

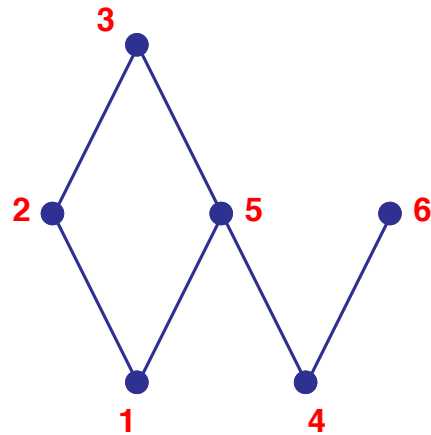
Aspects of Poset Topology

Michelle Wachs

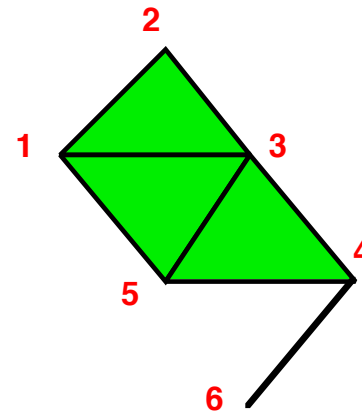
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poset \longrightarrow simplicial complex

Order complex $\Delta(P)$ of a poset P is the simplicial complex whose faces are the chains of P .



P



$\Delta(P)$

Part I. Applications and Connections

- enumerative combinatorics
- hyperplane & subspace arrangements
- complexity theory
- group theory
- Lie algebras
- knot theory
- commutative algebra
- algebraic geometry
- homotopy theory

Enumerative combinatorics

Möbius function: $\mu : P \times P \rightarrow \mathbb{Z}$ used in Möbius inversion.

For all $x < y$ in P ,

$$\mu(x, y) = \tilde{\chi}(\Delta(x, y))$$

Interval $(x, y) := \{z \in P : x < z < y\}$.

Reduced Euler characteristic

$$\tilde{\chi}(\Delta) := \sum_{i=-1}^{\dim(\Delta)} (-1)^i f_i(\Delta) = \sum_{i=-1}^{\dim(\Delta)} (-1)^i \tilde{\beta}_i(\Delta)$$

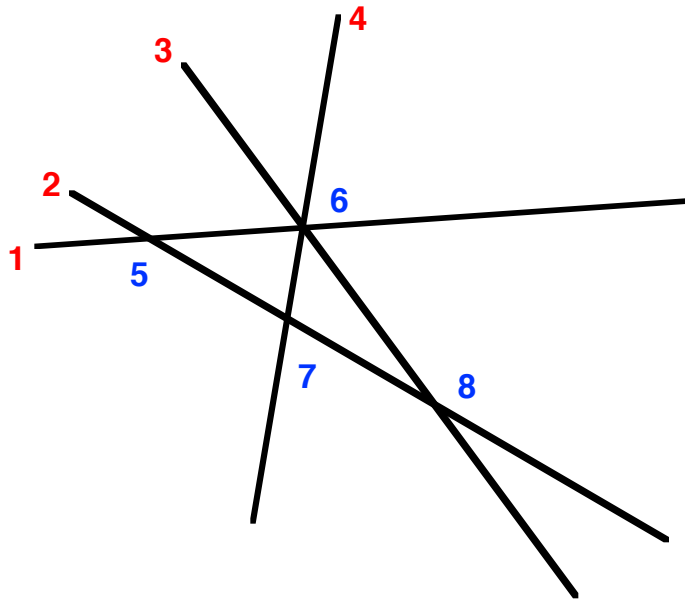
$f_i = \#$ i -dimensional faces of Δ

$$\tilde{\beta}_i(\Delta) = \dim \tilde{H}_i(\Delta; \mathbb{C})$$

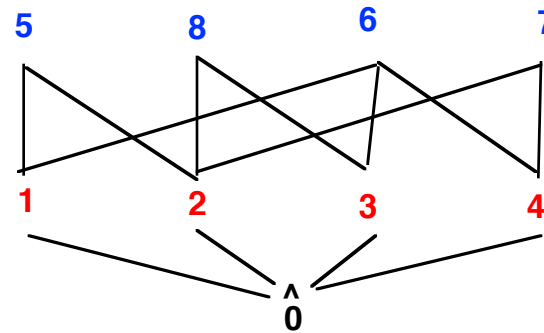
Hyperplane arrangements

Let \mathcal{A} be a finite collection of hyperplanes in \mathbb{R}^n

$L(\mathcal{A}) :=$ poset of intersections ordered by reverse inclusion



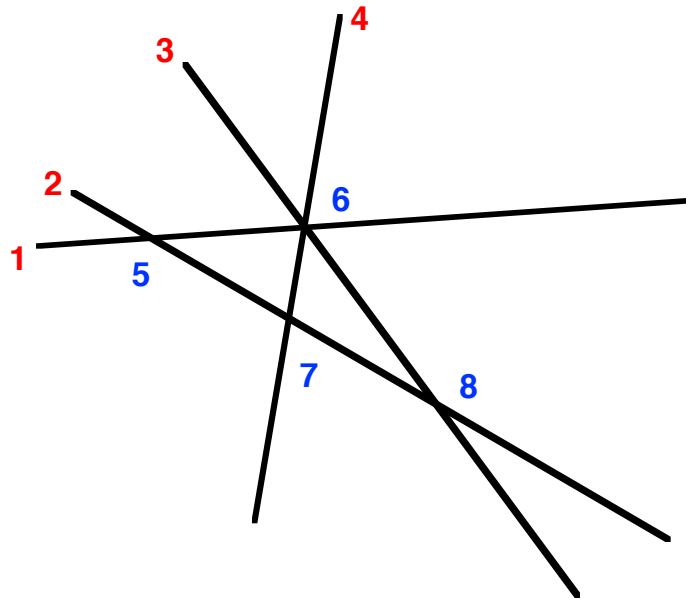
\mathcal{A}



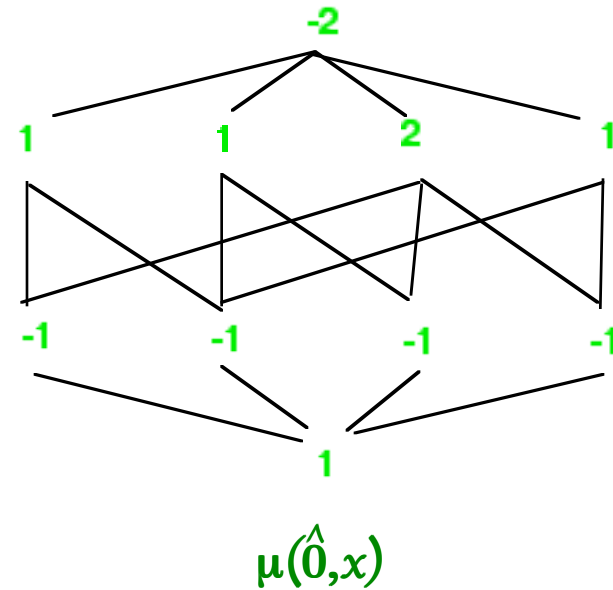
$L(\mathcal{A})$

Zaslavsky (1975): # bounded regions = $|\mu(L(\mathcal{A}) \cup \hat{1})|$

$$\# \text{ regions} = \sum_{x \in L(\mathcal{A})} |\mu(\hat{0}, x)|$$



\mathcal{A}

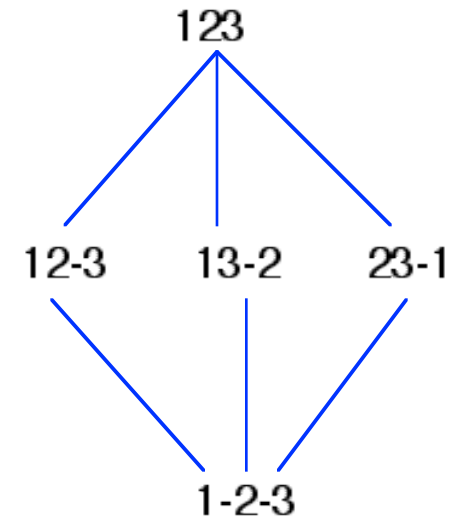
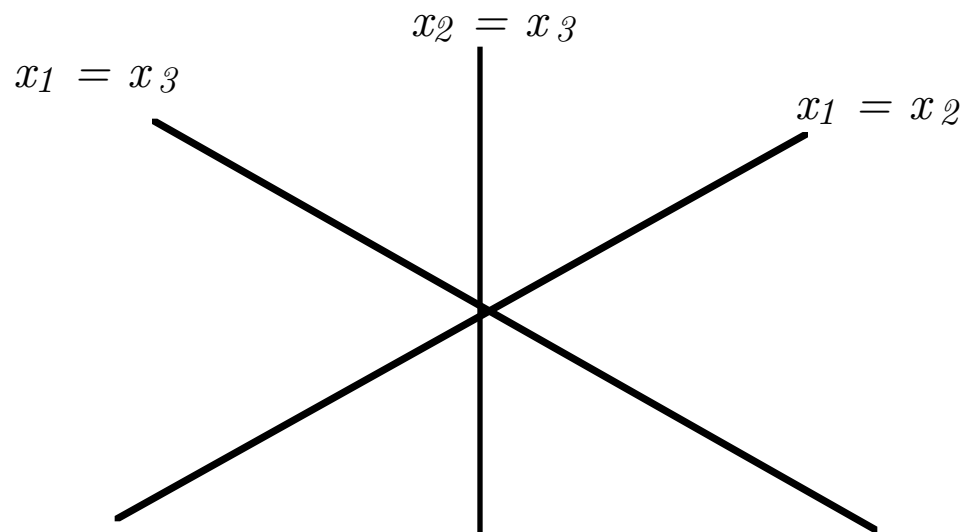


