Stanley's conjecture about the h-vector of a Matroid

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Why should we care about Matroids

Matroids arise naturally when we are studying:

- Linear independence of finite sets of vectors.
- Cycles of graphs.
- Matchings in bipartite graphs.
- Shellable simplicial complexes.
- Greedy algorithms.
- Algebraic independence in algebraic extensions.

Simplicial Complexes

Simplicial Complex

A **simplicial complex** is an ordered pair $\Delta = (E, I)$, $|E| < \infty$, $I \subseteq \mathcal{P}(E)$ such that:

i. $\emptyset \in I$

ii. If $B \subset A$ and $A \in I$, then $B \in I$.

The elements of I are called **faces**.

Basic concepts

- If $\Delta = (E, I)$ is a complex, then $P = (I, \subseteq)$ is a poset.
- A simplicial complex is called **pure** if all the maximal elements of P have the same size.
- For each $i \in \mathbb{Z}^{\geq -1}$ define f_i as the number of elements I that have $i + 1$ elements.
- The f-**vector** of a complex is given by $(f_{-1}, f_0, \ldots, f_k)$, where $k + 1$ is the size of a maximal set in I.
- The polynomial $f_\Delta(x) = \sum_{j=0}^{k+1} f_{j-1} x^{k+1-j} = \sum_{A \in I} x^{k+1-|A|}$ is called the f- **polynomial** of Δ.

Multicomplexes

We can extend the definitions pairs (E, I) where the elements of I are multisets whose ground set is a subset of E. The new defined structure is a multicomplex. The involved multisets are in bijection with the monomials in $|E|$ variables and with elements in $\mathbb{N}^E$. The biyection is natural.

Order ideal

Let A_n be the set of monomials in the variables $\{x_i\}_{i=1}^n$. An **order ideal** C of A_n is a finite subset of A_n such that if $m \in C$ and $m'|m$, then $m' \in C$. And order ideal **pure** if all the maximal monomials of the poset $(C, |)$ have the same degree.

O-sequences

O-sequence

A vector whose coordinates are natural numbers is an O-**sequence** if it is the f-vector of and order ideal. An O-sequence is called **pure** if the involved multicomplex ie pure.

Matroids

Matroid

A **matroid** is an ordered pair $M = (E, \mathcal{I})$, where E is a finite set and $\mathcal{I}$ is a subset of $\mathcal{P}(E)$ such that :

(I1) $\emptyset \in \mathcal{I}$.

(I2) If $A \in \mathcal{I}$ and $B \subset A$, then $B \in \mathcal{I}$.

(I3) If A, $B \in \mathcal{I}$ and $|A| > |B|$, then there exists $a \in A - B$ such that $B \cup \{a\} \in \mathcal{I}$.

The elements of $\mathcal{I}$ are called **independent** sets.

Example

Linear matroids

Let E be the columns of a matrix M and let $\mathcal{I} \subseteq \mathcal{P}(E)$ be such that $A \in \mathcal{I}$ if the columns corresponding to A are linearly independent. Then $M = (E, \mathcal{I})$ is a matroid. Matroids of this type are called **linear matroids**.

Example

A matroid in $\mathbb{R}^3$

Let the columns of a matrix be
$v_1 = (1,0,0)$, $v_2 = (1,1,0)$, $v_3 = (0,0,1)$, $v_4 = (2,0,0)$, $v_5 = (0,1,2)$, $v_6 = (0,0,0)$ and $v_7 = (1,1,1)$. The matroid $M = (E,\mathcal{I})$ induced has groundset $E = \{1,2,3,4,5,6,7\}$ and independents equal to:

$$
\begin{aligned}
\mathcal{I} \quad = \quad & \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{7\}, \\
& \{1,2\}, \{1,3\}, \{1,5\}, \{1,7\}, \{2,3\}, \{2,4\}\{2,5\}, \\
& \{2,7\}, \{3,4\}, \{3,5\}, \{3,7\}, \{4,5\}, \{4,7\}, \{5,7\}, \\
& \{1,2,3\}, \{1,2,5\}, \{1,2,7\}, \{1,3,5\}, \{1,3,7\}, \\
& \{1,5,7\}, \{2,3,4\}, \{2,3,5\}, \{2,3,7\}, \{2,5,7\} \\
& \{3,4,5\}, \{3,4,7\}, \{3,5,7\}, \{4,5,7\}\}
\end{aligned}
$$

Bases

Now we study the maximal elements of $\mathcal{I}$. They are called **bases**.

