus_{\lambda,\mu} S^{\lambda} \otimes S^{\mu}$$

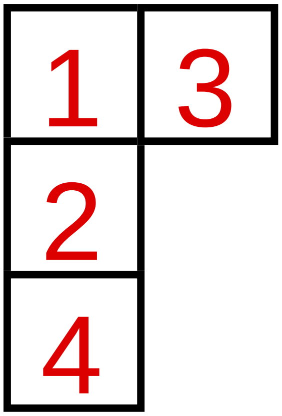
When $p = m \leq n$







From a pair of young tableaux of shape λ and μ we construct a cycle and a cocycle of dimension $m-1$ using the Robinson-Schensted correspondence.



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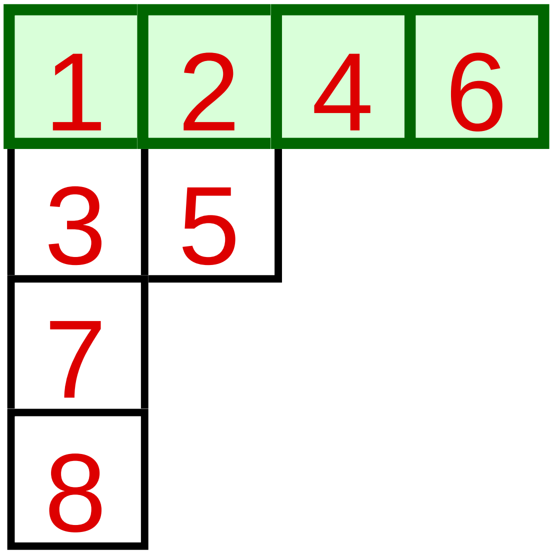

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Robinson-Schensted Correspondence

Let $[r] = \{1, 2, \ldots, r\}$
$[r]^n =$ the set of words of length n over $[r]$

Semistandard tableau of shape λ is a filling of λ with elements of $[r]$ so that rows are weakly increasing and columns are strictly increasing.

Let $w \in [r]^n$

$$w \quad \mapsto (P, Q)$$

where P is a semistandard tableau and Q is standard tableau of the same shape as P.

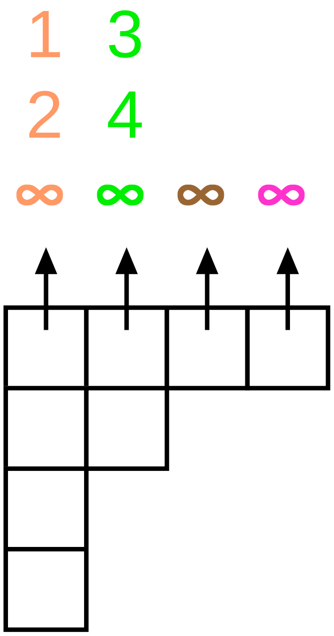
$$3\,4\,3\,1\,2\,4 \quad \longleftrightarrow \quad \left(\begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 3 & \multicolumn{1}{|c}{} \\ \cline{1-2} 4 & \multicolumn{1}{|c}{} & \\ \cline{1-1} \end{array} \;,\; \begin{array}{|c|c|c|} \hline 1 & 2 & 6 \\ \hline 3 & 5 & \multicolumn{1}{|c}{} \\ \cline{1-2} 4 & \multicolumn{1}{|c}{} & \\ \cline{1-1} \end{array} \right)$$

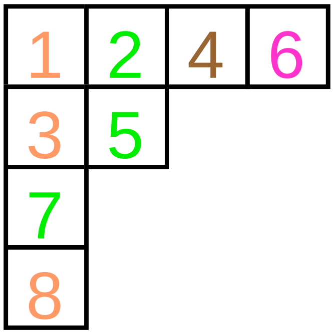
Cocycle:

$$\begin{array}{cccccccc} \infty & \infty & 2 & \infty & 4 & \infty & 3 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array}$$

$$\gamma(S,T) = (18, 23, 37, 45)$$

Cycle:





$M_{\{1,2\}\{1,3,8\}}$ and $M_{\{3,4\}\{2,5,7\}}$ are pseudomanifolds. So top homology is cyclic. Let

$$\begin{aligned}\tilde{H}_1(M_{\{1,2\}\{1,3,8\}}) &= \langle \alpha \rangle \\ \tilde{H}_1(M_{\{3,4\}\{2,5,7\}}) &= \langle \beta \rangle\end{aligned}$$

Define

$$\rho(S,T) = \alpha * \beta \in \tilde{H}_3(M_{4,8})$$

Shareshian & MW:

- $\{\rho(S,T) \mid (S,T)$ is a Friedman-Hanlon pair$\}$ is a basis for $\tilde{H}_{m-1}(M_{m,n})$
- $\{\gamma(S,T) \mid (S,T)$ is a Friedman-Hanlon pair$\}$ is a basis for the free part of $\tilde{H}^{m-1}(M_{m,n})$

Idea of Proof: We order the pairs of standard tableaux

$$(S_1, T_1), \ldots, (S_k, T_k)$$

so that the matrix

$$(\langle \rho(S_i, T_i), \gamma(S_j, T_j) \rangle)_{i,j=1,\ldots,t}$$

is triangular with 1's on the diagonal. From this we can establish the result.