MTH 510

Chapter 8 Notes

The definition of determinant of a matrix is given in 10.25. Recall

Proposition 1 (from MTH 210)**.** *Let A, B be $n \times n$ matrices. Then*

(1) $\det(AB) = \det(A)\det(B)$.
(2) *A is invertible if and only if* $\det A \neq 0$.
(3) *If A is invertible then* $\det A^{-1} = \frac{1}{\det A}$.
(4) *If A is triangular then* $\det A$ *equals the product of the diagonal entries.*
(5) *If A is invertible then* $\det(A^{-1}BA) = \det B$.

Assume V is finite dimensional. Let $T \in \mathcal{L}(V)$. Define $\det T := \det \mathcal{M}(T, \mathcal{B})$, where $\mathcal{B}$ is any basis for V (it follows from Part 5 that any basis will do). The characteristic polynomial $p_T(x)$ of T is defined to be $\det(xI - T)$. Note $\deg p_T(x) = \dim V$.

Theorem 2. *Let $T \in \mathcal{L}(V)$. Then the roots of $p_T(x)$ are precisely the eigenvalues of T.*

An eigenvalue λ of T is said to have multiplicity m if λ has multiplicity m as a root of $p_T(x)$. We let $m_T(\lambda)$ denote the multiplicity of λ.

Proposition 3. *Suppose $T \in \mathcal{L}(V)$ has a triangular matrix with respect to some basis. If the diagonal entries of the matrix are $d_1, \dots, d_n$, where $n = \dim V$, then*

$$p_T(x) = (x - d_1)\dots(x - d_n).$$

Theorem 4 (Cayley-Hamilton)**.** *For all $T \in \mathcal{L}(V)$, $p_T(T) = 0$.*

Proposition 5. *If $T \in \mathcal{L}(V)$ has a diagonal matrix with respect to some basis then for all eigenvalues λ,*

$$m_T(\lambda) = \dim \operatorname{null}(T - \lambda I).$$

Theorem 6. *Let $T \in \mathcal{L}(V)$ and let λ be an eigenvalue of T. Then*

$$m_T(\lambda) = \dim \operatorname{null}((T - \lambda I)^{\dim V}).$$

The set $\operatorname{null}((T - \lambda I)^{\dim V})$ is a subspace of V called the space of *generalized eigenvectors* corresponding to λ. An operator is called *nilpotent* if some power of it equals 0.

Proposition 7. *Let $T \in \mathcal{L}(V)$ and let λ be an eigenvalue of T. Then*

(1) $U := \operatorname{null}((T - \lambda I)^{\dim V})$ *is T-invariant.*
(2) $(T - \lambda I)|_U$ *is nilpotent.*

Theorem 8 (Decomposition Theorem - see Th. 8.23)**.** *Suppose $T \in \mathcal{L}(V)$ is such that $p_T(x)$ factors into linear factors. Let $\lambda_1, \dots, \lambda_m$ be the distinct eigenvalues of T and let $U_1, \dots, U_m$ be the corresponding spaces of generalized eigenvectors. Then*

$$V = U_1 \oplus \cdots \oplus U_m.$$

It follows from the Decomposition Theorem and the fact that each U_i is T-invariant that if $\mathcal{B}$ is the basis obtained by concatenating the bases $\mathcal{B}_1, \dots, \mathcal{B}_m$, where $\mathcal{B}_i$ is any basis for U_i, then $\mathcal{M}(T, \mathcal{B})$ is block diagonal with diagonal blocks

$$\mathcal{M}(T|_{U_1}, \mathcal{B}_1), \dots, \mathcal{M}(T|_{U_m}, \mathcal{B}_m).$$

We can use the fact that each $(T - \lambda_i I)|_{U_i}$ is nilpotent to choose the basis $\mathcal{B}_i$ so that the diagonal blocks have a nice form. Indeed, from Lemma 8.26 and the Decomposition Theorem we get,

Theorem 9 (see Th. 8.28)**.** *Suppose $T \in \mathcal{L}(V)$ is such that $p_T(x)$ factors into linear factors. Let $\lambda_1, \dots, \lambda_m$ be the distinct eigenvalues of T. Then there is a basis $\mathcal{B}$ for V such that $\mathcal{M}(T, \mathcal{B})$ is block diagonal with diagonal blocks $A_1, \dots, A_m$, where A_i is an $m_T(\lambda_i) \times m_T(\lambda_i)$ upper triangular matrix with diagonal entries equal to λ_i for each i.*

The following strengthening of Lemma 8.26 can be used to improve Theorem 9.

Lemma 10. *Let N be a nilpotent operator on V. Then there is a basis $\mathcal{B}$ of V such that $\mathcal{M}(N, \mathcal{B})$ is a block diagonal matrix with diagonal blocks $A_1, \dots, A_d$, where A_i is of the form*

$$\begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix}$$

and $d = \dim \operatorname{null} N$.

A *Jordan block* is a matrix of the form

$$\begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}$$

A block diagonal matrix is said to be in *Jordan form* if each diagonal block is a Jordan block. In class I gave the following example of a matrix in Jordan form

$$\begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & A_3 & 0 \\ 0 & 0 & 0 & A_4 \end{bmatrix}$$

where $A_1 = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix}$, $A_2 = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$, $A_3 = [7]$, $A_4 = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

Now the Decomposition Theorem and Lemma 10 yield,

Theorem 11. *Suppose $T \in \mathcal{L}(V)$ is such that $p_T(x)$ factors into linear factors. Then there exists a basis $\mathcal{B}$ of V such that $\mathcal{M}(T, \mathcal{B})$ is in Jordan form. Moreover the number of Jordan blocks with λ on the diagonal is* $\dim \operatorname{null}(T - \lambda I)$.

Theorem 12. *The Jordan form of $T \in \mathcal{L}(V)$ is unique up to rearrangement of the Jordan blocks.*