Theorem

All the bases of a matroid have the same size. That means that a matroid is a pure simplicial complex.

Theorem

A set $\mathcal{B} \subseteq \mathcal{P}(E)$ is the family of bases of a matroid if and only if

(B1) $\mathcal{B}$ is not empty.

(B2) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 - B_2$, then there is $y \in B_2 - B_1$ such that $(B_1 - x) \cup \{y\} \in \mathcal{B}$

Rank

There is also a rank function associated to a matroid. Let $r : \mathcal{P}(E) \to \mathbb{N}$ so that $r(A)$ is the size of a maximal independent contained in A.

Theorem

A function $r : \mathcal{P}(E) \to \mathbb{N}$ is the rank function of a Matroid if and only if:

(R1) For every $U \subseteq E$ the inequality $0 \leq r(U) \leq |U|$ holds.

(R2) If $W \subseteq U \subseteq E$, then $r(W) \leq r(U)$

(R3) If U, W are subsets of E then

$$r(U \cup W) + r(U \cap W) \leq r(U) + r(W)$$

Duality

The dual matroid

If r is the rank function of a matroid $M = (E, \mathcal{I})$, then the function $r^* : \mathcal{P}(E) \to \mathbb{N}$ such that

$$r^*(U) = |U| - r(E) + r(E - U)$$

is the rank function of a matroid M^* called the **dual matroid** of M.

Dual Bases

The set $\mathcal{B}^* = \{E - B \mid B \in \mathcal{B}\}$ is the set of bases of M^*.

Duality and linear matroids

The dual of a linear matroid

If $M = (E, i)$ is a linear matroid of rank r and $|E| = n$, then E is generated by a matrix of the form

$$A = \left(\begin{array}{c|c} I_r & D \end{array} \right)$$

Then the matroid M^* is generated by the columns of:

$$B = \left(\begin{array}{c|c} D^T & I_{n-r} \end{array} \right)$$

Example

The matrix from the example is equivalent to

$$\begin{pmatrix} 1 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 & 0 & 1 \end{pmatrix}$$

Thus the dual of the matroid is given by the columns of:

$$\begin{pmatrix} 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Deletion and contraction

Deletion

The **deletion** of e, denoted by $M \backslash e$, is the matroid whose ground set is $E - \{e\}$, and whose indepentent sets are the elements of $\mathcal{I} \cap \mathcal{P}(E - e)$.

Contraction

The **contraction** of e, denoted by M/e, is the dual matroid of the deletion of e in M^*. Written in symbols we have that $M/e = (M^* \backslash e)^*$.

TG Invariants

The Tutte-Grothendieck invariants are a great tool to construct objects recursively.

Tutte-Grothendieck invariants

A **Tutte-Grothendiek (o TG) invariant** is a map f from the class of matroids to a commutative ring with unity R, that satisfies the following conditions:

1. $f(M_1) = f(M_2)$ if $M_1 \cong M_2$
2. $f(M) = f(M \backslash e) + f(M/e)$ if e is neither a loop nor a coloop.
3. $f(M) = f(e) f(M \backslash e)$ if e is a loop or a coloop.

The Tutte polynomial

Theorem

There exists a unique TG invariant $T : Matroids \to \mathbb{Z}[x, y]$ such that $T(M_{coloop}; x, y) = x$ y $T(M_{loop}; x, y) = y$. Moreover, for every TG invariant f and every matroid M, the following equality holds:

$$f(M) = T(M; f(M_{coloop}), f(M_{loop}))$$

The Tutte polynomial

$T(M, x, y)$ is called the **Tutte polyonomial** of M and is given by:

$$T(M; x, y) := \sum_{A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{|A|-r(A)}$$

The dual polynomial

It is easy to show that $T(M^*, x, y) = T(M, y, x)$

Examples

We now introduce some nice TG invariants.

