SPATIAL AND TEMPORAL DYNAMICS OF A NONLOCAL VIRAL INFECTION MODEL*

GUANGYU ZHAO† AND SHIGUI RUAN‡

Abstract. Recent studies suggest that spatial heterogeneity plays an important role in the within-host infection of viruses such as HBV, HCV, and HIV. In this paper we propose a spatial model of viral dynamics on a bounded domain in which virus movement is described by a nonlocal (convolution) diffusion operator. The model is a spatial generalization of a basic ODE viral infection model that has been extensively studied in the literature. We investigate the principal eigenvalue of a perturbation of the nonlocal diffusion operator and show that the principal eigenvalue plays a key role similar to that of the basic reproduction number when it comes to determining the infection dynamics. Through analyzing the spectra of two matrix operators, it is shown that the model exhibits threshold dynamics. More precisely, if the principal eigenvalue is less or equal to zero, then the infection-free steady state is asymptotically stable while there is an infection steady state which is stable provided that the principal eigenvalue is greater than zero.

Key words. Nonlocal diffusion operator, spatial model, viral infection, principal eigenvalue, stability

AMS subject classifications. 35B36, 35J05, 35P15, 45K05

1. Introduction.

Infections with viruses, such as hepatitis B virus (HBV), hepatitis C virus (HCV), and human immunodeficiency virus (HIV), have caused very serious public health problems and economic burdens worldwide since infections with these viruses are chronic and incurable. Once entering the human body, the viral capsid protein binds to the specific receptors on the host cellular surface and injects its core. After an intracellular period associated with transcription, integration, and the production of capsid proteins, an infected cell releases hundreds of viruses that in turn infect other cells. Various mathematical models have been developed to describe the within-host dynamics of these viral infections, such as HBV (Nowak et al. [23]), HCV (Dixit et al. [11]), HIV (Nowak and Bangham [22], Nowak and May [24]), etc. The basic within-host viral infection model consists of three components: uninfected target cells, infected target cells and free virus, and is described by three ordinary differential equations (ODEs) (see Nowak and Bangham [22], Nowak and May [24], Perelson [25], Yang et al. [33]). Systems of ODEs have been long utilized as the mathematical models applied to experimental data on viral infections.

While ODE models have proven quite useful in both empirical studied and theoretical research, there is now ample evidence suggesting that spatial heterogeneity plays an important role in the within-host viral infection as well as the dynamics of the immune response (Graw and Perelson [16]). For example, HCV predominantly spreads among hepatocytes, which are epithelial cells that form tight junctions with their neighbors and are spatially organized within the liver. The results of Shulla and Randall [30] suggest a defined spatiotemporal regulation of HCV infection with highly varied replication efficiencies at the single cell level. As HIV mainly infects $CD4^+$ T cells which are most abundant and densely packed in secondary lymphoid

*Submitted to the editors DATE.

Funding: This work was partially supported by NSF grant no. DMS-1412454 and NSFC grant No. 11771168.

†Department of Mathematics and Statistics, Auburn University, Auburn, AL 36849, USA (gzz0021@auburn.edu).

‡Department of Mathematics, University of Miami, Coral Gables, FL 33146, USA (ruan@math.miami.edu).

organs, such as lymph nodes and the spleen, the spatial arrangement of cells might influence the infection dynamics and spatial conditions, such as the local availability of appropriate target cells, may strongly affect the outcome (Haase [18]). Thus, basic ODE models are not able to capture the spatial aspects of infection and spatial models may be preferred to ODE models (Graw and Perelson [16]).

