Topology of Posets of Partitions, Trees and Graphs

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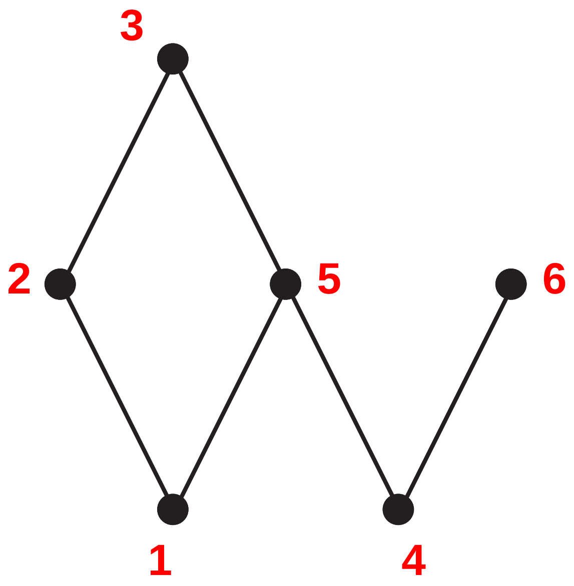
Order complex $\Delta(P)$ of a poset P is the simplicial complex whose faces are the chains of P.



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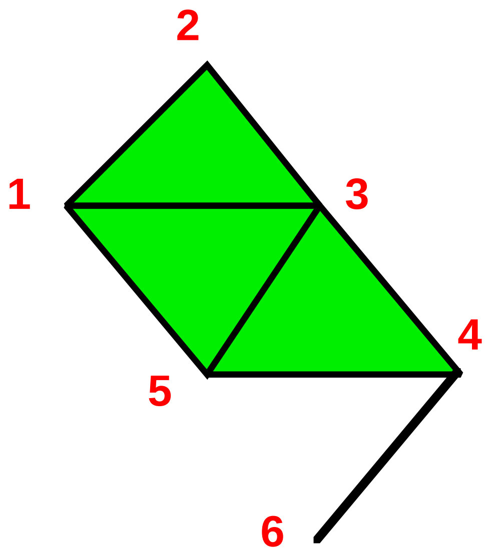


Δ(P)

$$\mu(\widehat{P}) = \tilde{\chi}(\Delta(P)), \quad \text{where } \widehat{P} = P \cup \{\hat{0}, \hat{1}\}$$

$$\tilde{\chi}(\Delta) := \sum_i (-1)^i f_i(\Delta) = \sum_i (-1)^i \dim \tilde{H}_i(\Delta)$$

I. SOME CONECTIONS

- hyperplane & subspace arrangements
- complexity theory
- group theory
- Lie theory
- knot theory
- commutative algebra
- algebraic geometry

Complexity of Graph Properties

Let p be an (isomorphism invariant) graph property.

An algorithm for deciding whether a graph G with n nodes has property p checks the entries of the adjacency matrix until a determination can be made.

p is said to be evasive if best algorithm needs to check all $\binom{n}{2}$ entries (in the worst case).

Examples:

- property of having at least 2 edges
- property of being connected
- property of containing a perfect matching
- property of having degree at most b

Evasiveness Conjecture (Karp) Every nontrivial monotone graph property is evasive.

Kahn-Saks-Sturtevant (1984) Evasiveness conjecture true when n is a prime power.

The proof makes use of topology of $\Delta(P)$, where P is the poset of nonempty graphs having monotone graph property p ordered by inclusion of edge sets.

- $\Delta(P)$ not acyclic $\Longrightarrow$ p is evasive.
- Fixed point theorem of Oliver (1975) is used to show $\Delta(P)$ is not acyclic when n is a prime power.

Complexity of the k-equal problem

Given n numbers $x_1, x_2, \ldots, x_n \in \mathbb{R}$, how many comparisons does the best algorithm need (in the worst case) to decide whether k of the numbers are equal?

Let $c(n,k)$ be the number of comparisons needed.

Björner & Lovász (1994)

