

Eulerian quasisymmetric functions and poset topology, Part I

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arXiv:0805.2416

POSET TOPOLOGY \implies ENUMERATIVE COMBINATORICS

Björner-Welker

Rees product of posets

q -Eulerian polynomials

Eulerian Permutation Statistics

For σ in symmetric group \mathfrak{S}_n

Descent set: $\text{DES}(\sigma) := \{i \in [n] : \sigma(i) > \sigma(i+1)\}$

Excedance set: $\text{EXC}(\sigma) := \{i \in [n] : \sigma(i) > i\}$

$$\text{des}(\sigma) := |\text{DES}(\sigma)|$$

$$\text{exc}(\sigma) := |\text{EXC}(\sigma)|$$

$$\sigma = 3.25.4.1$$

$$\text{DES}(\sigma) = \{1, 3, 4\}$$

$$\text{des}(\sigma) = 3$$

$$\sigma = 32541$$

$$\text{EXC}(\sigma) = \{1, 3\}$$

$$\text{exc}(\sigma) = 2$$

Eulerian Permutation Statistics

Eulerian polynomial

$$A_n(t) := \sum_{\sigma \in \mathfrak{S}_n} t^{\text{des}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{exc}(\sigma)}$$

\mathfrak{S}_3	des	exc
123	0	0
132	1	1
213	1	1
231	1	2
312	1	1
321	2	1

$$A_3(t) = 1 + 4t + t^2$$

Eulerian Permutation Statistics

Eulerian polynomial

$$A_n(t) := \sum_{\sigma \in \mathfrak{S}_n} t^{\text{des}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{exc}(\sigma)}$$

Exponential generating function:

$$\sum_{n \geq 0} A_n(t) \frac{z^n}{n!} = \frac{1-t}{e^{z(t-1)} - t}$$

Mahonian Permutation Statistics

$$\text{inv}(\sigma) := |\{(i, j) : 1 \leq i < j \leq n \ \& \ \sigma(i) > \sigma(j)\}|$$

$$\text{maj}(\sigma) := \sum_{i \in \text{DES}(\sigma)} i$$

$$\text{inv}(32541) = 6$$

$$\text{maj}(32541) = \text{maj}(3.25.4.1) = 1 + 3 + 4 = 8$$

Mahonian Permutation Statistics

q -analogs

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\text{inv}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}(\sigma)} = [n]_q!$$

where $[n]_q := 1 + q + \dots + q^{n-1}$ and $[n]_q! := [n]_q [n-1]_q \dots [1]_q$

\mathfrak{S}_3	inv	maj
123	0	0
132	1	2
213	1	1
231	2	2
312	2	1
321	3	3

$$1 + 2q + 2q^2 + q^3 = (1 + q + q^2)(1 + q)$$

q-Eulerian polynomials

$$A_n^{\text{inv,des}}(q, t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\text{inv}(\sigma)} t^{\text{des}(\sigma)}$$

$$A_n^{\text{maj,des}}(q, t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}(\sigma)} t^{\text{des}(\sigma)}$$

$$A_n^{\text{inv,exc}}(q, t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\text{inv}(\sigma)} t^{\text{exc}(\sigma)}$$

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q-Eulerian polynomials

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$$A_n^{\text{maj,exc}}(q, t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}(\sigma)} t^{\text{exc}(\sigma)}$$

Theorem (Stanley 1976)

$$\sum_{n \geq 0} A_n^{\text{inv,des}}(q, t) \frac{z^n}{[n]_q!} = \frac{1-t}{\text{Exp}_q(z(t-1)) - t}$$

where

$$\text{Exp}_q(z) := \sum_{n \geq 0} \frac{q^{\binom{n}{2}} z^n}{[n]_q!}$$

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Theorem (Shareshian & MW 2006)

$$\sum_{n \geq 0} A_n^{\text{maj,exc}}(q, t) \frac{z^n}{[n]_q!} = \frac{(1-tq) \exp_q(z)}{\exp_q(z tq) - tq \exp_q(z)}$$

where

$$\exp_q(z) := \sum_{n \geq 0} \frac{z^n}{[n]_q!}$$

Symmetric Function Generalization

$$\sum_{n \geq 0} A_n^{\text{maj,exc}}(q, t) \frac{z^n}{[n]_q!} = \frac{(1 - tq) \exp_q(z)}{\exp_q(z tq) - tq \exp_q(z)}$$