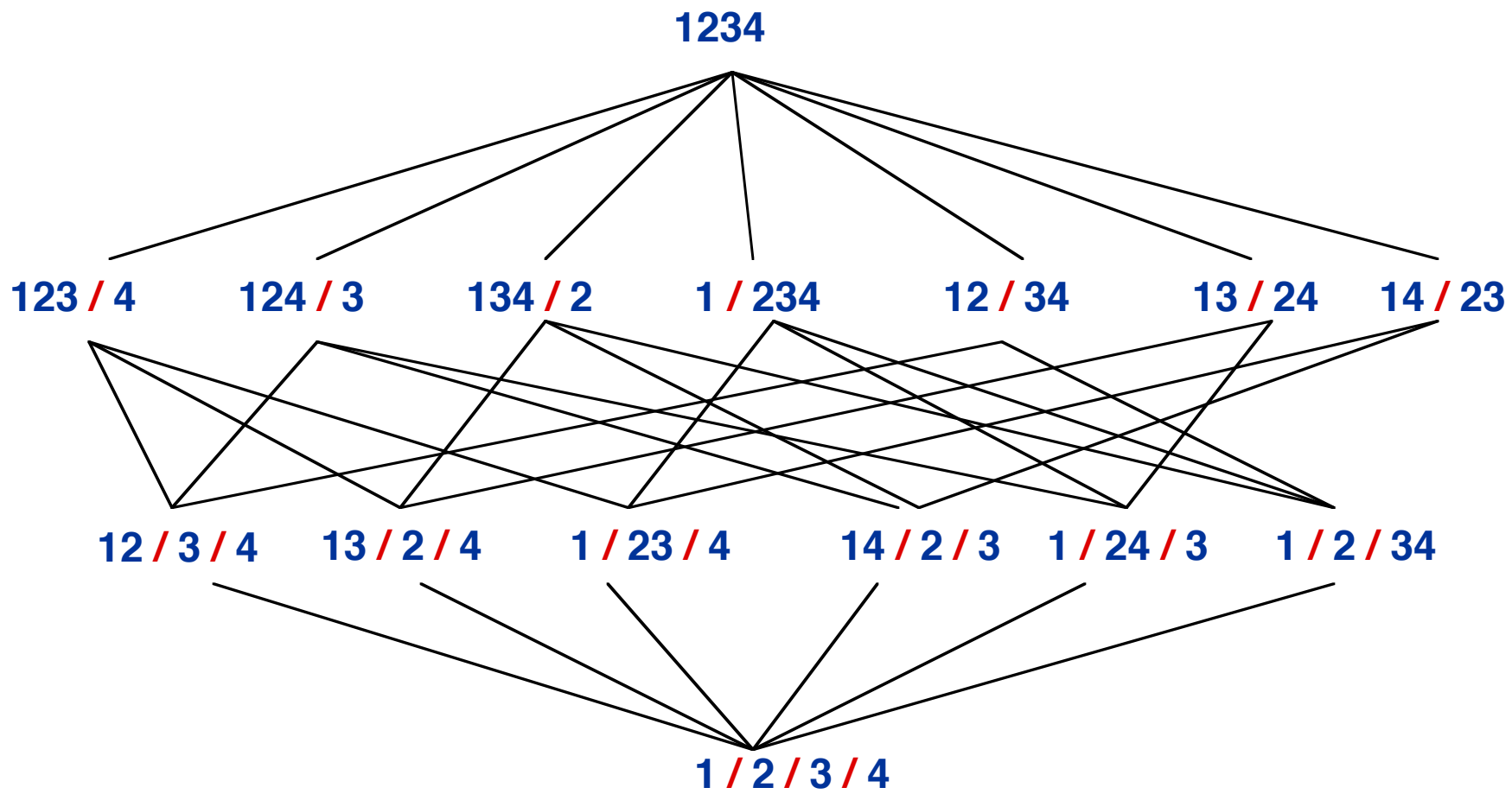
$\mu(\hat{0}, x)$

Braid Arrangement $\mathcal{A}_n := \{x_i = x_j : 1 \leq i < j \leq n\}$

$$L(\mathcal{A}_n) = \Pi_n$$

Partition Lattice $\Pi_n :=$ poset of partitions of $[n] := \{1, 2, \dots, n\}$
ordered by merging blocks





Orlik & Solomon (1980): Let \mathcal{A} be a hyperplane arrangement in \mathbb{C}^n . Complement $M_{\mathcal{A}} = \mathbb{C}^n - \cup \mathcal{A}$. Then $\forall i$, $H^i(M_{\mathcal{A}})$ is free and

$$\beta_i(M_{\mathcal{A}}) = \sum_{\substack{x \in L_{\mathcal{A}} \\ \dim^{\mathbb{C}}(x) = n - i}} |\mu(\hat{0}, x)|$$

Orlik & Solomon (1980): Let \mathcal{A} be a hyperplane arrangement in \mathbb{C}^n . Complement $M_{\mathcal{A}} = \mathbb{C}^n - \cup \mathcal{A}$. Then $\forall i$, $H^i(M_{\mathcal{A}})$ is free and

$$\beta_i(M_{\mathcal{A}}) = \sum_{\substack{x \in L_{\mathcal{A}} \\ \dim^{\mathbb{C}}(x) = n - i}} |\mu(\hat{0}, x)|$$

Goresky & MacPherson (1988): Let \mathcal{A} be a **subspace** arrangement in \mathbb{R}^n . Complement $M_{\mathcal{A}} = \mathbb{R}^n - \cup \mathcal{A}$. Then $\forall i$

$$\tilde{H}^i(M_{\mathcal{A}}) \cong \bigoplus_{x \in L_{\mathcal{A}} \setminus \{\hat{0}\}} \tilde{H}_{n - \dim x - 2 - i}(\Delta(\hat{0}, x)).$$

Complexity Theory

Given n numbers $x_1, x_2, \dots, x_n \in \mathbb{R}$, how many **comparisons** c does the best algorithm need (in the worst case) to decide whether

- k of the numbers are equal
- each element appears an even number of times
- each element appears at least k times

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Björner & Lovász (1994)

$$c \geq \sum_{x \in P} \sum_j \tilde{\beta}_j(\hat{O}, x)$$

Complexity Theory

Given n numbers $x_1, x_2, \dots, x_n \in \mathbb{R}$, how many **comparisons** c does the best algorithm need (in the worst case) to decide whether

- k of the numbers are equal $P = \Pi_n^{=k}$
- each element appears an even number of times
- each element appears at least k times

Björner & Lovász (1994)

$$\begin{aligned} c &\geq \sum_{x \in P} \sum_j \tilde{\beta}_j(\hat{0}, x) \\ &\geq n \log_3 \frac{n}{3k} \end{aligned}$$

Complexity Theory

Given n numbers $x_1, x_2, \dots, x_n \in \mathbb{R}$, how many **comparisons** c does the best algorithm need (in the worst case) to decide whether

- k of the numbers are equal $P = \Pi_n^{=k}$
- each element appears an even number of times $P = \Pi_n^{\text{even}}$
- each element appears at least k times $P = \Pi_n^{\geq k}$

Björner & Lovász (1994)

$$c \geq \sum_{x \in P} \sum_j \tilde{\beta}_j(\hat{0}, x)$$

Complexity of Graph Properties

Let p be a monotone graph property (isomorphism invariant and closed under addition or removal of edges).

Examples:

- property of having **at least 2 edges**
- property of being **connected**
- property of containing a **perfect matching**
- property of having **degree at most b**

An algorithm for deciding whether a graph G with n nodes has property p checks the entries of the adjacency matrix until a determination can be made.

p is said to be **evasive** if best algorithm needs to check all $\binom{n}{2}$ entries (in the worst case).

Evasiveness Conjecture (Karp): Every nontrivial monotone graph property is evasive.

Kahn-Saks-Sturtevant (1984) Evasiveness conjecture true when n is a prime power.

The proof makes use of topology of $\Delta(P)$, where P is the poset of nonempty graphs on node set $[n] := \{1, 2, \dots, n\}$ having monotone graph property p , ordered by inclusion of edge sets.

- $\tilde{H}_*(\Delta(P)) \neq 0 \implies p$ is evasive.
- Fixed point theorem of Oliver (1975) is used to show $\tilde{H}_*(\Delta(P)) \neq 0$ when n is a prime power.

Group theory and group cohomology

Let G be a finite group.

$\mathcal{S}_p(G) :=$ poset of nontrivial p -subgroups of G

Brown (1975): $H^*(G)_{(p)} \cong H_G^*(\Delta(\mathcal{S}_p(G)))_{(p)}$

Quillen Conjecture (1978): $\Delta(\mathcal{S}_p(G))$ is contractible $\iff G$ has a nontrivial normal p -subgroup.

Part II. Tools

- lexicographical shellability
- bases and generating sets
- discrete Morse theory
- discrete Hodge theory
- group actions on posets
- recursive techniques
- computer generated data
- poset operations and maps

Lexicographic shellability - Björner and MW

Let P be a poset with a minimum $\hat{0}$ and a maximum $\hat{1}$.