- $T_{\mathcal{B}} : Matroids \to \mathbb{Z}$ that assigns the number of bases to each matroid. It is given by the evaluation$T(M, 1, 1)$ of the Tutte polynomial.
- $T_{\mathcal{I}} : Matroids \to \mathbb{Z}$ that assigns the number of indepentents to each matroid. It is given by the evaluation $T(M, 2, 1)$ of the Tutte polynomial.
- $T_f : Matroides \to \mathbb{Z}[x]$ that assigns the f-polynomial to each matroid. It is given by the evaluation $T(M, x+1, 1)$ of the Tutte polynomial.

Shellable complexes

Shellings

A **shelling** of a pure simplicial complex Δ is a linear order of it's maximal faces $F_1, F_2, \ldots, F_t$ such that for any pair of numbers $1 \leq i < j \leq t$ there exist $x \in F_j$ and $k < j$ such that $F_i \cap F_j \subseteq F_j \cap F_k = F_j - x$. A complex is **shellable** if it has a shelling.

Example

$\Delta = ([4], I)$, $A \in I \iff |A| \leq 2$. The order $\{1,2\} < \{1,3\} < \{1,4\} < \{2,3\} < \{2,4\} < \{3,4\}$ is a shelling. If the two smallest terms are $\{1,2\}$ and $\{3,4\}$, then the order is not a shelling.

A partition of the simplicial complex

$F_1 < \cdots < F_t$ is a shelling. Let $\mathcal{R}_j = \{x \in F_j | F_j - x \in \Delta_{j-1}\}$ and let $[\mathcal{R}_j, F_j] = \{G \subseteq F_j \,|\, \mathcal{R}_j \subseteq G\}$.

Lemma

We have that $\Delta_{j+1} - \Delta_j = [\mathcal{R}_{j+1}, F_{j+1}]$. In other words, the set $\{[\mathcal{R}_j, F_j] \,|\, 1 \leq j \leq t\}$ is a partition of I.

Corollary

If d is the dimension Δ, then

$$f_k = \sum_{j=1}^{t} \binom{d + 1 - |\mathcal{R}_j|}{k + 1 - |\mathcal{R}_j|}$$

The h-vector

h-polynomial

The h-**polynomial** of a shellable complex Δ is given by

$$h_\Delta(x) = \sum_{j=1}^{t} x^{|F_j - \mathcal{R}_j|} = \sum_{j=0}^{d+1} h_j x^{d+1-j}$$

h-vector

The vector $(h_0, h_1, \ldots, h_{d+1})$ of coefficients of the h-polynomial is called the h-**vector** of Δ.

h-vector vs. f-vector

Lemma

(1) $h_\Delta(x+1) = f_\Delta(x)$

(2) $h_i = \sum_{k=i}^{d+1} (-1)^{k-1} \binom{d+1-k}{k-i} f_k$

A nice fact about matroids is that they are shellable so their h-vector is non-negative.

Corollary

The h-polynomial of a matroid is a TG invariant, given by the evaluation $T(M, x, 1)$ of the Tutte polynomial.

Stanley's conjecture

Conjecture

The h-vector of a matroid is a pure O-sequence.

Example

For the old example: $h_M = (1, 3, 5, 5)$. Let

$$C = \{1, x, y, z, x^2, xy, xz, y^2, z^2, x^3, x^2y, x^2z, y^3, z^3\}$$

Note that C is an pure order ideal, thus it's f-vector is a pure O-sequence. Thus the conjecture is true for M.

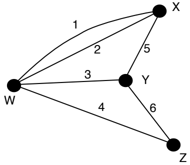
Graphical matroids

Each graph induces a matroid naturally. We allow graphs to have loops and multiedges.