Symmetric Function Generalization

$H(z) := \sum_{n \geq 0} h_n z^n$, where h_n is the n th complete homogeneous symmetric function in x_1, x_2, \dots .

$$\frac{(1-t)H(z)}{H(zt) - tH(z)}$$

$$\begin{array}{l} \downarrow \\ x_i := q^{i-1} \\ z := z(1-q) \\ t := tq \end{array}$$

$$\sum_{n \geq 0} A_n^{\text{maj,exc}}(q, t) \frac{z^n}{[n]_q!} = \frac{(1-tq) \exp_q(z)}{\exp_q(ztq) - tq \exp_q(z)}$$

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$$\begin{array}{ccc} ? & = & \frac{(1-t)H(z)}{H(zt) - tH(z)} \\ \downarrow \begin{array}{l} x_i := q^{i-1} \\ z := z(1-q) \\ t := qt \end{array} & & \downarrow \begin{array}{l} x_i := q^{i-1} \\ z := z(1-q) \\ t := tq \end{array} \\ \sum_{n \geq 0} A_n^{\text{maj, exc}}(q, t) \frac{z^n}{[n]_q!} & = & \frac{(1-tq) \exp_q(z)}{\exp_q(ztq) - tq \exp_q(z)} \end{array}$$

Symmetric Function Generalization

For $\sigma \in \mathfrak{S}_n$, let $\bar{\sigma}$ be obtained by placing bars above each **excedance**.

$$\bar{5}3\bar{1}4\bar{6}2$$

View $\bar{\sigma}$ as a word over ordered alphabet

$$\{\bar{1} < \bar{2} < \dots < \bar{n} < 1 < 2 < \dots < n\}.$$

Define

$$\text{DEX}(\sigma) := \text{DES}(\bar{\sigma})$$

$$\text{DEX}(531462) = \text{DES}(\bar{5}.3\bar{1}4.\bar{6}2) = \{1, 4\}$$

$$\sum_{i \in \text{DEX}(\sigma)} i = \text{maj}(\sigma) - \text{exc}(\sigma)$$

Symmetric Function Generalization

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Symmetric Function Generalization

For $T \subseteq [n - 1]$, quasisymmetric function

$$F_T(x_1, x_2, \dots) := \sum_{\substack{s_1 \geq \dots \geq s_n \\ i \in T \Rightarrow s_i > s_{i+1}}} x_{s_1} \dots x_{s_n}$$

From theory of quasisymmetric functions we have

$$F_T(1, q, q^2, \dots) = \frac{q^{\sum T}}{(1 - q)(1 - q^2) \dots (1 - q^n)}$$

Hence

$$F_{\text{DEX}(\sigma)}(1, q, q^2, \dots) = \frac{q^{\text{maj}(\sigma) - \text{exc}(\sigma)}}{(1 - q)(1 - q^2) \dots (1 - q^n)}$$

Symmetric Function Generalization

By setting $x_i := q^{i-1}$ and $z := z(1 - q)$ in

$$\sum_{n \geq 0} \sum_{\sigma \in \mathfrak{S}_n} F_{\text{DEX}(\sigma)} t^{\text{exc}(\sigma)} z^n$$

we get

$$\sum_{n \geq 0} \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}(\sigma) - \text{exc}(\sigma)} t^{\text{exc}(\sigma)} \frac{z^n}{[n]_q!}$$