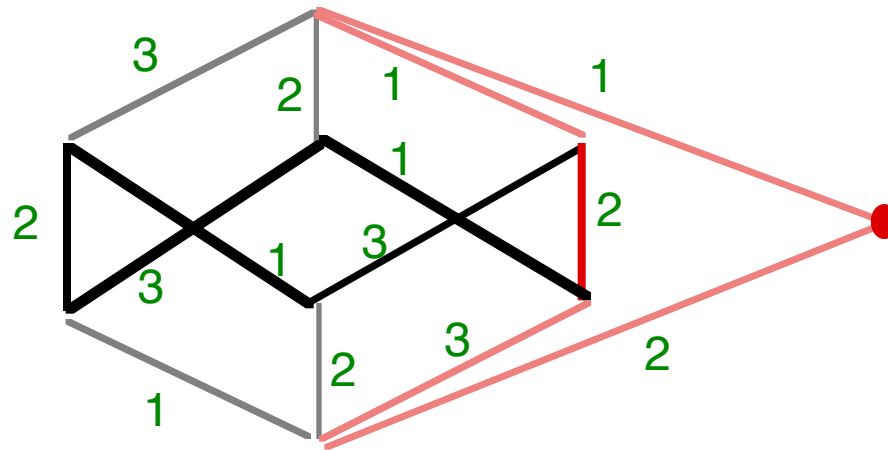
An **EL-labeling** of P is a labeling of the edges of the Hasse diagram of P so that the **lexicographically first** maximal chain of each interval is the **only increasing** maximal chain of the interval.

Björner and MW: If P has an EL -labeling and $\bar{P} = P \setminus \{\hat{0}, \hat{1}\}$ then

(ii) $\Delta(\bar{P})$ has the homotopy type of a wedge of spheres.

$\# i$ -spheres = $\#$ decreasing maximal chains of length $i + 2$

(iii) The decreasing chains form a basis for top cohomology.



$$S^1 \vee S^0$$

EL-labeling for partition lattice Π_n : If y is obtained from x by merging blocks B_1 and B_2 . Label edge $x \prec y$ with $\max B_1 \cup B_2$

Increasing chain from $\hat{0}$ to $\hat{1}$: (We write only the nonsingleton block.)

$$\hat{0} \prec \{1, 2\} \prec \{1, 2, 3\} \prec \cdots \prec \{1, 2, \dots, n\} = \hat{1}$$

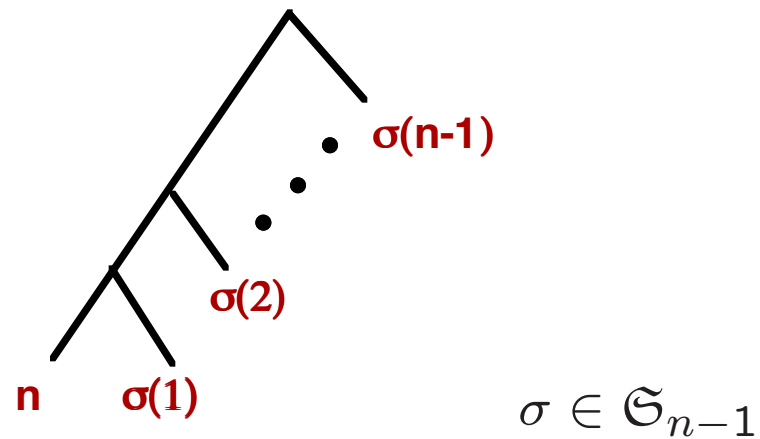
Decreasing chains from $\hat{0}$ to $\hat{1}$:

$$\hat{0} \prec \{n, \sigma(1)\} \prec \{n, \sigma(1), \sigma(2)\} \prec \cdots \prec \{n, \sigma(1), \sigma(2), \dots, \sigma(n-1)\} = \hat{1}$$

for all $\sigma \in \mathfrak{S}_{n-1}$

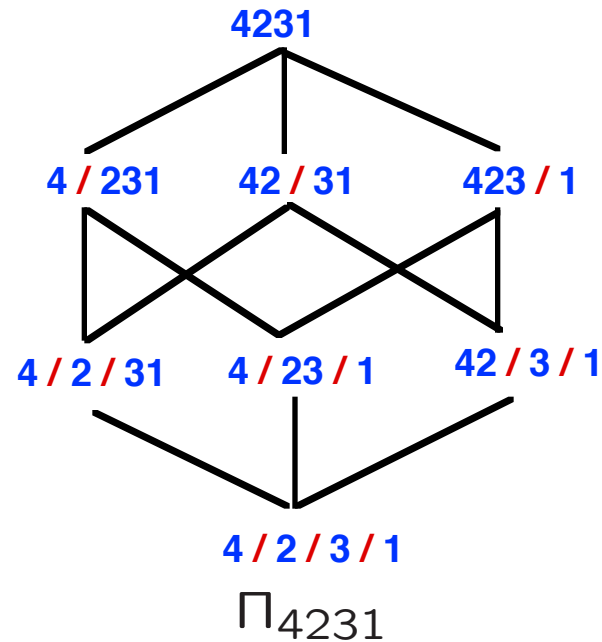
So $\Delta(\bar{\Pi}_n)$ has the homotopy type of a wedge of $(n-1)!$ $(n-3)$ -spheres.

Basis for top cohomology - **comb basis**

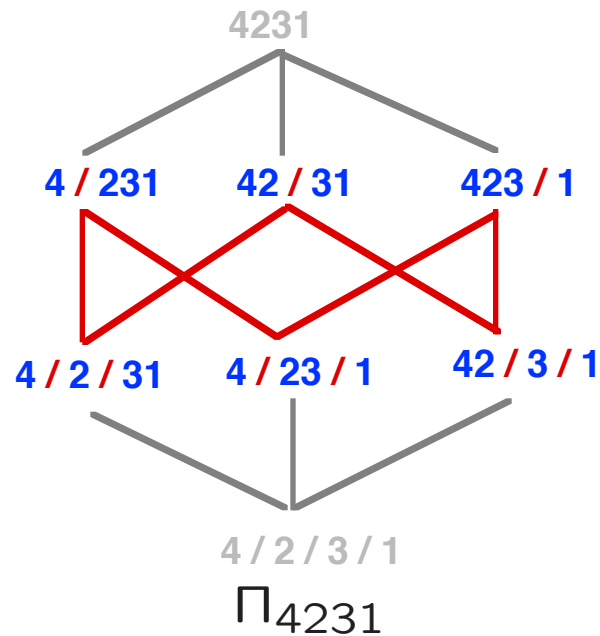


$$\{n, \sigma(1)\} \triangleleft \{n, \sigma(1), \sigma(2)\} \triangleleft \cdots \triangleleft \{n, \sigma(1), \sigma(2), \dots, \sigma(n-2)\}$$

Dual basis for homology - splitting basis. For each $\sigma \in \mathfrak{S}_n$, let Π_σ be the subposet of Π_n obtained by splitting σ in all possible ways.



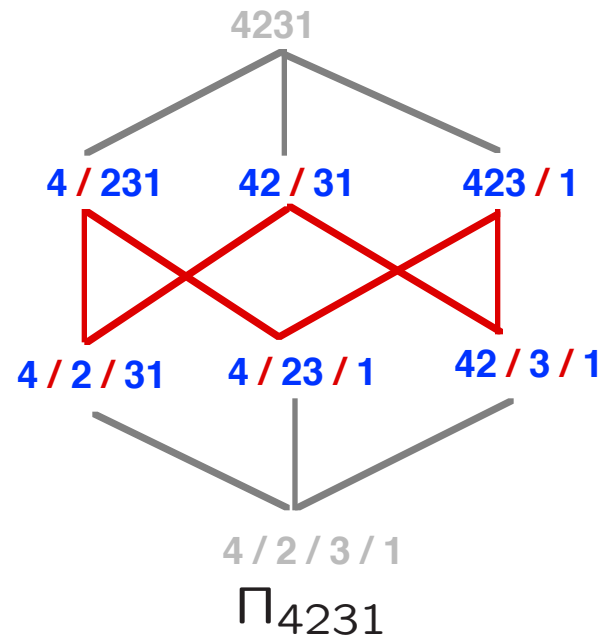
Dual basis for homology - splitting basis. For each $\sigma \in \mathfrak{S}_n$, let Π_σ be the subposet of Π_n obtained by splitting σ in all possible ways.



Π_σ is isomorphic to the lattice of subsets of $[n - 1] \Rightarrow \Delta(\bar{\Pi}_\sigma)$ is homeomorphic to an $(n - 3)$ -sphere.

