Combinatorics of polytopes that approximate smooth convex bodies

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1. Introduction

Let P be a simplicial d-polytope.

- Let $\mathbf{f} = (f_{-1}, f_0, \dots f_{d-1})$ be such that $f_{-1} = 1$ and f_i is the number of faces F of P with $\dim F = i$
- Let $\mathbf{h} = (h_0, h_1, \dots, h_d)$ be given by
$$\sum_{i=0}^{d} h_i t^i = \sum_{i=0}^{d} f_{i-1} t^i (1-t)^{d-i}$$
The Dehn-Sommerville relations say that $h_i = h_{d-i}$.
- Let $\mathbf{g} = (g_0, g_1, \dots, g_{\lfloor \frac{d}{2} \rfloor})$ be given by $g_0 = h_0$ and $g_i = h_i - h_{i-1}$.

Theorem (g-theorem)**.** *An integer vector $(g_0, g_1, \dots, g_{\lfloor \frac{d}{2} \rfloor})$ is the $\mathbf{g}$-vector of a simplicial polytope P if and only if the following conditions hold:*

- $g_0 = 1$
- $g_k \geq 0$ *for* $k = 1, 2, \dots \lfloor \frac{d}{2} \rfloor$.
- $g_k \geq \partial^k g_{k+1}$ *for* $k = 1, 2, \dots, \lfloor \frac{d}{2} \rfloor - 1$.

Q: The g-theorem's proof does not use any information from the metric properties of P. Can we extract more information about g by imposing some metric restrictions on P?

2. Smooth convex bodies and the Hausdorff metric

A smooth convex body M is a compact convex subset of $\mathbb{R}^d$ with non empty interior and such that the boundary ∂M is C^1.
For $A, B \subseteq \mathbb{R}^d$ bounded, the Hausdorff distance between them is given by:
$$\delta^H(A, B) := \max \left\{ \sup_{a \in A} \inf_{b \in B} |a - b| \, , \, \sup_{b \in B} \inf_{a \in A} |b - a| \right\}$$
This distance defines a metric on the family of bounded subsets of $\mathbb{R}^d$.

3. Kalai's conjecture

Conjecture 3.1 (Kalai [1])**.** *Let M be a smooth convex body and let $\{P_n\}_{n=1}^{\infty}$ be a sequence of simplicial polytopes that converges to M in the Hausdorff metric. Then the following holds:*

a. For $k = 1, 2, \dots, \lfloor \frac{d}{2} \rfloor$
$$\lim_{n \to \infty} g_k(P_n) = \infty$$
b. For $k = 1, 2, \dots, \lfloor \frac{d}{2} \rfloor - 1$
$$\lim_{n \to \infty} g_k(P_n) - \partial^k(g_{k+1}(P_n)) = \infty$$

4. Stackedness and the short g-vector

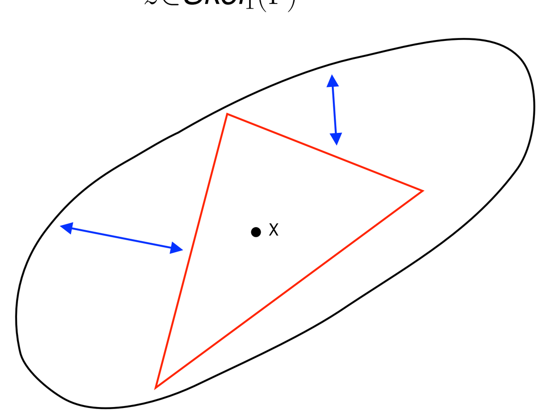
Let P be a simplicial d-Polytope with boundary complex Δ and let $2 \leq r \leq \lfloor \frac{d}{2} \rfloor$. Let $\Delta(r-1)$ be the simplicial complex generated by the set
$$\{\Gamma \subset V(P) \,:\, |\Gamma| = d + 1\, , \, \mathrm{Skel}_{(d-r)} 2^{\Gamma} \subset \Delta\}$$
A polytope is $(r-1)$-stacked if $\Delta(r-1)$ is a triangulation of P.
Theorem 4.1 (Murai-Nevo [2])**.** *The following are equivalent:*