$$\max\{n-1, n\log_3 \frac{n}{3k}\} \le c(n,k) \le 8nlog_3\frac{n}{k}$$

The proof makes use of topology of $\Delta(P)$ where

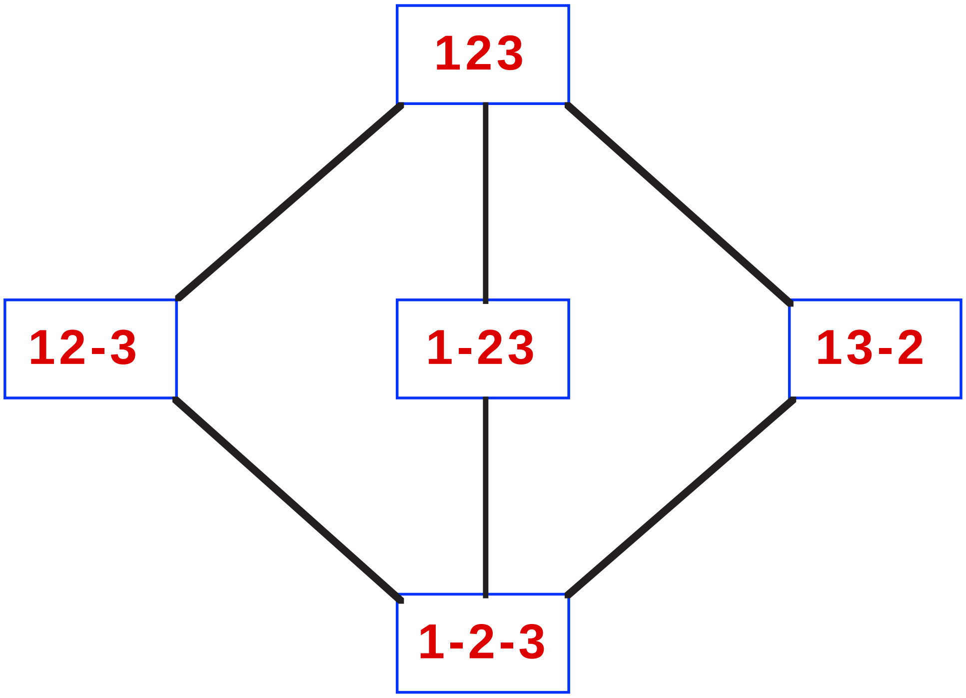
$P =$ poset of partitions of $[n] := \{1, 2, \ldots, n\}$ with no blocks of sizes $2, 3, ..., k-1$.

Key result:

$$c(n,k) \ge \sum_{x \in P} \sum_{j} \dim \tilde{H}_j(\bar{P}_{<x})$$

II. INTERESTING EXAMPLES

Π_n = lattice of partitions of $[n]$ ordered by refinement



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1-2-3

Π_n is the intersection lattice of the braid hyperplane arrangement

$$\mathcal{A}_n = \{x_i = x_j | 1 \leq i \leq j \leq n\}$$

$|\mu(\Pi_n)| = (n-1)!$

$\Delta(\Pi_n \setminus \{\hat{0}, \hat{1}\})$ is

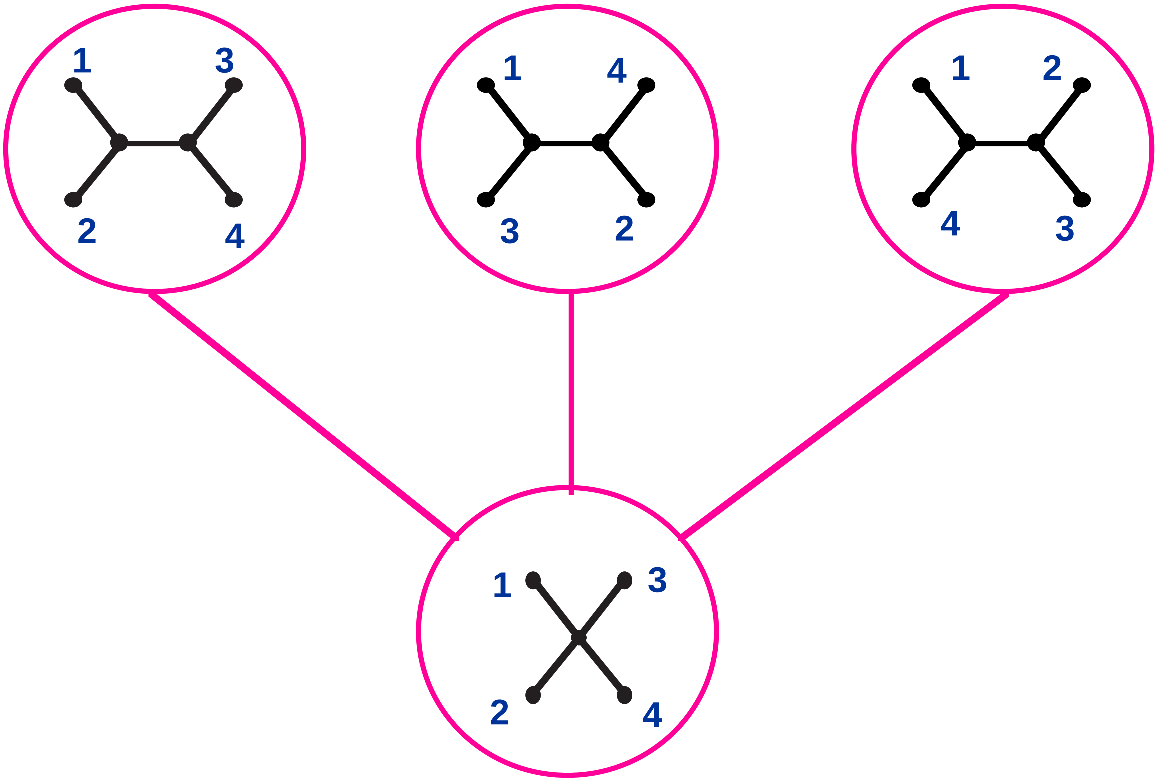
- $(n-3)$-dimensional
- has the homotopy type of a wedge of $(n-1)!$ $(n-3)$-spheres.

Proof Technique: Lexicographical shellability

Vogtmann (1990)

$\mathcal{T}_n$ = poset of trees on leaf set $[n]$ where all internal nodes have degree ≥ 3.