Let ρ_σ be the fundamental cycle of $\Delta(\bar{\Pi}_\sigma)$

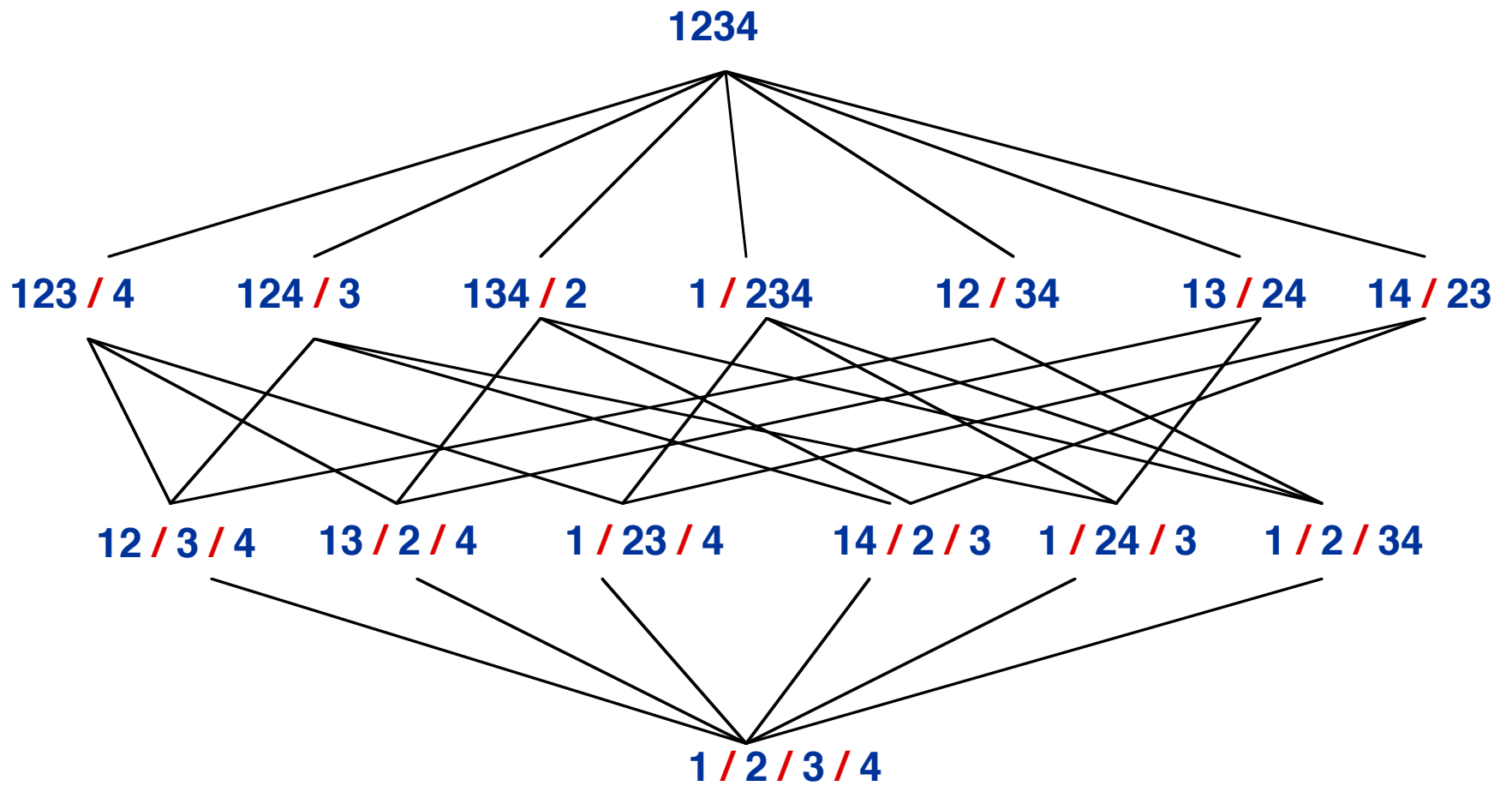
Dual basis for homology - splitting basis. For each $\sigma \in \mathfrak{S}_n$, let Π_σ be the subposet of Π_n obtained by splitting σ in all possible ways.