- $g_r(P) = 0$
- *P is $(r-1)$-stacked.*

Let $\tilde{\mathbf{h}} := \left(\tilde{h}_0, \tilde{h}_1, \dots, \tilde{h}_{\lfloor \frac{d-1}{2} \rfloor}\right)$, where
$$\tilde{h}_k := \sum_{v \in V(P)} h_k(\mathrm{Lk}(v))$$
It is well known that
$$\tilde{h}_{k-1} := k h_k + (d + 1 - k) h_{k-1}$$
Now let $\tilde{\mathbf{g}} := \left(\tilde{g}_0, \tilde{g}_1, \dots, \tilde{g}_{\lfloor \frac{d-1}{2} \rfloor}\right)$ where we put $\tilde{g}_0 := \tilde{h}_0$ and for $k = 1, \dots, \lfloor \frac{d-1}{2} \rfloor$:
$$\tilde{g}_k = \tilde{h}_k - \tilde{h}_{k-1} = \sum_{v \in V(P)} g_k(\mathrm{Lk}(v))$$
A similar computation to the one done in [3] for g_2 leads to
$$\tilde{g}_k = (k+1) g_{k+1} + (d + 1 - k) g_k \leq K(g_k)$$
Then we get that at most $K(g_k)$ vertices v can satisfy $g_k(\mathrm{Lk}(v)) > 0$, so all but $K(g_r)$ vertices have $(r-1)$-stacked links. So $\Delta(k-1)$ covers at least $|V(P)| - K(g_k)$ vertices.

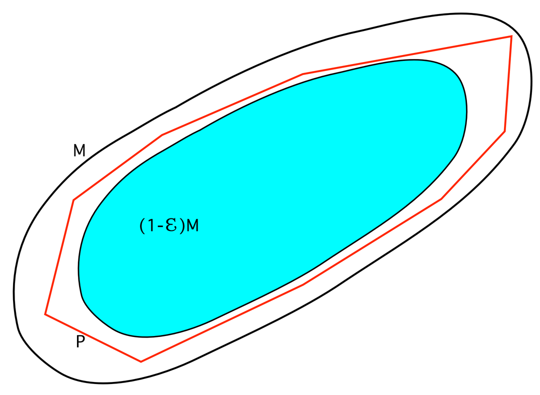
5. Metric properties of polytopes and approximations

Lemma 5.1. *Let M be a smooth convex body and let x be a point in the interior of M. Let A be family of all simplices that contain x. Then*
$$\Psi(x) := \inf_{\Gamma \in A} \sup_{z \in \mathrm{Skel}_1(\Gamma)} \inf_{m \in \partial M} |z - m| > 0$$

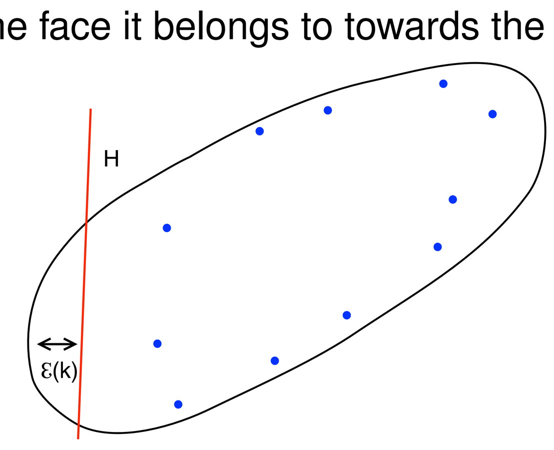


Sketch. First we construct a compact space A' of matrices that contains a subset that parametrizes A. This can be done by putting the vertices of the simplices as columns of the matrix and assuming $x = 0$. Then it is easy to construct a map $f : A' \to \mathbb{R}$ such that f is continuous and such that for $\Gamma \in A$ we have $f(\Gamma) = sup_{z \in \mathrm{Skel}_1(\Gamma)} \inf_{m \in \partial M} |z - m|$: For each pair of columns take the distance between the line that joins them and ∂M and then take the maximum of those distances.
Since A' is compact f has a minimum attained by some X. $f(X)$ is clearly non-negative and if it is zero, then ∂M contains the graph of the polytope whose vertices are the columns of X. This contradicts the smoothness of ∂M. □

Corollary 5.2. *Let M be a smooth convex body and let U be a closed set with $U \subset \mathrm{int}(M)$. Then*
$$\inf_{x \in U} \Psi(x) > 0$$
Lemma 5.3. *Let M be a smooth convex body and let $\{P_n\}_{n=1}^{\infty}$ be a sequence of polytopes that converges to M.*
$$\lim_{n \to \infty} \sup_{z \in \mathrm{Skel}_1(\partial P_n)} \inf_{m \in \partial M} |z - m| = 0$$
Sketch. Assume again that $0 \in \mathrm{int}(M)$. A simple continuity argument shows that for $\epsilon > 0$ there is $\delta > 0$ such that if P is a polytope with $\delta^H(P, M) < \delta$ then $(1 - \epsilon)M \subsetneq P$. Then $\partial P_n \subseteq M \backslash (1 - \epsilon)M$ and this yields the result. □