Over the past few yeas, much effort has been made to combine an ODE model with spatial aspects in modeling of viral dynamics. Under the assumption that target cells and infected cells were stationary while virus particles were capable of migrating from one grid site to a neighboring site, Funk et al. [15] used a discrete ordinary differential equation model to study the interactions of target cells, infected cells, and viral load at anatomical sites where each grid site represents different anatomical sites inside the host. Through simulation of viral spread by such a spatially discrete model of viral dynamics, it was shown that overall infection dynamics are altered, and that models not accounting for spatial aspects might underestimate the genuine infection dynamics. Strain et al. [31] introduced a cellular automaton model of viral propagation based on the known biophysical properties of HIV including the competition between viral lability and Brownian motion. Wang and Wang [32] generalized Funk et al.'s model by assuming that the hepatocytes cannot move under normal conditions and neglected their mobility, whereas virus particles, i.e., virions, can move freely and their motion follows a Fickian diffusion, and proposed a spatial HBV model of two ODEs coupled with a parabolic PDE for the virus particles, and proved the existence of traveling waves.

Meanwhile, there is an increasing interest in nonlocal diffusion problems modeled by nonlocal (convolution) diffusion operators such as

$$L_0 v := d \int_\Omega J(x-y)[v(y)-v(x)]dy,$$

where $v \in X$ and X is a proper Banach space (see Andreu et al. [1], Bates et al. [4], Bates and Zhao [5, 6], Cortazar et al. [9], Coville [10], Du et al. [12], Green et al. [17], Hutson et al. [20], Kao et al. [21], Rawal and Shen [26] and references therein). As shown in Bates et al. [4], $J(x-y)$ is viewed as the probability distribution of jumping from location y to location x; namely the convolution $\int_\Omega J(x-y)u(t,y)dy$ is the rate at which individuals are arriving at position x from other places and $\int_\Omega J(y-x)u(t,x)dy$ is the rate at which they are leaving location x to travel to other sites. Such models with nonlocal diffusion operators have been used to study problems in materials science (Bates [3]) and epidemiology (Ruan [28]).

In this paper, we propose a spatial model of viral dynamics with a nonlocal (convolution) diffusion operator describing the spatial spread of virions between cells. Let $w(t,x), u(t,x)$, and $v(t,x)$ denote the densities of target cells, infected cells, and free virions, respectively, at time t and in location $x \in \Omega \subset \mathbb{R}^n$ $(n \geq 1)$, where Ω is a bounded and connected domain. $d > 0$ is a constant that stands for the diffusion coefficient of free virions, $J(\cdot)$ is a linear dispersal kernel which gives probabilities of rate of motion of virions from location y to location x. Target cells are produced at a rate $s(x)$ and die at a rate b. Target cells become infected cells at an infection rate $c(x)$ and infected cells die at a constant rate a, new virions generated from infected cells have an average lifetime of $1/q$, at rate p per cell. The nonlocal viral infection model takes the following form:

$$(1)\qquad \begin{cases} \dfrac{\partial w(t,x)}{\partial t} = s(x) - bw(t,x) - c(x)w(t,x)v(t,x), \\ \dfrac{\partial u(t,x)}{\partial t} = -au(t,x) + c(x)w(t,x)v(t,x), \\ \dfrac{\partial v(t,x)}{\partial t} = d\displaystyle\int_\Omega J(x-y)[v(t,y) - v(t,x)]dy - qv(t,x) + pu(t,x) \end{cases}$$

for $(t,x) \in \mathbb{R}^+ \times \Omega$. When $d = 0$, and w, u, v, and s and c are all independent of x, system (1) becomes the basic ODE model of viral dynamics proposed by Nowak and Bangham [22], Nowak and May [24], Perelson [25], etc. Hence, model (1) may be viewed as a spatial generalization of the ODE model of Nowak and Bangham [22] and a counterpart of the spatially discrete model of Funk et al. [15] in which virus movement is spatially continuous.

This paper is organized as follows: In section 2, some preliminaries are given. In section 3, we consider positive stationary solutions of (1), which represent infection steady states. We show that the existence of infection steady states hinges upon the sign of the principal eigenvalue of a nonlocal operator. More precisely, when the principal eigenvalue is less than or equal to zero, the only non-negative steady state of (1) is the infection-free steady state, which is stable. While (1) has a unique infection steady state if the principal eigenvalue is great than zero and this steady state is stable. In section 4, we study the dependence of infection steady states on the dispersal rate d. In section 5, we investigate the asymptotical stability of the infection-free steady state in invariant regions. Numerical simulations are presented in section 6. Finally, a brief discussion is given in section 7.