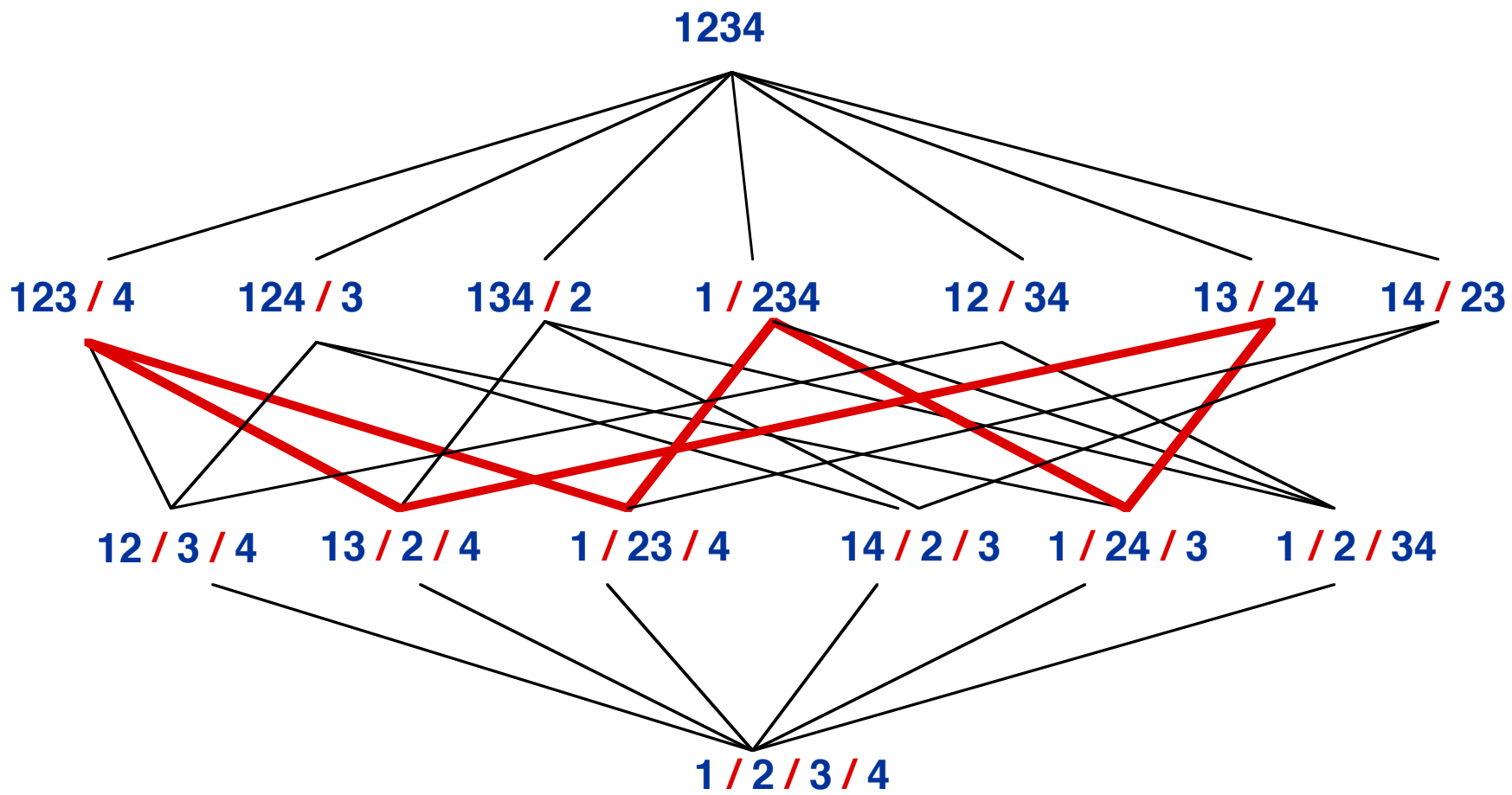


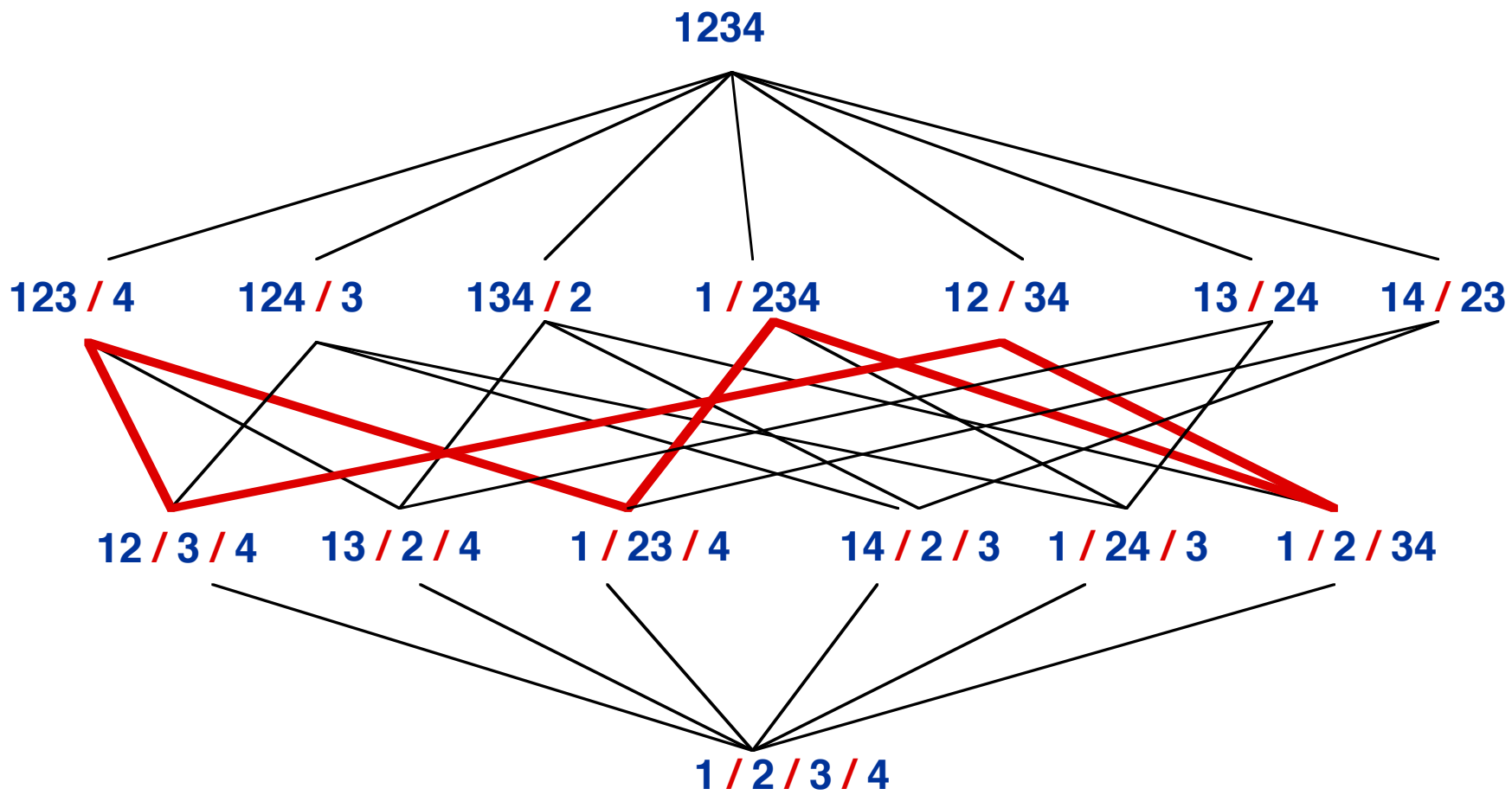
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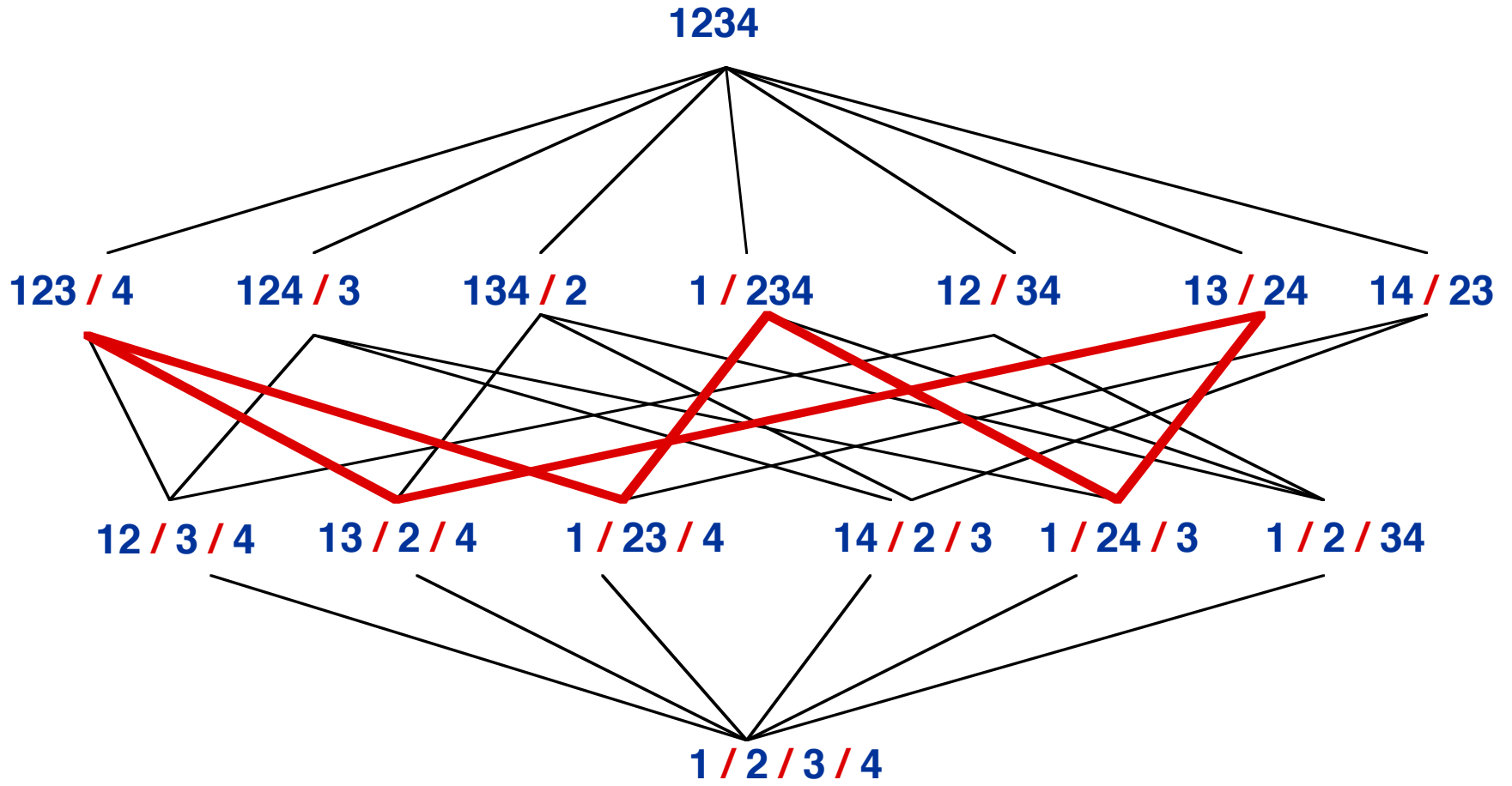
Let ρ_σ be the fundamental cycle of $\Delta(\bar{\Pi}_\sigma)$

Basis $\{\rho_\sigma : \sigma \in \mathfrak{S}_n, \sigma(1) = n\}$









Symmetric group \mathfrak{S}_n acts on partition lattice Π_n

$$(2, 3) [134 - 58 - 267] = [124 - 58 - 367]$$

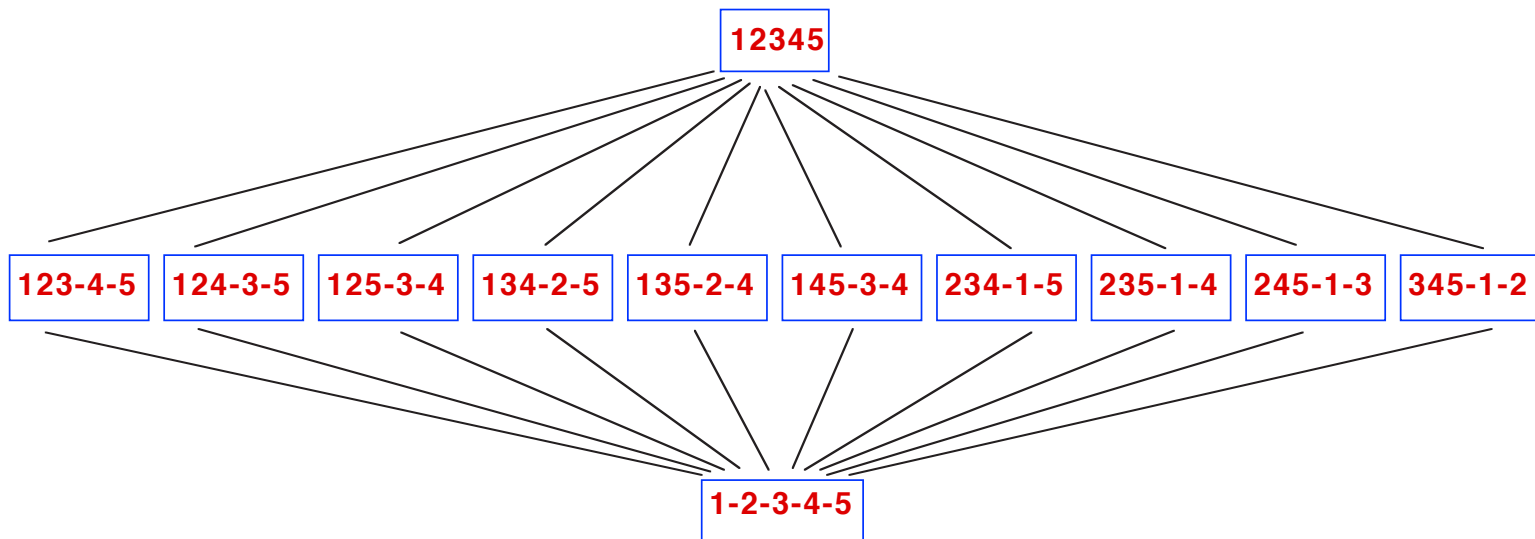
Stanley (1982) As symmetric group representations

$$\tilde{H}_{n-3}(\bar{\Pi}_n) \cong_{\mathfrak{S}_n} e^{2\pi i/n} \uparrow_{C_n}^{\mathfrak{S}_n} \otimes \text{sgn}_n$$

Isomorphic to multilinear component of free Lie algebra - Barcelo,
MW

The odd block size partition poset

Π_n^{odd} = subposet of Π_n consisting of partitions with odd block sizes.



$\Delta(\Pi_5^{\text{odd}})$ has homotopy type of a wedge of 9 0-spheres.