$T < T'$ if T can be obtained from T' by contracting internal edges.



$|\mu(\mathcal{T}_n \cup \{\hat{1}\})| = (n-2)!$

$\Delta(\mathcal{T}_n \setminus \{\hat{0}\})$ is

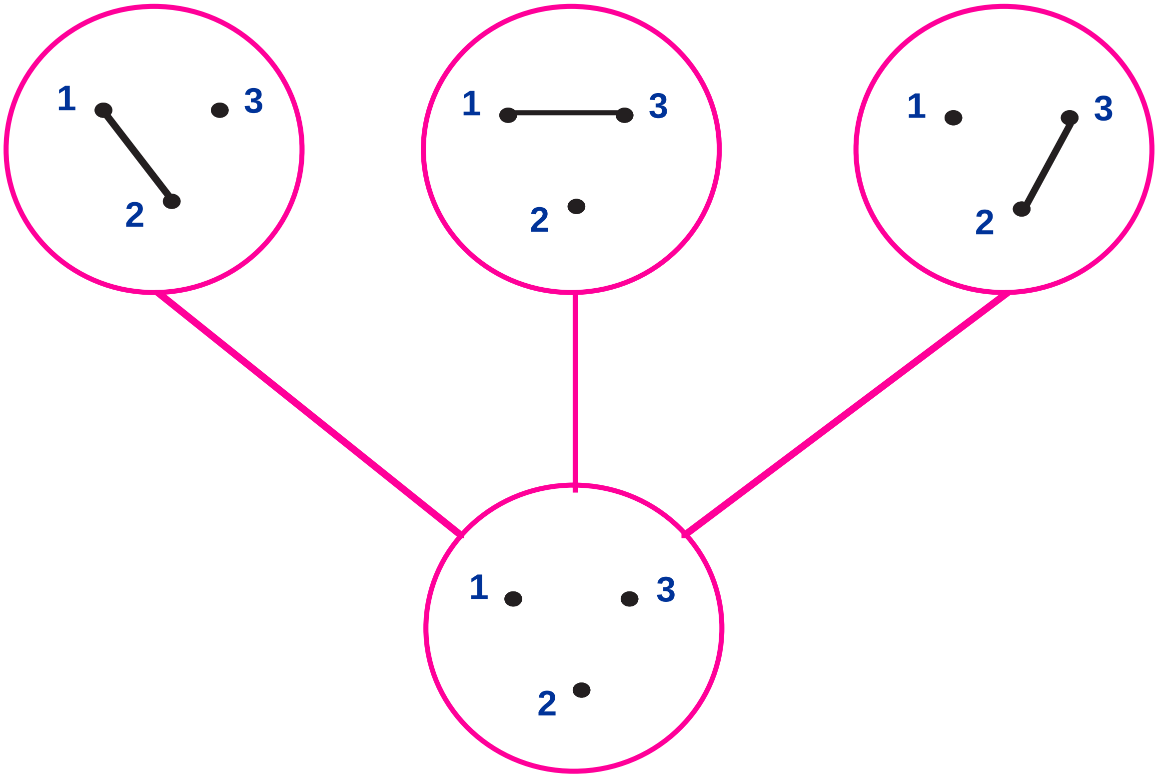
- $(n-4)$-dimensional
- has the homotopy type of a wedge of $(n-2)!$ $(n-4)$-spheres.

Can also use shellability theory to prove this.

So $\Delta(\mathcal{T}_{n+1} \setminus \{\hat{0}\}) \simeq \Delta(\Pi_n \setminus \{\hat{0}, \hat{1}\})$

Vassiliev - knot theory (1990)

NC_n = poset of not connected graphs on node set $[n]$ ordered by inclusion of edge sets



$|\mu(\mathsf{NC}_n \cup \{\hat{1}\})| = (n-1)!$

$\Delta(\mathsf{NC}_n \setminus \{\hat{0}\})$ is

- $(\binom{n-1}{2} - 1)$-dimensional
- has the homotopy type of a wedge of $(n-1)!$ $(n-3)$-spheres.

Proof Technique: Quillen Fiber Lemma

$$f : \mathsf{NC}_n \setminus \{\hat{0}\} \to \Pi_n \setminus \{\hat{0}, \hat{1}\}$$

Blocks of $f(G)$ are node sets of the connected components of G

So $\Delta(\Pi_n \setminus \{\hat{0}, \hat{1}\})$, $\Delta(\mathcal{T}_{n+1} \setminus \{\hat{0}\})$, & $\Delta(\mathsf{NC}_n \setminus \{\hat{0}\})$ are all homotopy equivalent.

Let $\bar{\Pi}_n = \Pi_n \setminus \{\hat{0}, \hat{1}\}$, $\bar{\mathcal{T}}_n = \mathcal{T}_n \setminus \{\hat{0}\}$,
$\overline{\mathrm{NC}}_n = \mathrm{NC}_n \setminus \{\hat{0}\}$

Symmetric group $\mathfrak{S}_n$ acts on Π_n, $\mathcal{T}_n$, NC_n

Theorem. As $\mathfrak{S}_n$-modules

$$\tilde{H}_{n-3}(\bar{\Pi}_n) \cong \mathrm{sgn} \otimes \mathrm{Lie}_n \tag{1}$$

$$\tilde{H}_{n-3}(\bar{\mathcal{T}}_{n+1}) \downarrow_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}} \cong \mathrm{sgn} \otimes \mathrm{Lie}_n \tag{2}$$

$$\tilde{H}_{n-3}(\overline{\mathrm{NC}}_n) \cong \mathrm{sgn} \otimes \mathrm{Lie}_n \tag{3}$$

where $\mathrm{Lie}_n = e^{2\pi i/n} \uparrow_{C_n}^{\mathfrak{S}_n}$.