Graphic matroid

Let $G = (V, E, i)$ be a graph, define $\mathcal{I}(G)$ as the family of sugraphs that contain no cycles. Then $M(G) = (E, \mathcal{I}(G))$ is a matroid. Matroids of this type are called **graphical matroids**.

Example



The independets of the graph are given by:

$$\begin{aligned}\mathcal{I}(G) \quad = \quad & \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\} \\ & \{1,3\}, \{1,4\}, \{1,5\}, \{1,6\}, \{2,3\}, \{2,5\}, \{2,6\}, \\ & \{3,4\}, \{3,5\}, \{3,6\}, \{4,5\}\{4,6\}, \{5,6\}, \\ & \{1,3,4\}, \{1,3,6\}, \{1,4,5\}, \{1,4,6\}, \{1,5,6\}, \\ & \{2,3,4\}, \{2,3,6\}, \{2,4,5\}, \{2,4,6\}, \{2,5,6\}, \\ & \{3,4,5\}, \{3,5,6\}, \{4,5,6\}\}\end{aligned}$$

Special matrices

Incidence matrix

For a graph $G = (V, E, i)$, let $<$ be an order of V. For every edge e, such that $i(e) = \{a, b\}$,with $a < b$, let $v_e \in \mathbb{F}^V$ be equal to $e_a - e_b$. If $|i(e)| = 1$, let $v_e = 0$. The **incidence matrix** of G is the matrix whose columns are $\{v_e\}_{e \in E}$.

Laplacian Matrix

The **Laplacian Matrix** Q_G of a graph G is the matrix in $\mathcal{M}_{V\times V}(\mathbb{F})$ such that $Q_{v,v'} = -\nu(v, v')$ if $v \neq v'$ and $Q_{v,v} = exdeg(v)$.

Theorem

$Q_G = A_G A_G^T$ and ker$(Q_G) = \ker A_G^T = (1)_{v\in V}$

Example

The incidence matrix if $W > X > Y > Z$ is:

$$A_G = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & 0 & -1 \end{pmatrix}$$

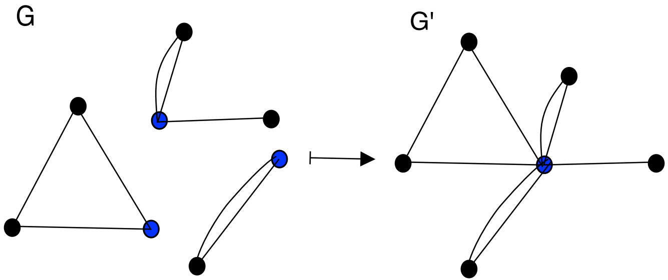
The Laplacian of G is:

$$Q_G = \begin{pmatrix} 4 & -2 & -1 & -1 \\ -2 & 3 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$$

Connected graphs

Theorem

For every graph $G = (V, E, i)$ there exists a connected graph $G' = (V', E', i')$ such that $M(G) \cong M(G')$.



Chip firing game

It is played in a connected graph $G = (V, E, i)$. The idea is to choose a special vertex q and give chips to all the other vertices. The special vertex is a bank and the other spend the coins following certain rules. Formally, we define the game as follows. This version of the chip firing game is also known as the dollar game.

q-configurations

A q-**configuration** of G is a function $\theta : V \to \mathbb{Z}$ such that $\theta(v) \geq 0$ if $v \neq q$ and $\theta(q) = -\sum_{v \neq q} \theta(v)$.

Chip firing game

Firings

A vertex $v \neq q$ is **ready** in θ if $\theta(v) \geq deg(v)$. q is **ready** if no other vertex is. A **firing** of a vertex v that is ready is a configuration θ' such that $\theta'(v') = \theta(v') + \nu(v, v')$ if $v' \neq v$ and $\theta'(v) = \theta(v) - exdeg(v)$.

Secuencias legales

A **legal sequence** from θ to θ' is a non empty sequence of firings that may start in θ to reach θ'. In case such a legal sequence exists, we write $\theta \to \theta'$. We denote legal sequences with $\mathcal{X}$ and $\mathcal{X}(v)$ denotes de number of times v is fired in $\mathcal{X}$.