Björner, Stanley (1983): $\Delta(\bar{\Pi}_{2n-1}^{\text{odd}})$ has the homotopy type of a wedge of

$$(2n - 3)!!^2 := (1 \cdot 3 \cdots (2n - 3))^2$$

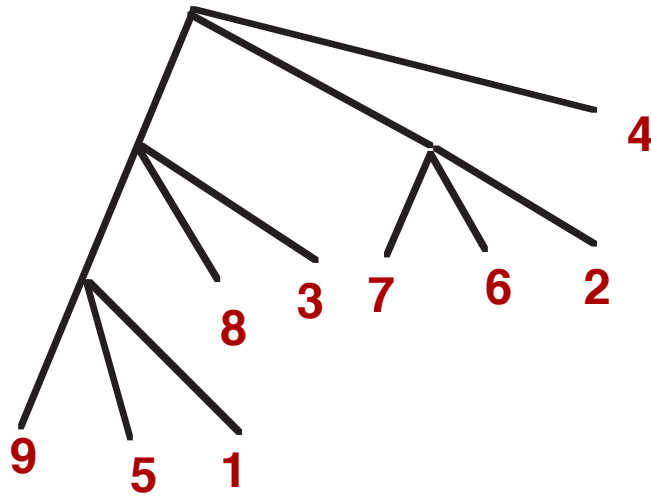
spheres of dimension $n - 3$.

Calderbank-Hanlon-Robinson (1986) computed representation of symmetric group on homology

Hanlon-MW (1995): Lie k -algebras, brush basis for cohomology

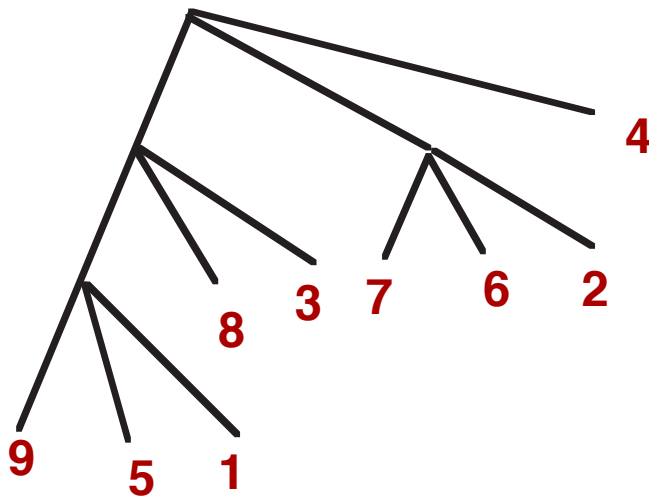
For any node x , let $m(x) =$ largest leaf in the tree rooted at x .

A k -brush is a k -ary tree on a totally ordered leaf set such that for each node y , the child of y with the smallest m value is a leaf.



A 2-brush is a comb.

The number of 3-brushes on leaf set $[2n + 1]$ is $(2n - 1)!!^2$.



$$951/8/3/7/6/2/4 < 95183/7/6/2/4 < 95183/762/4$$

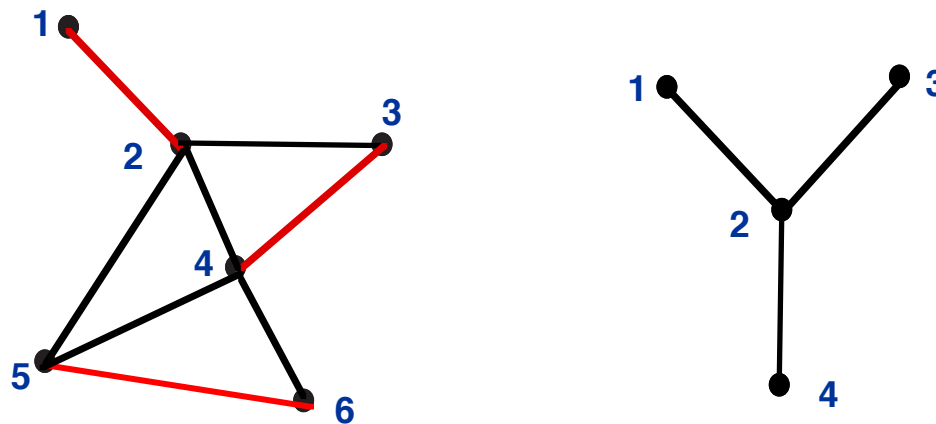
Hanlon & MW (1995): The 3-brushes form a basis for cohomology $\tilde{H}^{n-3}(\Delta(\bar{\Pi}_{2n-1}^{\text{odd}}))$

Is there a splitting basis for homology of $\bar{\Pi}_{2n-1}^{\text{odd}}$?

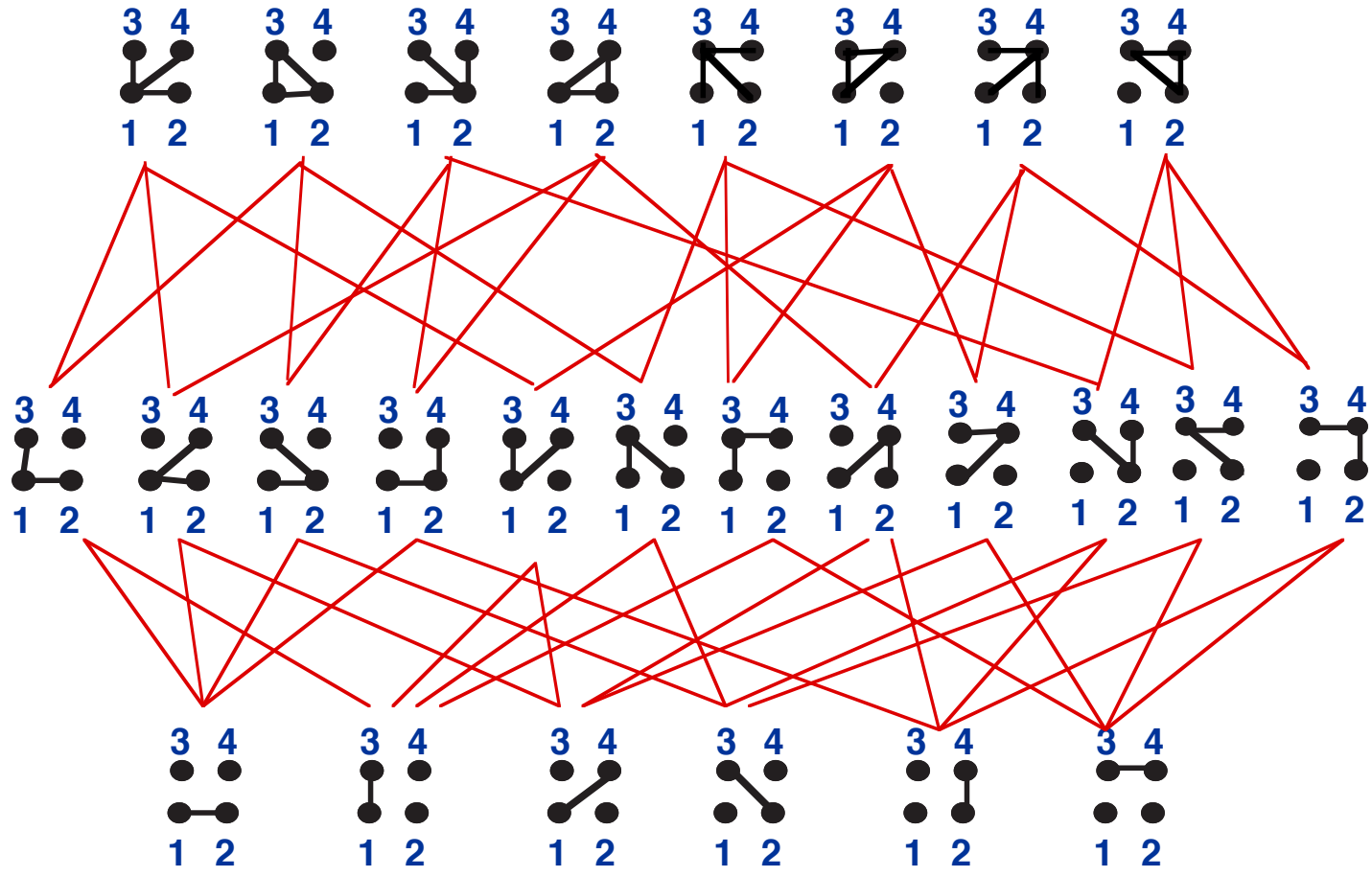
What do the cycles look like?

The no-perfect matching poset - Linusson, Shareshian & Welker (2003)

NPM_{2n} = poset of nonempty graphs on node set $[2n]$ with **no** perfect matching.



Order relation is inclusion of edge sets.



GAP homology software package of Dumas, Heckenback, Saunders and Welker \Rightarrow **Betti numbers**

$n = 2$ 0, 0, **1**
 $n = 3$ 0, 0, 0, 0, 0, **9**, 0, 0, 0, 0
 $n = 4$ 0, 0, 0, 0, 0, 0, 0, 0, **225**, 0, 0, 0, 0, 0, 0, 0, ...
 $n = 5$ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, **11025**, 0, 0, 0, 0, ...