Now set $t := qt$ to get

$$\sum_{n \geq 0} A_n^{\text{maj,exc}}(q, t) \frac{z^n}{[n]_q!}$$

Symmetric Function Generalization

$H(z) := \sum_{n \geq 0} h_n z^n$, where h_n is the n th complete homogeneous symmetric function in x_1, x_2, \dots

$$\sum_{n \geq 0} \sum_{\sigma \in \mathfrak{S}_n} F_{\text{DEX}(\sigma)} t^{\text{exc}(\sigma)} r^{\text{fix}(\sigma)} z^n = \frac{(1-t)H(rz)}{H(zt) - tH(z)}$$

$$\begin{array}{l} \downarrow \\ x_i := q^{i-1} \\ z := z(1-q) \\ t := qt \end{array}$$

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$$\sum_{n \geq 0} A_n^{\text{maj,exc,fix}}(q, t, r) \frac{z^n}{[n]_q!} = \frac{(1-tq) \exp_q(rz)}{\exp_q(ztq) - tq \exp_q(z)}$$

Symmetric Function Generalization

Theorem (Shareshian and MW, 2006)

Let

$$Q_{n,j,k} := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{exc}(\sigma) = j \\ \text{fix}(\sigma) = k}} F_{\text{DEX}(\sigma)}$$

Then

$$\sum_{n \geq 0} \sum_{j=0}^{n-1} \sum_{k=0}^n Q_{n,j,k} t^j r^k z^n = \frac{(1-t)H(rz)}{H(zt) - tH(z)}$$

Another specialization of $\sum Q_{n,j,k} t^j r^k z^n = \frac{(1-t)H(rz)}{H(zt)-tH(z)}$

Two ways to specialize the quasisymmetric functions:

$$F_T(1, q, q^2, \dots) = \frac{q^{\sum T}}{(q; q)_n}$$

$$\sum_{m \geq 0} F_T(1, q, \dots, q^{m-1}) p^m = \frac{p^{|T|+1} q^{\sum T}}{(p; q)_{n+1}}.$$

where

$$(a; q)_n := \begin{cases} 1 & \text{if } n = 0 \\ (1-a)(1-aq) \cdots (1-aq^{n-1}) & \text{if } n \geq 1. \end{cases}$$

Another specialization of $\sum Q_{n,j,k} t^j r^k z^n = \frac{(1-t)H(rz)}{H(zt)-tH(z)}$

First specialization: $\sum \text{DEX}(\sigma) = \text{maj}(\sigma) - \text{exc}(\sigma)$.

Theorem (Shareshian and MW, 2006)

$$\sum_{n \geq 0} A_n^{\text{maj,exc,fix}}(q, t, r) \frac{z^n}{[n]_q!} = \frac{(1-tq) \exp_q(rz)}{\exp_q(ztq) - tq \exp_q(z)}.$$

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Second specialization:

$$|\text{DEX}(\sigma)| = \begin{cases} \text{des}(\sigma) & \text{if } \sigma(1) = 1 \\ \text{des}(\sigma) - 1 & \text{otherwise} \end{cases}$$

Another specialization of $\sum Q_{n,j,k} t^j r^k z^n = \frac{(1-t)H(rz)}{H(zt)-tH(z)}$

First specialization: $\sum \text{DEX}(\sigma) = \text{maj}(\sigma) - \text{exc}(\sigma)$.