(1) Stanley (1982)

(2) Robinson and Whitehouse (1996)

(3) Vassiliev (1993)

The Hopf trace formula is used to prove (1) and (2) and the Quillen Fiber Lemma is used to prove (3)

The Whitehouse Module

Robinson and Whitehouse (1996) As $\mathfrak{S}_{n+1}$-modules

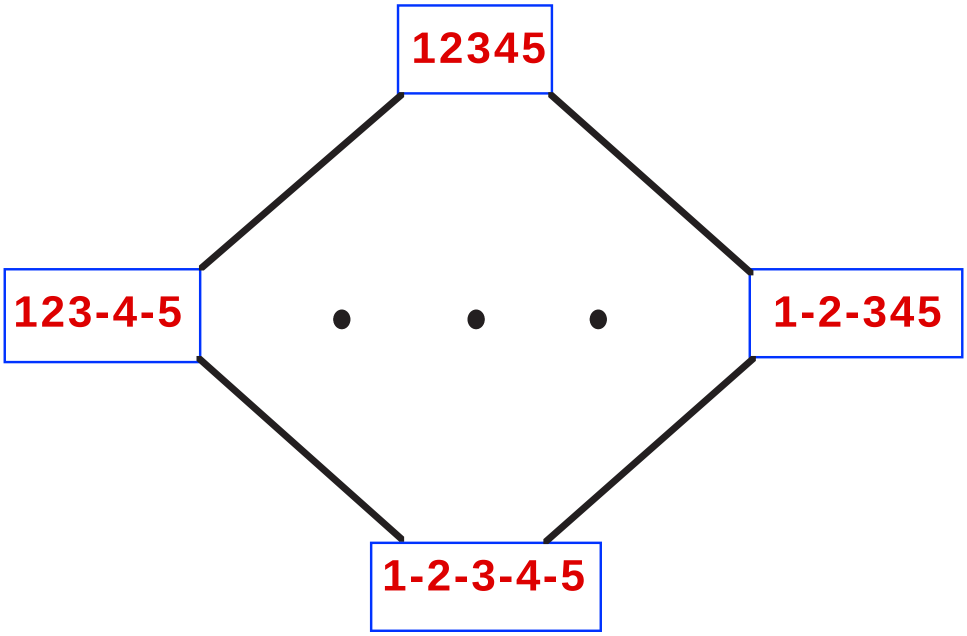
$$\tilde{H}_{n-3}(\bar{\mathcal{T}}_{n+1}) \cong \mathrm{sgn} \otimes (\mathrm{Lie}_n \uparrow_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}} - \mathrm{Lie}_{n+1})$$

Other occurrences of Whitehouse module:

- Cyclic Lie operad - Kontsevich (1993)
- Nonmodular partition poset - Sundaram (1999)
- Bounded block size partition poset - Sundaram (1998)
- Kernel of Varchenko operator - Hanlon & Stanley (1998)
- Not 2-connected graph complex - Vassiliev (1997), Babson, Björner, Linusson, Shareshian, Welker (1997), Turchin (1997).

k-analogues

Π_n^k = subposet of Π_n consisting of partitions with block sizes $\equiv 1 \bmod k$



$|\mu(\Pi^2_{2n+1})| = (2n-1)!!^2 := (1 \cdot 3 \cdots (2n-1))^2$

$|\mu(\Pi^k_{kn+1})| =$ b(n,k) $:=$ # k-brushes on leaf set $[kn+1]$. (Hanlon & Wachs 1995)

$\Delta(\bar{\Pi}^k_{kn+1})$ has the homotopy type of a wedge of $(n-2)$-spheres (Björner 1985)

Calderbank-Hanlon-Robinson (1986):

$$\mathrm{ch}(\tilde{H}_{n-2}(\bar{\Pi}^k_{kn+1})) = (-1)^n \left[\sum_{r\geq 0} h_{kr+1}\right]^{-1}\Bigg|_{\mathrm{deg}\,n}$$

ch( ) := Frobenius characteristic,
h_i := complete homogenous symmetric function
[ ]$^{-1}$:= plethystic inverse.

Hanlon & Wachs (1995): As $\mathfrak{S}_{kn+1}$-modules

$$\tilde{H}_{n-2}(\bar{\Pi}^k_{kn+1}) \cong \mathrm{Lie}^k_{kn+1},$$

where Lie^k_{kn+1} is the representation of $\mathfrak{S}_{kn+1}$ on the multilinear component of the evenly generated free Lie k-algebra

$\mathcal{T}^k_{nk+2}$ = subposet of $\mathcal{T}_{nk+2}$ consisting of trees in which all internal nodes have degree $\equiv 2 \bmod k$.