GAP homology software package of Dumas, Heckenback, Saunders and Welker \Rightarrow **Betti numbers**

$$\begin{aligned}n = 2 & \quad 0, 0, \mathbf{1^2} \\n = 3 & \quad 0, 0, 0, 0, 0, \mathbf{(1 \cdot 3)^2}, 0, 0, 0, 0 \\n = 4 & \quad 0, 0, 0, 0, 0, 0, 0, 0, \mathbf{(1 \cdot 3 \cdot 5)^2}, 0, 0, 0, 0, 0, 0, \dots \\n = 5 & \quad 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \mathbf{(1 \cdot 3 \cdot 5 \cdot 7)^2}, 0, 0, \dots\end{aligned}$$

GAP homology software package of Dumas, Heckenback, Saunders and Welker \Rightarrow **Betti numbers**

$$\begin{array}{l}
 n = 2 \quad 0, 0, \mathbf{1^2} \\
 n = 3 \quad 0, 0, 0, 0, 0, \mathbf{(1 \cdot 3)^2}, 0, 0, 0, 0 \\
 n = 4 \quad 0, 0, 0, 0, 0, 0, 0, 0, \mathbf{(1 \cdot 3 \cdot 5)^2}, 0, 0, 0, 0, 0, 0, \dots \\
 n = 5 \quad 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \mathbf{(1 \cdot 3 \cdot 5 \cdot 7)^2}, 0, 0, \dots
 \end{array}$$

Linusson, Shareshian & Welker: $\Delta(\text{NPM}_{2n})$ has the homotopy type of a wedge of

$$\mathbf{(2n - 3)!!^2} := (1 \cdot 3 \cdots (2n - 1))^2$$

spheres of dimension $(3n - 4)$.

Proof technique: discrete Morse theory and Gallai-Edmonds structure theorem

Björner-Stanley + Linusson-Shareshian-Welker implies

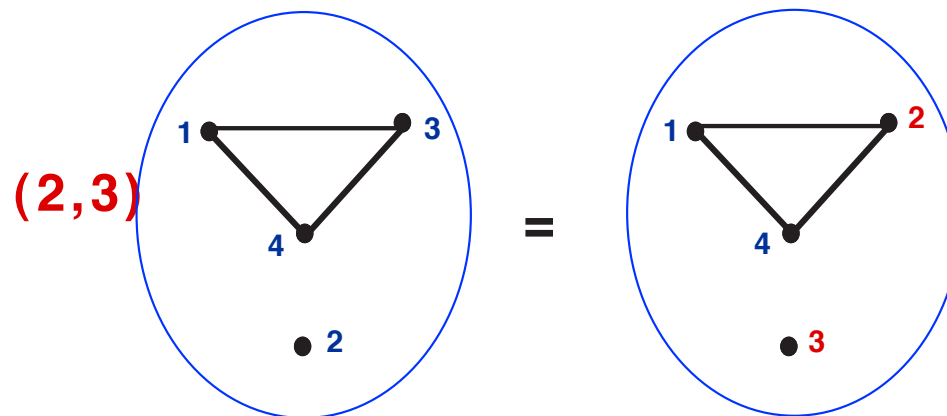
$$\tilde{H}_{n-3}(\bar{\Pi}_{2n-1}^{\text{odd}}) \cong \tilde{H}_{3n-4}(\text{NPM}_{2n})$$

Question: Is there an equivariant isomorphism?

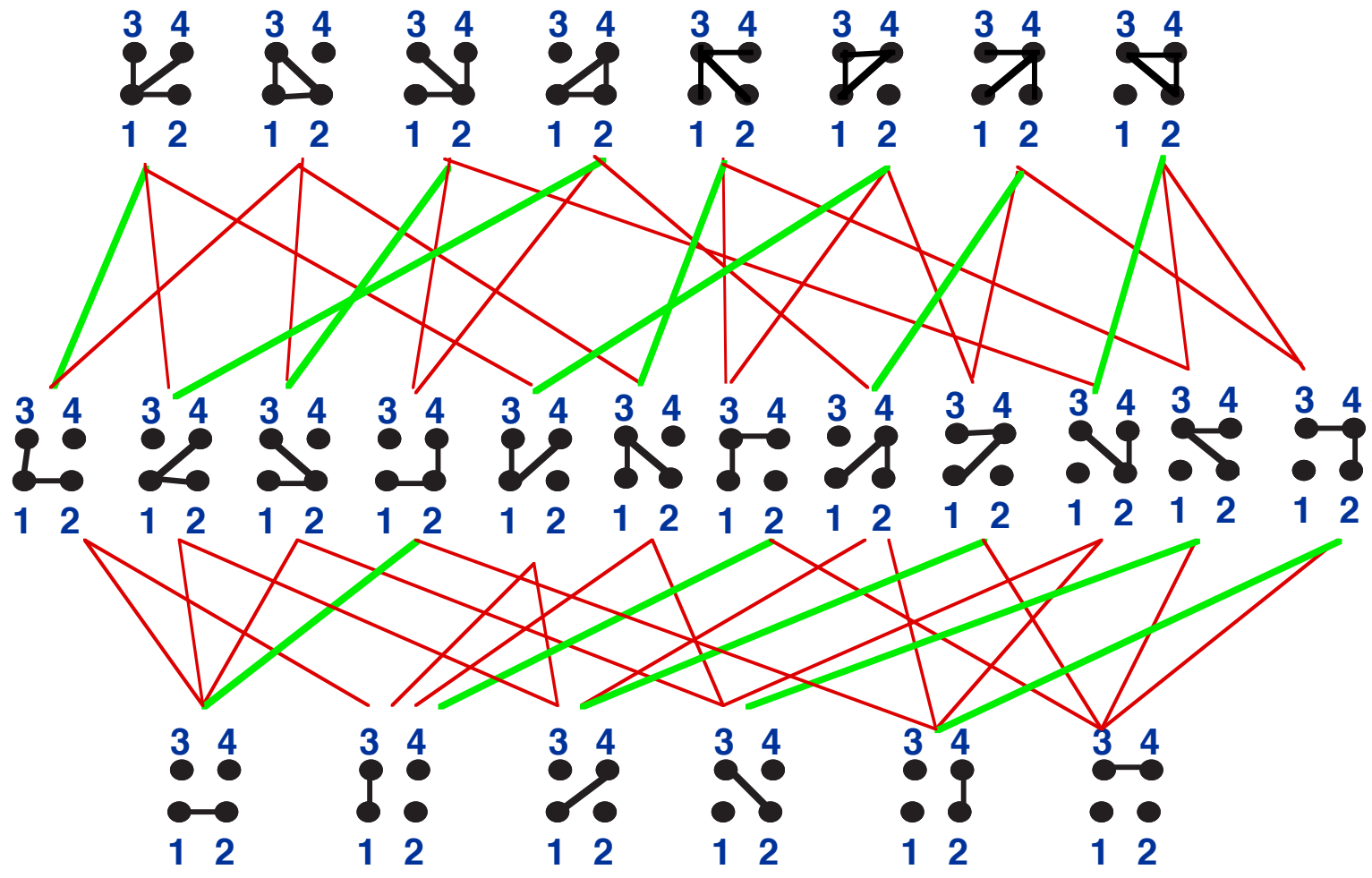
Shareshian & MW: As \mathfrak{S}_{2n-1} -modules

$$\tilde{H}_{3n-4}(\text{NPM}_{2n}) \downarrow_{\mathfrak{S}_{2n-1}}^{\mathfrak{S}_{2n}} \cong \text{sgn} \otimes \tilde{H}_{n-3}(\Pi_{2n-1}^{\text{odd}})$$

Symmetric group \mathfrak{S}_{2n} acts on the poset NPM_{2n} by relabeling graph nodes.



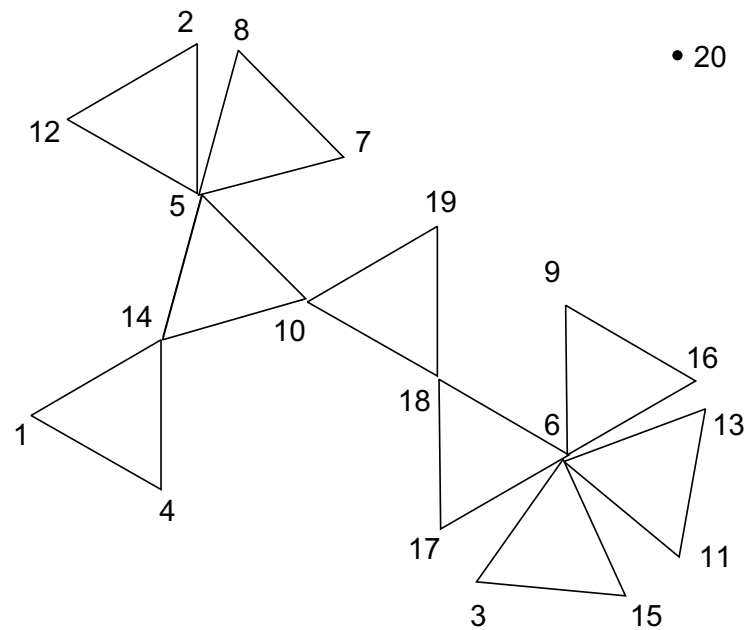
Proof technique: discrete Morse theory and bases



Discrete Morse Theory - Formin (1998)

Let P have a Morse matching with c_i critical points of rank i for each i . Then $\Delta(P)$ has the homotopy type of a CW-complex with c_i cells of dimension i .

Linusson, Shareshian, Welker: critical cells are **critical triangle trees on leaf set $[2n - 1]$** together with isolated node $2n$.