Theorem (Shareshian and MW, 2006)

$$\sum_{n \geq 0} A_n^{\text{maj,exc,fix}}(q, t, r) \frac{z^n}{[n]_q!} = \frac{(1-tq) \exp_q(rz)}{\exp_q(ztq) - tq \exp_q(z)}.$$

Second specialization:

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Theorem (Foata-Han (2007), Gessel-Reutenauer (1993) t=1)

$$\begin{aligned} \sum_{n \geq 0} A_n^{\text{maj,des,exc,fix}}(q, p, t, r) \frac{z^n}{(p; q)_{n+1}} &= \\ &= \sum_{m \geq 0} p^m \frac{(1-tq)(z; q)_m (ztq; q)_m}{((z; q)_m - tq(ztq; q)_m)(zr; q)_{m+1}}, \end{aligned}$$

Steps of the Proof of $\sum Q_{n,j,k} t^j r^k z^n = \frac{(1-t)H(rz)}{H(zt)-tH(z)}$

1. Modification of bijection of Gessel and Reutenauer, which takes **compatible pairs** to **ornaments**, is used to give alternative characterization of $Q_{n,j,k}$
2. Bijection from **ornaments** to **banners**, using Lyndon decomposition, is used to give another characterization of $Q_{n,j,k}$
3. Generalization of bijection of Stembridge is used to show $Q_{n,j,k}$ satisfies **recurrence relation**, which yields generating function.

Steps of the Proof of $\sum Q_{n,j,k} t^j r^k z^n = \frac{(1-t)H(rz)}{H(zt)-tH(z)}$

- \implies
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Step 1 - Bicolored necklaces and ornaments

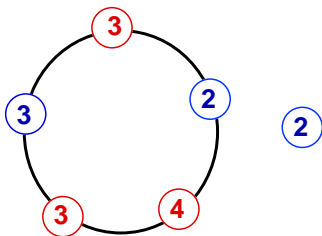
A **bicolored necklace** is a primitive circular word over alphabet

$$\{1, 1, 2, 2, \dots\}$$

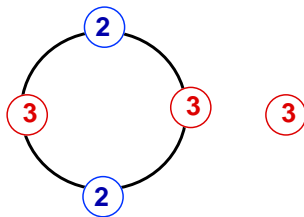
such that if size > 1

- a **blue** letter is followed by letter greater than or equal in value
- a **red** letter is followed by a letter less than or equal in value

Necklaces of size 1 are **blue**.



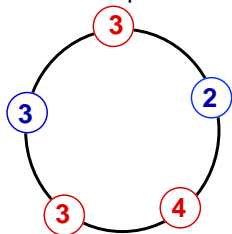
necklaces



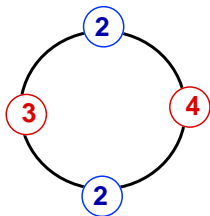
not necklaces

Step 1 - Bicolored necklaces and ornaments

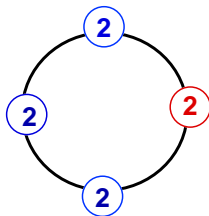
An **ornament** of type λ is a multiset of necklaces whose necklace sizes form partition λ



type = (5, 4, 4)



weight = $x_2^7 x_3^4 x_4^2$



Let $\mathcal{R}_{\lambda,j}$ = set of ornaments of type λ with j red letters.

Step 1 - Bicolored necklaces and ornaments

Theorem (Gessel and Reutenauer (1993))

For $\lambda \vdash n$, define

$$Q_\lambda := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \lambda(\sigma) = \lambda}} F_{\text{DES}}(\sigma)$$

Then

$$Q_\lambda = \sum_{R \in \mathcal{R}_\lambda} \text{wt}(R)$$

Step 1 - Bicolored necklaces and ornaments

Theorem (Shareshian and MW (2006))

For $\lambda \vdash n$, define

$$Q_{\lambda,j} := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \lambda(\sigma) = \lambda \\ \text{exc}(\sigma) = j}} F_{\text{DEX}(\sigma)}$$

Then

$$Q_{\lambda,j} = \sum_{R \in \mathcal{R}_{\lambda,j}} \text{wt}(R)$$

Step 1 - Bicolored necklaces and ornaments

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Then

$$Q_{\lambda,j} = \sum_{R \in \mathcal{R}_{\lambda,j}} \text{wt}(R)$$

$$F_{\text{DEX}(\sigma)} = \sum_{\substack{s_1 \geq \dots \geq s_n \\ i \in \text{DEX}(\sigma) \Rightarrow s_i > s_{i+1}}} X_{s_1} \dots X_{s_n}$$