Hanlon (1996) As $\mathfrak{S}_{nk+1}$-modules

$$\tilde{H}_{n-2}(\bar{\mathcal{T}}^k_{nk+2}) \downarrow^{\mathfrak{S}_{nk+2}}_{\mathfrak{S}_{nk+1}} \quad \cong \quad \tilde{H}_{n-2}(\bar{\Pi}^k_{nk+1})$$

As $\mathfrak{S}_{nk+2}$-modules

$$\tilde{H}_{n-2}(\bar{\mathcal{T}}^k_{nk+2}) \cong$$

$$\tilde{H}_{n-2}(\bar{\Pi}^k_{nk+1}) \uparrow^{\mathfrak{S}_{nk+2}}_{\mathfrak{S}_{nk+1}} - \tilde{H}_{n-1}(\bar{\Pi}^k_{nk+2})$$

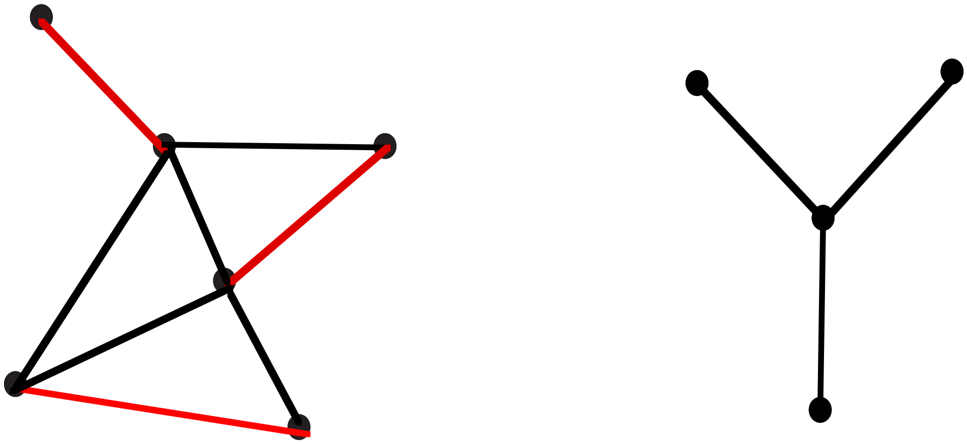
Trappmann & Ziegler(1997), Wachs (1997) $\Delta(T^k_{nk+2})$ has the homotopy type of a wedge of $b(n,k)$ $(n-2)$-spheres.

Wachs (1997) k-Cyclic Lie operad is another occurrence of $\tilde{H}_{n-2}(\bar{\mathcal{T}}^k_{nk+2})$.

A monotone graph property

PM_{2n} = poset of graphs on node set $[2n]$ that contain a perfect matching.

NPM_{2n} = poset of graphs on node set $[2n]$ with no perfect matching.



Linusson, Shareshian & Welker (2000)

$\Delta(\overline{\mathrm{NPM}_{2n}})$ has the homotopy type of a wedge of $(2n-3)!!^2$ $(3n-4)$-spheres.

Shareshian & Wachs (2002): As $\mathfrak{S}_{2n-1}$-modules

$$\tilde{H}_{3n-4}(\overline{\mathsf{NPM}}_{2n}) \downarrow^{\mathfrak{S}_{2n}}_{\mathfrak{S}_{2n-1}} \;\cong\; \mathsf{sgn} \otimes \tilde{H}_{n-3}(\bar{\Pi}^2_{2n-1})$$

$$\tilde{H}_{2n^2-4n+1}(\overline{\mathsf{PM}}_{2n}) \downarrow^{\mathfrak{S}_{2n}}_{\mathfrak{S}_{2n-1}} \;\cong\; \mathsf{sgn} \otimes \tilde{H}_{n-3}(\bar{\Pi}^2_{2n-1})$$

Conjecture: As $\mathfrak{S}_{2n}$-modules

$$\tilde{H}_{3n-4}(\overline{\mathsf{NPM}}_{2n}) \;\cong\; \mathsf{sgn} \otimes \tilde{H}_{n-3}(\bar{\mathcal{T}}^2_{2n})$$

$$\tilde{H}_{2n^2-4n+1}(\overline{\mathsf{PM}}_{2n}) \;\cong\; \mathsf{sgn} \otimes \tilde{H}_{n-3}(\bar{\mathcal{T}}^2_{2n})$$

III Proof Techniques

Let $M(P)$ be the vector space generated by maximal chains of poset P.

$\tilde{H}_{\text{top}}(P)$ is a subspace of $M(P)$.

$\tilde{H}^{\text{top}}(P)$ is a quotient of $M(P)$.

If $\mathfrak{S}_n$ acts on P then

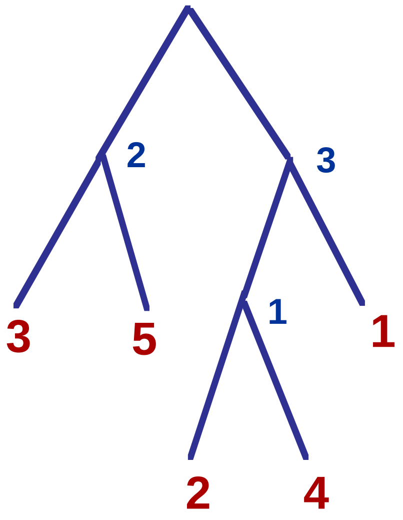
$$\tilde{H}_{\text{top}}(P) \cong \tilde{H}^{\text{top}}(P)$$

as $\mathfrak{S}_n$-modules.