The number of critical triangle trees on node set $[2n - 1]$ is **$(2n - 3)!!^2$** .

Let FC_{2n-1} be the poset of factor critical graphs on node set $[2n-1]$

Shareshian and MW:

$$(a) \quad \tilde{H}_{3n-4}(\text{NPM}_{2n}) \downarrow_{\mathfrak{S}_{2n-1}}^{\mathfrak{S}_{2n}} \cong \text{sgn} \otimes \tilde{H}^{\text{top}}(\text{FC}_{2n-1})$$

$$(b) \quad \tilde{H}^{\text{top}}(\text{FC}_{2n-1}) \cong \tilde{H}_{n-3}(\Pi_{2n-1}^{\text{odd}})$$

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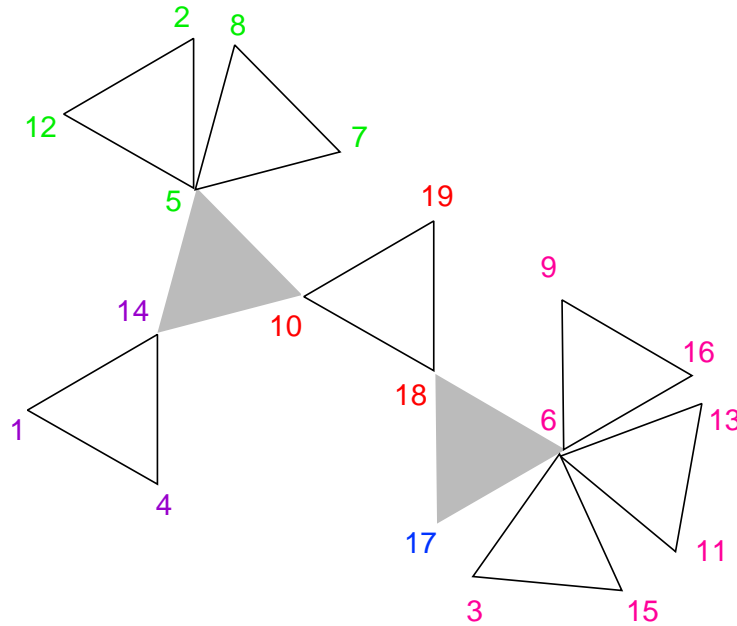
To prove (b): Basis for $\tilde{H}^{\text{top}}(\text{FC}_{2n-1})$,

$$\{\gamma_T \mid T \in \text{CTT}_{2n-1}\},$$

$\text{CTT}_{2n-1} :=$ set of critical triangle trees on node set $[2n-1]$.

Triangle splitting basis for $\tilde{H}_{n-3}(\Pi_{2n-1}^{\text{odd}})$

For $T \in \text{CTT}_{2n-1}$, **split** T by choosing some triangles of T and removing all the edges of these triangles. This disconnects the graph. The components induce a partition in Π_{2n-1}^{odd} .



$$1, 4, 14 - 12, 5, 2, 8, 7 - 10, 18, 19 - 17 - 3, 15, 11, 13, 16, 9, 6$$

Let Π_T be the subposet consisting of all the partitions obtained by splitting T .

Π_T is isomorphic to the lattice of subsets of $[n - 1]$. So $\Delta(\bar{\Pi}_T)$ is homeomorphic to an $(n - 3)$ -sphere.

$\rho_T :=$ fundamental cycle of $\Delta(\bar{\Pi}_T)$

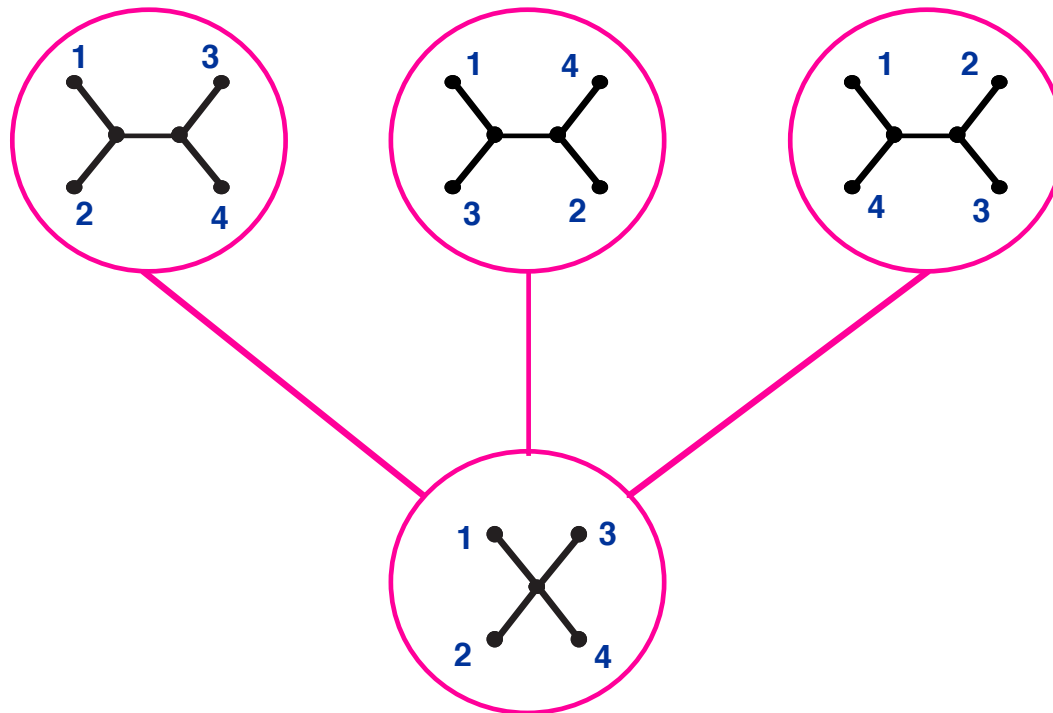
- $\{\rho_T \mid T \in \text{CTT}_{2n-1}\}$ is a basis for $\tilde{H}_{n-3}(\bar{\Pi}_{2n-1}^{\text{odd}})$
- The bijection $\rho_T \mapsto \gamma_T$ determines an \mathfrak{S}_{2n-1} -module isomorphism

$$\tilde{H}_{n-3}(\bar{\Pi}_{2n-1}^{\text{odd}}) \rightarrow \tilde{H}^{\text{top}}(\text{FC}_{2n-1})$$

Tree poset Vogtmann (1990)

\mathcal{T}_n = poset of trees on leaf set $[n]$ where all internal nodes have degree ≥ 3 .

$T < T'$ if T can be obtained from T' by contracting internal edges.



$\Delta(\mathcal{T}_n)$ has the homotopy type of a wedge of $(n-2)!$ $(n-4)$ -spheres.

So $\Delta(\mathcal{T}_{n+1}) \simeq \Delta(\Pi_n)$

Robinson and Whitehouse (1996)

As \mathfrak{S}_n -modules

$$\tilde{H}_{n-3}(\mathcal{T}_{n+1}) \downarrow_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}} \cong \tilde{H}_{n-3}(\Pi_n)$$

As \mathfrak{S}_{n+1} -modules

$$\tilde{H}_{n-3}(\mathcal{T}_{n+1}) \cong \tilde{H}_{n-3}(\Pi_n) \uparrow_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}} - \tilde{H}_{n-2}(\Pi_{n+1})$$

Other occurrences of Whitehouse module:

- Cyclic Lie operad - Kontsevich (1993)
- Not 2-connected graph complex - Vassiliev (1997), Babson, Björner, Linusson, Shareshian, Welker (1997), Turchin (1997)
- Kernel of Varchenko operator - Hanlon & Stanley (1998)
- Bounded block size partition poset - Sundaram (1998)
- Nonmodular partition poset - Sundaram (1999)

Analogue of Whitehouse Module - Hanlon (1996)

$\mathcal{T}_{2n}^{\text{even}}$ = subposet of \mathcal{T}_{2n} consisting of trees in which all internal nodes have even degree.

- As \mathfrak{S}_{2n-1} -modules

$$\tilde{H}_{n-3}(\mathcal{T}_{2n}^{\text{even}}) \downarrow_{\mathfrak{S}_{2n-1}}^{\mathfrak{S}_{2n}} \cong \tilde{H}_{n-3}(\Pi_{2n-1}^{\text{odd}})$$

- As \mathfrak{S}_{2n} -modules

$$\tilde{H}_{n-3}(\mathcal{T}_{2n}^{\text{even}}) \cong \tilde{H}_{n-3}(\Pi_{2n-1}^{\text{odd}}) \uparrow_{\mathfrak{S}_{2n-1}}^{\mathfrak{S}_{2n}} - \tilde{H}_{n-2}(\Pi_{2n}^{\text{odd}})$$

Another Occurrence: Cyclic Lie 3-Operad - MW (1998)

Shareshian & MW:

$$\tilde{H}_{3n-4}(\text{NPM}_{2n}) \downarrow_{\mathfrak{S}_{2n-1}}^{\mathfrak{S}_{2n}} \cong \tilde{H}_{n-3}(\mathcal{T}_{2n}^{\text{even}}) \downarrow_{\mathfrak{S}_{2n-1}}^{\mathfrak{S}_{2n}}$$

Conjecture: As \mathfrak{S}_{2n} -modules

$$\tilde{H}_{3n-4}(\text{NPM}_{2n}) \cong \tilde{H}_{n-3}(\mathcal{T}_{2n}^{\text{even}})$$