Step 1 - Bicolored necklaces and ornaments

For $\lambda \vdash n$ and $j = 0, \dots, n-1$, let

$\text{Com}_{\lambda,j} := \{(\sigma, s) : \sigma \in \mathfrak{S}_n, \lambda(\sigma) = \lambda, \text{exc}(\sigma) = j, (\sigma, s) \text{ is compat.}\}$

Bijection $\phi : \text{Com}_{\lambda,j} \rightarrow \mathcal{R}_{\lambda,j}$

Let $\sigma = 45162387$ and $s = (7, 7, 7, 5, 5, 4, 2, 2)$.

- 1 write σ in cycle form,

$$\sigma = (1, 4, 6, 3)(2, 5), (7, 8).$$

- 2 color letters that are followed (cyclicly) by larger letters **red** and letters that are singletons or are followed by smaller letters **blue**,

$$(1, 4, 6, 3)(2, 5)(7, 8).$$

- 3 replace each i by s_i , we have the ornament

$$(7, 5, 4, 7)(7, 5)(2, 2).$$

Step 1 - Bicolored necklaces and ornaments

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Steps of the Proof of $\sum Q_{n,j,k} t^j r^k z^n = \frac{(1-t)H(rz)}{H(zt)-tH(z)}$

1. Modification of bijection of Gessel and Reutenauer which takes **compatible pairs** to **ornaments** is used to give alternative characterization of $Q_{n,j,k}$
- \implies 2. Bijection from **ornaments** to **banners** using Lyndon decomposition is used to give another alternative characterization of $Q_{n,j,k}$
3. Generalization of bijection of Stembridge is used to show $Q_{n,j,k}$ satisfies **recurrence relation**, which yields generating function.

Step 2 - Banners

A *banner* is a word over alphabet

$$\{1, 1, 2, 2, \dots\}$$

such that

- blue letter is followed by letter greater than or equal in value or is last
- red letter is followed by a letter less than or equal in value

Example:

22757547

Step 2 - Banners

A **Lyndon word** over an ordered alphabet is a word that is strictly lexicographically larger than all its circular rearrangements.

The **Lyndon factorization** of a word over an ordered alphabet is a factorization into a weakly lexicographically increasing sequence of Lyndon words.

The **Lyndon type** $\lambda(w)$ of a word w is the partition whose parts are the lengths of the words in its Lyndon factorization.

Use the ordering $1 < 1 < 2 < 2 < \dots$ for our alphabet.
Example:

$$\lambda(22757547) = \lambda(22 \cdot 75 \cdot 7547) = 4, 2, 2$$

Step 2 - Banners

Theorem (Shareshian and MW)

Let $\mathcal{B}_{\lambda,j}$ = the set of banners of Lyndon type λ with j red letters.
Then there is a weight-preserving bijection

$$\mathcal{B}_{\lambda,j} \rightarrow \mathcal{R}_{\lambda,j}.$$

Consequently

$$Q_{\lambda,j} = \sum_{b \in \mathcal{B}_{\lambda,j}} \text{wt}(b)$$

$$22757547 \mapsto 22 \cdot 75 \cdot 7547 \mapsto (2, 2)(7, 5)(7, 5, 4, 7)$$

Steps of the Proof of $\sum Q_{n,j,k} t^j r^k z^n = \frac{(1-t)H(rz)}{H(zt)-tH(z)}$

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Step 3 - Recurrence Relation