Π_n

Wachs (1993):

Maximal chains of Π_n can be represented by binary trees on leaf set $[n]$ together with an increasing labeling of the internal nodes.



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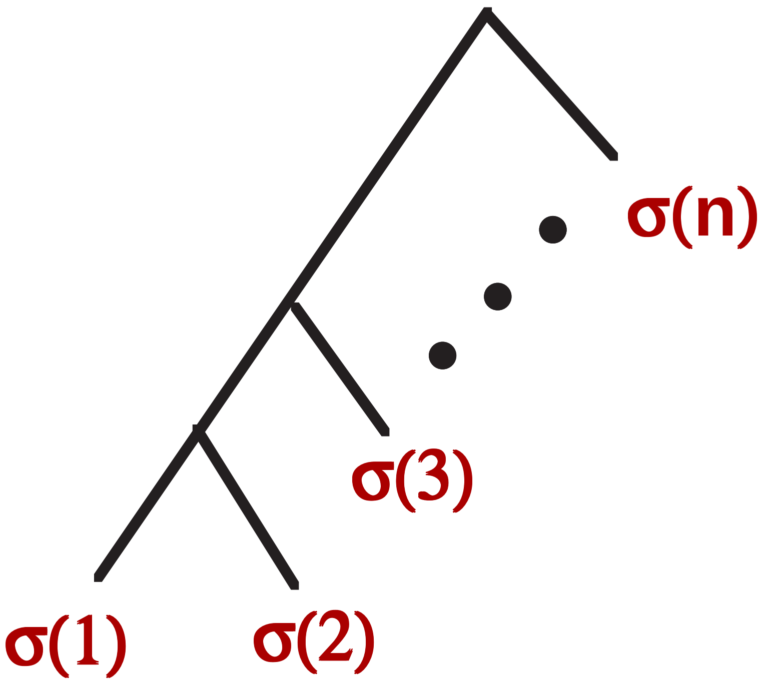
Remark: This tree representation of maximal chains can be used to give an explicit $\mathfrak{S}_n$-module isomorphism $\tilde{H}^{\text{top}}(\bar{\Pi}_n) \cong \tilde{H}^{\text{top}}(\bar{\mathcal{T}}_{n+1})$.

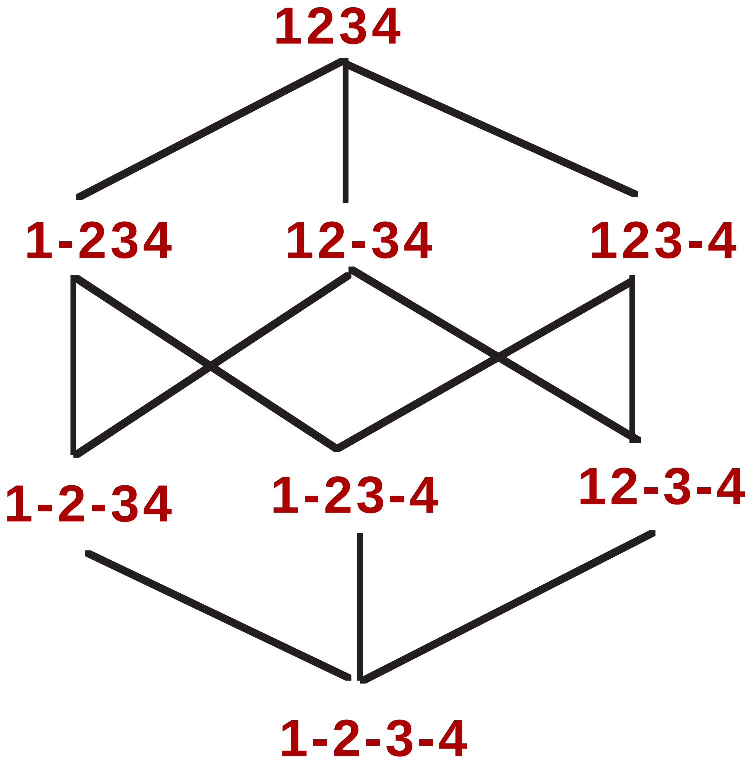
Comb basis for $\tilde{H}^{\mathrm{top}}(\bar{\Pi}_n)$: For $\sigma \in \mathfrak{S}_n$, let c_σ be the maximal chain



$$\tau c_\sigma = c_{\tau\sigma}$$

Theorem $\{c_\sigma \mid \sigma \in \mathfrak{S}_n, \sigma(1) = 1\}$ is a basis for $\tilde{H}^{\mathrm{top}}(\bar{\Pi}_n)$

Splitting basis for $\tilde{H}_{\text{top}}(\bar{\Pi}_n)$: For $\sigma \in \mathfrak{S}_n$, let Π_σ be the subposet of Π_n consisting of partitions obtained by splitting the word $\sigma(1)\sigma(2)\cdots\sigma(n)$.