An algebraic interpretation

Note that if $\theta \to \theta'$ using a sequence $\mathcal{X}$, then $\theta' = \theta - Q_G(\mathcal{X}(v))_{v\in V}$, where Q_G is the Laplacian matrix. In particular, if $\theta = \theta'$, then $(\mathcal{X}(v))_{v\in V}$ lies in the kernel of Q_G, so it is a scalar (actually integer) multiple of $(1)_{v\in V}$. This means that every vertex is fired the same number of times.

Special configurations

- A q-configuration θ is **stable** if q is ready.
- A q-configuration θ is **recurrent** if $\theta \to \theta$, that is, there exists a legal sequence that begins and ends in θ.
- A q configuration θ is **critical** if it is both, stable and recurrent.

Teorema

If we start to play the chip firing game at any point we eventually reach a critical configuration.

Some legal sequences

Theorem

Let θ be a q-configuration. θ is critical if and only if it is stable and there exists a legal sequence $\mathcal{X}$ of θ such that $\mathcal{X}(v) = 1$ for all $v \in V(G)$.

This theorem is useful to compute critical configurations. It has also nice theoretic applications, some of which will be discussed later.

The structure of the critical configurations

The weight of a configuration

The **weight** $w(\theta)$ of a configuration θ is the amount of chips in θ, that is, $w(\theta) = -\theta(q) = \sum_{v \in V-q} \theta(v)$.

Minimal configurations

Order the configurations partially by comparing the coordinates different from q. All the minimal configurations have the same weight:

$$w(c) = |E| - \deg(q) = \frac{\sum_{v \in V} exdeg(v)}{2} + \sum_{v \in V} indeg(v) - \deg(q)$$

The structure of the critical configurations

An order property

Let c be a critical configuration and let c' be a stable configuration such that $c(v) \leq c'(v)$ for $v \neq q$. Then c' is critical.

An order ideal

Let $\mathcal{C}$ be the set of critical configurations ignoring the coordinate q. Let C be a configuration such that $C(v) = \deg(v) - 1$ for $v \neq q$. Define $f : \mathcal{C} \to \mathbb{N}^{|V|-1}$ so that $f(c) = C - c$. The set $S_G = \{\mathbf{x}^{f(c)} \mid c \in \mathcal{C}\}$ is a pure order ideal.

The critical polynomial

The critical polynomial

The **level** $lev(c)$ of a critical configuration c is defined as $lev(c) := \deg(\mathbf{x}^{f(c)}) = w(C) - w(c)$ and let $\Gamma(G) := |E| - |V| + 1$. The **critical polynomial** $P_q(G, y)$ of G is defined as

$$P_q(G,y) := \sum_{c \in \mathcal{C}} y^{\Gamma(G) - lev(c)} := \sum_{i=0}^{\Gamma(G)} c_i x^{\Gamma(G)-i}$$

The conjecture for cographical matroids

Theorem

The critical polynomial is independent of the choice of q. Moreover, we have that $P_q(G, y) = T(G; 1, y)$.

This implies that $P_q(G, y) = T(M(G)^*; y, 1) = h_{M(G)^*}(y)$, thus

Theorem

$h_i^* = c_i$, so the h-vector of $M(G)^*$ is a pure O-sequence.

Example

In the graph we worked before the minimal critical configurations of $\mathcal{C}$ are $(1,1,0)$, $(1,0,1)$, $(0,2,0)$ and $(0,1,1)$. Furthermore, the configuration C is $(2,2,1)$. It follows that the maximal elements of the set $f(\mathcal{C})$ are $(1,1,1)$, $(1,2,0)$, $(2,0,1)$ y $(2,1,0)$. So we get that:

$$S_G = \{1, x, y, z, x^2, xy, xz, y^2, yz, x^2y, x^2z, xy^2, xyz\}$$

$M(G)$ is self dual, thus the f-vector of $M(G)^*$ is given $(1,6,13,13)$, and computing the h-vector we obtain it is equal to $(1,3,5,4)$.

The End