Using ornaments one can easily show that the formula

$$\sum_{n \geq 0} \sum_{j=0}^{n-1} \sum_{k=0}^n Q_{n,j,k} t^j r^k z^n = \frac{(1-t)H(rz)}{H(zt) - tH(z)}$$

is equivalent to

$$\sum_{n \geq 0} \sum_{j=0}^{n-1} Q_{n,j,0} t^j z^n = \frac{1-t}{H(zt) - tH(z)}$$

which is equivalent to recurrence relation

$$Q_{n,j,0} = \sum_{\substack{0 \leq m \leq n-2 \\ j+m-n < i < j}} Q_{m,i,0} h_{n-m}$$

Step 3 - Recurrence Relation

$$Q_{n,j,0} = \sum_{\substack{0 \leq m \leq n-2 \\ j+m-n < i < j}} Q_{m,i,0} h_{n-m}$$

Let \mathcal{B}_n^0 be the set of banners of length n whose Lyndon type has no parts of size 1.

Let \mathcal{M}_n be the set of pairs (ω, i) , such that ω is a weakly increasing sequence of n positive integers and $i \in [n-1]$.

Lemma (Shareshian and MW)

There is a bijection

$$\gamma : \mathcal{B}_n^0 \rightarrow \bigcup_{0 \leq m \leq n-2} \mathcal{B}_m^0 \times \mathcal{M}_{n-m},$$

such that if $\gamma(b) = (b', i)$ then $wt(b) = wt(b')wt(\omega)$ and $bar(b) = bar(b') + i$.

Symmetry and unimodality of Eulerian polynomials

$$A_3(t) = 1 + 4t + t^2$$

$$A_4(t) = 1 + 11t + 11t^2 + t^3$$

$$A_5(t) = 1 + 26t + 66t^2 + 26t^3 + t^4$$

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$$A_3^{\text{maj,exc}}(q, q^{-1}t) = 1 + (2 + q + q^2)t + t^2$$

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$$A_5^{\text{maj,exc}}(q, q^{-1}t) = 1 + a(q)t^2 + b(q)t^3 + a(q)t^3 + t^4$$

$$a(q) < b(q)$$

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$$a(q) < b(q)$$

Theorem (Shareshian & MW)

$A_n^{\text{maj,exc}}(q, q^{-1}t)$ is a symmetric & unimodal polynomial in t .

Symmetry and unimodality of Eulerian polynomials

Let

$$A_{\lambda}^{\text{maj,des,exc}}(q, p, t) := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \lambda(\sigma) = \lambda}} q^{\text{maj}(\sigma)} p^{\text{des}(\sigma)} t^{\text{exc}(\sigma)}$$

Theorem (Shareshian & MW)

$A_{\lambda}^{\text{maj,des,exc}}(q, p, q^{-1}t)$ is a symmetric polynomial in t .

Conjecture

$A_{\lambda}^{\text{maj,des,exc}}(q, p, q^{-1}t)$ is a unimodal polynomial in t .

Eulerian quasisymmetric functions

$$Q_{n,j} := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{exc}(\sigma) = j}} F_{\text{DEX}(\sigma)}$$

$$Q_{n,j,k} := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{exc}(\sigma) = j \\ \text{fix}(\sigma) = k}} F_{\text{DEX}(\sigma)}$$

$$Q_{\lambda,j} := \sum_{\substack{\lambda(\sigma) = \lambda \\ \text{exc}(\sigma) = j}} F_{\text{DEX}(\sigma)}$$

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Theorem (Shareshian and MW)

$Q_{\lambda,j}$ is a symmetric function for all $\lambda \vdash n$

Eulerian quasisymmetric functions

Theorem (Shareshian and MW)

- $Q_{n,j} = Q_{n,n-1-j}$
- $Q_{n,j,k} = Q_{n,n-k-j,k}$
- $Q_{\lambda,j} = Q_{\lambda,n-k-j}$, where λ has k parts equal to 1.
- $Q_{n,j}$ and $Q_{n,j,k}$ are *h-positive*
- $Q_{n,j} - Q_{n,j-1}$ is *h-positive* for all $j = 1, \dots, \lfloor \frac{n-1}{2} \rfloor$.
- $Q_{n,j,k} - Q_{n,k,j-1}$ is *h-positive* for all $j = 1, \dots, \lfloor \frac{n-k}{2} \rfloor$.