Π_σ is isomorphic to the lattice of subsets of $[n-1]$. So $\Delta(\bar{\Pi}_\sigma)$ is homeomorphic to an $(n-3)$-sphere.

Let $\rho_\sigma =$ fundamental cycle of $\Delta(\bar{\Pi}_\sigma)$

Theorem. $\{\rho_\sigma \mid \sigma \in \mathfrak{S}_n, \sigma(1) = 1\}$ is a basis for $\tilde{H}_{\mathsf{top}}(\bar{\Pi}_n)$.

The splitting basis and the comb basis are dual bases.

Remark: Clearly $\tau\rho_\sigma = \rho_{\tau\sigma}$ and $\tau c_\sigma = c_{\tau\sigma}$. Hence the splitting basis shows that $\tilde{H}_{\mathsf{top}}(\bar{\Pi}_n) \downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}$ is the regular representation and the comb basis shows this for $\tilde{H}^{\mathsf{top}}(\bar{\Pi}_n) \downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}$.

Π^k_{nk+1}

Hanlon & Wachs (1993):

Maximal chains of Π^k_{nk+1} can be represented by $(k+1)$-ary trees on leaf set $[nk+1]$ together with an increasing labeling of the internal nodes.

Brush basis for $\tilde{H}^{n-2}(\bar{\Pi}^k_{nk+1})$ is the set of maximal chains represented by certain $(k+1)$-ary trees called k-brushes.

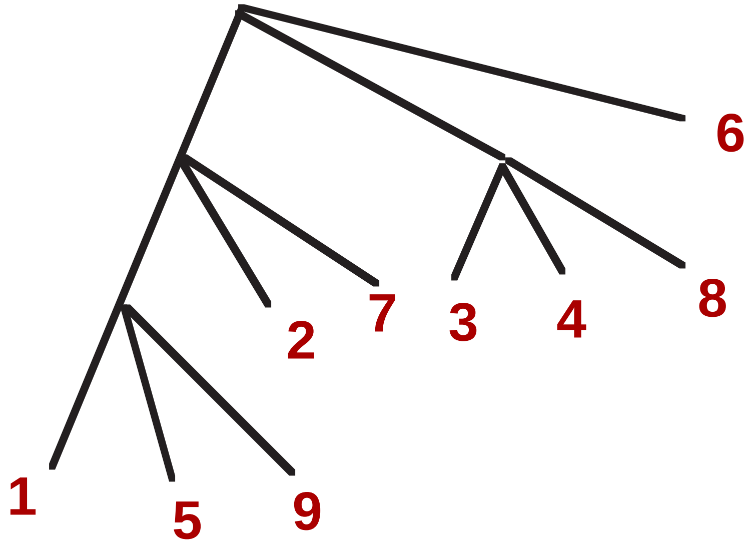
Remark: This tree representation of maximal chains and the brush basis were used to show

$$\tilde{H}^{\mathrm{top}}(\bar{\Pi}^k_{nk+1}) \cong \mathrm{Lie}^k_{nk+1}$$

as $\mathfrak{S}_{nk+1}$-modules.

For any node x in a rooted tree T on leaf set $[nk+1]$, let $m(x)$ be the smallest leaf in the tree rooted at x.

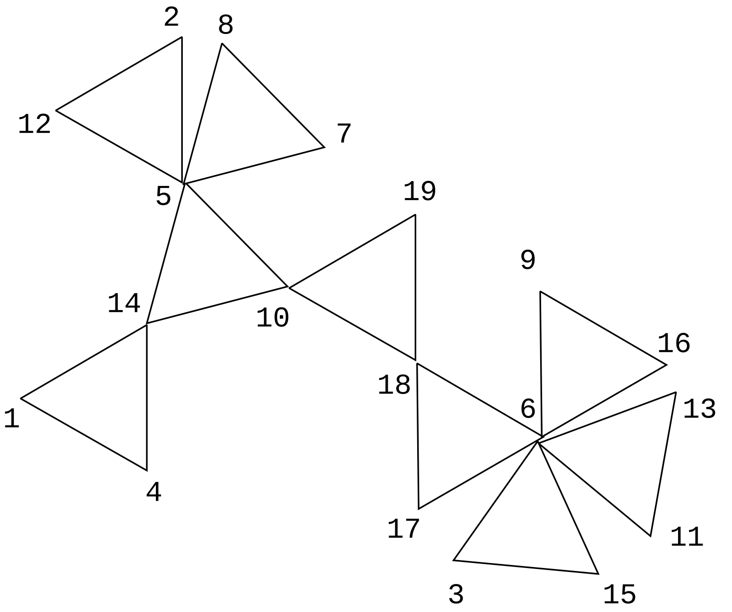
A k-brush is a $(k+1)$-ary tree on leaf set $[nk+1]$ such that for each internal node y, the child of y with the largest m value is a leaf.



A 1-brush is a comb.
The number of 2-brushes on leaf set $[2n+1]$ is $(2n-1)!!^2$.

