PRACTICE PROBLEMS FOR THE FINAL EXAM

1. Let C be the portion of the helix $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 4t\,\mathbf{k}$ between $t = 0$ and $t = \pi$, and let $\mathbf{F} = x\,\mathbf{i} + z\,\mathbf{k}$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

2. Compute work done by vector field $\mathbf{F} = (-yx^2 + e^{x^2})\,\mathbf{i} + (xy^2 - e^{y^2})\,\mathbf{j}$ in moving an object along the circle $x^2 + y^2 = 1$ once in the counterclockwise direction.

3. Let $\mathbf{F} = ye^{xy}\,\mathbf{i} + (xe^{xy} - z^3)\,\mathbf{j} + (2\sin z - 3yz^2)\,\mathbf{k}$. (a) Evaluate curl F. (b) Find a function f such that $\mathbf{F} = \nabla f$. (c) Find work done by $\mathbf{F}$ along the straight line segment from $(0, 5, 0)$ to $(1, 0, \pi)$.

4. Find the work done by the force field $\mathbf{F}(x, y, z) = z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$ in moving a particle from the point $(3, 0, 0)$ to the point $(0, \pi/2, 3)$ (a) along a straight line and (b) along the helix $x = 3\cos t$, $y = t$, $z = 3\sin t$. Is this force field conservative? Justify your answer.

5. Consider the vector field

$$\mathbf{F} = \frac{-y}{x^2 + y^2}\,\mathbf{i} + \frac{x}{x^2 + y^2}\,\mathbf{j}\,.$$

(a) Evaluate directly the line integral of $\mathbf{F}$ along the unit circle, once around in the counterclockwise direction. Is $\mathbf{F}$ conservative?
(b) Compute the curl of $\mathbf{F}$. Why does your answer not contradict Green's theorem?

6. Let C_1 be the unit circle $x^2 + y^2 = 1$ and C_2 the concentric circle of radius two. Orient both C_1 and C_2 counterclockwise. Suppose that $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ is a vector field in plane such that

$$\int_{C_1} \mathbf{F} \cdot \mathbf{n}\, ds = 10 \quad \text{and} \quad \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 17.$$

(a) If $\mathbf{F}$ is smooth in the plane, compute $\iint_D \operatorname{div} \mathbf{F}\, dA$, where D is the domain defined by the inequality $x^2 + y^2 \le 1$.
(b) If $\mathbf{F}$ is smooth on the annulus bounded by C_1 and C_2 and $Q_x = P_y$ everywhere on the annulus, compute $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$.

7. Find the center of mass:

(a) of the thin wire of density 1 bent into the shape of cycloid $\mathbf{r}(t) = (t - \sin t)\,\mathbf{i} + (1 - \cos t)\,\mathbf{j}$, $0 \le t \le 2\pi$.

(b) of the part of the spherical surface $x^2 + y^2 + z^2 = 25$ above the plane $z = 3$, assuming that its density at (x, y, z) equals z.

8. Evaluate surface integral $\iint_S x^2 y\, dS$ over the portion S of the cylinder $x^2 + y^2 = 4$ between the planes $z = 0$ and $z = 3$.

9. Find the flux of $\mathbf{F} = x\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$ across the hemisphere $x^2 + y^2 + z^2 = 25$, $y \ge 0$, oriented in the direction of the positive y–axis.

10. Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + xz\,\mathbf{k}$ and C is the triangle with the vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ oriented counterclockwise as viewed from above.

11. Use the divergence theorem to calculate the outward flux of $\mathbf{F}(x, y, z) = x^3\,\mathbf{i} + y^3\,\mathbf{j} + z^3\,\mathbf{k}$ across the surface of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = 2$.

12. Compute the outward flux of $\mathbf{F}(x, y, z) = (x^2 + y^2 + z^2)^{-3/2}(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$ across the ellipsoid $4x^2 + 9y^2 + 6z^2 = 36$. (Hint: wouldn't it be easier to compute the flux across a sphere?)

SOLUTIONS

1. $8\pi^2$. **2.** $\pi/2$. **3.** (a) 0; (b) $e^{xy} - yz^3 - 2\cos z + C$; (c) 4.

4. (a) $(3\pi - 9)/2$. (b) $-3\pi/4$. The force field is not conservative because $\int_C \mathbf{F} \cdot d\mathbf{r}$ is not path independent.

5. (a) 2π, $\mathbf{F}$ is not conservative. (b) curl $\mathbf{F} = 0$. This does not contradict Green's Theorem because $\mathbf{F}$ is not continuous at the origin.

6. (a) 10; (b) 17. **7.** (a) $(\pi, 4/3)$; (b) $(0, 0, 49/12)$. **8.** 0. **9.** $250\pi/3$.

10. $-1/2$. **11.** 11π. **12.** 4π.