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$Q_{(6),3} = 2h_{(4,2)} - h_{(4,1,1)} + h_{(3,2,1)} + h_{(5,1)}$ *not h-positive*.

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$$Q_{(6),3} = 3s_{(6)} + 3s_{(5,1)} + 3s_{(4,2)} + s_{(3,3)} + s_{3,2,1}$$

Eulerian quasisymmetric functions

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$Q_{(6),3} = 2h_{(4,2)} - h_{(4,1,1)} + h_{(3,2,1)} + h_{(5,1)}$ *not h-positive*.

$$Q_{(6),3} = 3s_{(6)} + 3s_{(5,1)} + 3s_{(4,2)} + s_{(3,3)} + s_{3,2,1}$$

Conjecture

For all λ and j , $Q_{\lambda,j}$ is Schur positive. Moreover $Q_{\lambda,j} - Q_{\lambda,j-1}$ is *Schur-positive* for all $j = 1, \dots, \lfloor \frac{n-k}{2} \rfloor$.

Occurrences of $Q_{n,j,0}$ and $Q_{n,j}$

- [MacMahon \(1915\)](#), [Askey-Ismael\(1976\)](#): Enumerator for multiset derangements with j excedences
- [Carlitz-Scoville-Vaughan \(1976\)](#), [Dollhopf-Goulden-Greene \(2006\)](#), [Stanley \(1995, 2006\)](#): Enumerator for words with no adjacent repeats, with j descents
- [Procesi, Stanley \(1989\)](#): Frobenius characteristic of representation of symmetric group on degree $2j$ cohomology of the toric variety associated with Coxeter complex of type A
- [Shareshian-MW \(2005\)](#): Frobenius characteristic of representation of symmetric group on top homology of the j th interval of the Rees product of truncated Boolean algebra and chain.

Connection with Toric Varieties

Let X_n be the toric variety associated with the Coxeter complex of \mathfrak{S}_n . The action of \mathfrak{S}_n on X_n induces a representation of \mathfrak{S}_n on $H^{2j}(X_n)$.

Theorem (Procesi, Stanley 1989)

$$\sum_{n \geq 0} \sum_{j=0}^{n-1} \text{ch} H^{2j}(X_n) t^j z^n = \frac{(1-t)H(z)}{H(zt) - tH(z)},$$

where ch is the Frobenius characteristic,

Theorem (Shareshian and MW)

$$\text{ch} H^{2j}(X_n) = Q_{n,j} := \sum_{k=0}^n Q_{n,j,k}$$

Representation on Rees product

Action of \mathfrak{S}_n on boolean algebra B_n induces an action of \mathfrak{S}_n on maximal open intervals $I_{n,j} := (\hat{0}, ([n], j))$ of Rees product $(B_n - \{\hat{0}\}) * C_n$, which induces a representation of \mathfrak{S}_n on $\tilde{H}_{n-2}(I_{n,j})$

Theorem (Shareshian and MW)

$$1 + \sum_{n \geq 1} \sum_{j=0}^{n-1} \text{ch} \tilde{H}_{n-2}(I_{n,j+1}) t^j z^n = \frac{(1-t)E(z)}{E(zt) - tE(z)},$$

where $E(z) = \sum_{n \geq 0} e_n z^n$ and e_n is the n th elementary symmetric function.

Corollary

$$\text{ch} \tilde{H}_{n-2}(I_{n,j+1}) = \omega Q_{n,j}$$

and

$$H^{2j}(X_n) \cong_{\mathfrak{S}_n} \tilde{H}_{n-2}(I_{n,j+1}) \otimes \text{sgn}$$