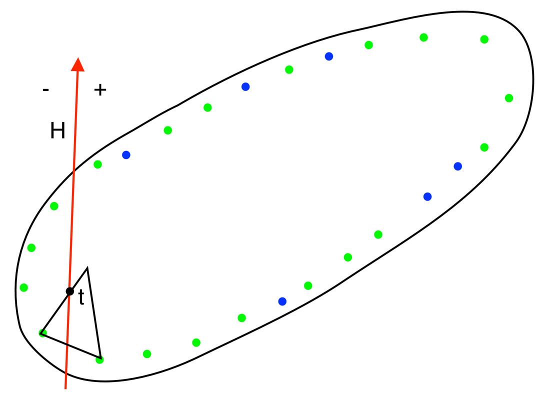


Lemma 5.4. *Let M be a smooth convex body and let $k > 0$ be an integer. There exists a real number $\epsilon(k) > 0$ such that for every $A \subset M$ with $|A| \leq k$ there exists an affine hyperplane H_A with $A \subseteq H_A^+$ and such that*
$$\sup_{x \in M \cap H_A} \inf_{y \in \partial M \cap H_A^-} |x - y| \geq \epsilon(k)$$
Sketch. It is easy to check that there exists $t(k)$ such that if P is a polytope with $P \subset M$ and has at most k vertices then $\sup_{x \in \partial P} \inf_{y \in \partial M} |x - y| > t$. To do this just parametrize all polytopes with k vertices in the columns of a matrix as in 5.1. Then $\epsilon(k) = \frac{t(k)}{2}$ works by taking the affine span of the point in ∂P whose distance to ∂M is at least $t(k)$ and translating a supporting hyperplane of the face it belongs to towards the boundary. □



6. Proof of part a. of the conjecture if $2k < d$

Now we prove the conjecture using the machinery we have developed. Assume that $\{P_n\}_{n=1}^{\infty}$ is a family of polytopes that converges to a smooth convex body M and that there is a constant $g \geq 0$ such that $g_k(P_n) \leq g$ for all n. Let Δ_n be the boundary complex of P_n.
Let $\epsilon := \epsilon(K(g))$ and let $M_\epsilon = \{x \in M \,:\, \inf_{y \in \partial M} |x - y| \geq \epsilon\}$ then M_ϵ is compact and contained in the interior of M. We claim that for n sufficiently large there is a simplex Γ_n of $\Delta_n(r-1)$ such that $\Gamma_n \cap M_\epsilon \neq \emptyset$. Each P_n has at most $K(g)$ vertices whose link is not $(r-1)$-stacked, so there is a hyperplane H_n such that $\mathrm{int}(H_n^+)$ contains all the non-stacked vertices and
$$\sup_{x \in H_n \cap M} \inf_{y \in \partial M \cap H_n^-} |x - y| > \epsilon$$
Let V_n be the set of vertices of P_n in $H_n^- \cap \partial M$ and let $N(V_n)$ be the set of vertices of the union of the stars of elements of V_n. Let $\Sigma_n = \Delta_n(r-1)|_{N(V_n)}$. Since all the vertices in V_n have $(r-1)$-stacked links in P_n, the vertex set of Σ_n is indeed $N(V_n)$. We now claim that $P_n \cap H_n = \Sigma_n \cap H_n$.
It suffices to show that LHS⊆RHS. For this pick a generic point u in $P_n \cap H_n$, a generic line $\ell \subseteq H_n$ with $u \in \ell$ and a parametrisation $\gamma : [0, 1] \to \ell \cap P_n$. Assume that $\ell \cap P_n \not\subset \Sigma_n$. Let $t = \inf\{s \in [0, 1] \,:\, \gamma(s) \notin \Sigma_n\}$. Note that $\gamma(0) \in \partial(P_n)$, so $\gamma(0) \in \Sigma_n$ and since Σ_n is closed $\gamma(t) \in \Sigma_n$. By genericity $\gamma(t)$ belongs to the relative interior of a $(d-1)$-face and that face is connected to two d-faces of Σ_n, so it is interior in Σ_n which implies that t is not the infimum. Generic points are dense in $P_n \cap H_n$, so the inclusion is true by taking closures.
It follows that $\Sigma_n \cap H_n = P_n \cap H_n$, and if n is large enough, $M_\epsilon \subseteq P_n$, so $\emptyset \neq M_\epsilon \cap H_n \subseteq P_n \cap H_n = \Sigma_n \cap H_n$, so Σ_n contains a simplex that intersects M_ϵ. This implies that $\mathrm{Skel}_1(P_n)$ has to be far from ∂M, 5.2, and contradicts the fact that $\{P_n\}_{n=1}^{\infty}$ converges to M by 5.3.



References

[1] Gil Kalai. Some aspects of the combinatorial theory of convex polytopes. In *Polytopes: abstract, convex and computational (Scarborough, ON, 1993)*, volume 440 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 205–229. Kluwer Acad. Publ., Dordrecht, 1994.

[2] Satoshi Murai and Eran Nevo. On the generalized lower bound conjecture for polytopes and spheres. *Acta Math.*, 210(1):185–202, 2013.

[3] Ed Swartz. Topological finiteness for edge-vertex enumeration. *Adv. Math.*, 219(5):1722–1728, 2008.