NPM$_{2n}$

Linusson, Shareshian, Welker (2000):
Discrete Morse theory $\Longrightarrow$ $\Delta(\mathrm{NPM}_{2n})$ has the homotopy type of a wedge of m $(3n-4)$-spheres where $m =$ number of critical triangle trees on node set $[2n-1]$.



The number of critical triangle trees on node set $[2n-1]$ is $(2n-3)!!^2$.

It follows from Linusson, Shareshian and Welker's results that

$$\tilde{H}_{3n-4}(\overline{\mathsf{NPM}}_{2n}) \downarrow^{\mathfrak{S}_{2n}}_{\mathfrak{S}_{2n-1}} \;\cong\; \mathsf{sgn} \otimes \tilde{H}_{\mathsf{top}}(\mathsf{FC}_{2n-1})$$

FC_{2n-1} is the poset of factor critical graphs on node set $[2n-1]$.

The critical triangle trees index a basis for $\tilde{H}^{\mathsf{top}}(\mathsf{FC}_{2n-1})$,

$$\{\gamma_T \mid T \in \mathsf{CTT}_{2n-1}\}$$

CTT_{2n-1} = set of critical triangle trees on node set $[2n-1]$.

Shareshian & Wachs: nice bijection between 2-brushes and critical triangle trees

Is the isomorphism

$$\tilde{H}^{\mathsf{top}}(\bar{\Pi}^2_{2n-1}) \to \tilde{H}^{\mathsf{top}}(\overline{\mathsf{FC}}_{2n-1})$$

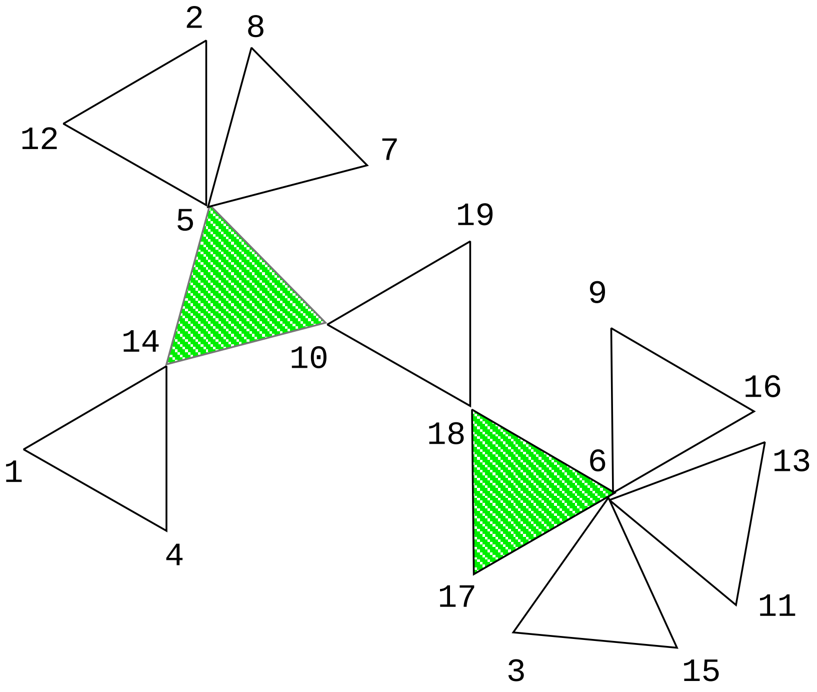
defined by the bijection between 2-brushes and critical triangle trees $\mathfrak{S}_{2n-1}$-equivariant?

No!

We find a basis for homology of Π^2_{2n-1} that yields an $\mathfrak{S}_{2n-1}$-equivariant isomorphism

Triangle splitting basis:

Let $T \in \mathsf{CTT}_{2n-1}$. We split T by choosing some triangles of T and removing all the edges of these triangles. This disconnects the graph. The components induce a partition in Π^2_{2n-1}.



$$\pi = \quad 12, 2, 5, 8, 7 - 1, 4, 14 - 10, 19, 18 - 17 - 3, 15, 6, 11, 13, 9, 16$$

Let Π_T be the subposet consisting of all the partitions obtained by splitting T.

Π_T is isomorphic to the lattice of subsets of $[n-1]$. So $\Delta(\bar{\Pi}_T)$ is homeomorphic to an $(n-3)$-sphere.

$\rho_T :=$ fundamental cycle of $\bar{\Pi}_T$

Shareshian and Wachs:

- $\{\rho_T \mid T \in \mathsf{CTT}_{2n-1}\}$ is a basis for $\tilde{H}_{\mathsf{top}}(\bar{\Pi}^2_{2n-1})$
- The bijection $\rho_T \mapsto \gamma_T$ determines an $\mathfrak{S}_{2n-1}$-module isomorphism

$$\tilde{H}_{\mathsf{top}}(\bar{\Pi}^2_{2n-1}) \to \tilde{H}^{\mathsf{top}}(\overline{\mathsf{FC}}_{2n-1})$$

Triangle splitting basis generalizes to a splitting basis for $\tilde{H}_{\mathrm{top}}(\bar{\Pi}^{k}_{kn+1})$.

Is there a generalization of the perfect matching poset whose homology is isomorphic to $\tilde{H}_{\mathrm{top}}(\bar{\Pi}^{k}_{kn+1})$ for general k?