2. Preliminaries. We first list a set of notions that will be used in the rest of the paper. Let Y be a complex Banach Space and $\mathcal{L}(Y)$ be the space of bounded linear operators on Y with the usual operator norm. Let $A \in \mathcal{L}(Y)$ be a closed linear operator on Y. Denote the *resolvent* and *spectrum* of A by

$$\rho(A) = \{\lambda \in \mathbb{C} \mid \ker(\lambda I - A) = \{0\}, (\lambda I - A)^{-1} \in \mathcal{L}(Y)\} \text{ and } \sigma(A) = \mathbb{C} \setminus \rho(A),$$

respectively. The *point spectrum* of A is defined by

$$\sigma_p(A) = \{\lambda \in \mathbb{C} \mid \ker(\lambda I - A) \setminus \{0\} \neq \varnothing\}.$$

An operator is *semi-Fredholm* if it has closed range and its kernel or cokernel is finite-dimensional. The *discrete, essential, continuous, and residual spectra* of A are defined by

$$\sigma_d(A) = \{\lambda \in \mathbb{C} \mid \lambda \in \sigma_p(A) \text{ is isolated and } \dim \bigcup_{k=1}^{\infty} \ker(\lambda I - A)^k < \infty\},$$

$$\sigma_{\text{ess}}(A) = \{\lambda \in \mathbb{C} \mid \lambda I - A \text{ is not semi-Fredholm}\} (= \sigma(A) \backslash \sigma_d(A) \text{ if } A \text{ is self-adjoint}),$$

$$\sigma_c(A) = \{\lambda \in \mathbb{C} \mid \ker(\lambda I - A) = \{0\}, (\lambda I - A)^{-1} \text{ is unbounded with } \overline{\mathcal{R}(\lambda I - A)} = Y\},$$

and

$$\sigma_r(A) = \{\lambda \in \mathbb{C} \mid \ker(\lambda I - A) = \{0\} \text{ with } \overline{\mathcal{R}(\lambda I - A)} \neq Y\},$$

respectively. Following Appell et al. [2], we also write the *compression spectrum* of A as

$$\sigma_{\rm co}(A) = \{\lambda \in \mathbb{C} \mid \overline{\mathcal{R}(\lambda I - A)} \neq Y\},$$

and the *approximate point spectrum* of A as

$$\sigma_q(A) = \{\lambda \in \mathbb{C} \mid \text{there exists a Weyl sequence for } \lambda I - A\},$$

where a sequence $\{z_n\} \in Y$ is called a *Weyl sequence* for A if $\|z_n\|_Y = 1$ and $\|Az_n\|_Y \to 0$ as $n \to \infty$.

In the following, given that $r \in C(\overline{\Omega})$, we define $L_r : C(\overline{\Omega}) \to C(\overline{\Omega})$ by

$$(L_r z)(x) := d\int_\Omega J(x-y)[z(y) - z(x)]dy + r(x)z(x). \tag{2}$$

Let $C_c(\mathbb{R}^n)$ denote the space of continuous functions in $\mathbb{R}^n$ with compact support. We start it by presenting the following lemma.

LEMMA 2.1. *Assume that $J \in C_c(\mathbb{R}^n)$ is a non-negative radial function with $J(0) > 0$ and $r \in C(\overline{\Omega})$, where $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a bounded and connected domain. Let $b(x) = r(x) - d\int_\Omega J(x-y)dy$. Suppose that there exists a bounded sub-domain $\Omega' \subset \overline{\Omega}$ such that $[\kappa - b(x)]^{-1} \notin L^1(\Omega')$, where $\kappa = \sup_{x\in\Omega} b(x)$. Then L_r possesses a principal eigenpair (μ_r, ϕ_r) with $\phi_r \in C(\overline{\Omega})$ and $\phi_r > 0$. Moreover, there holds*

$$\mu_r = -\inf_{\varphi \in L^2(\Omega), \varphi \neq 0} \frac{\frac{d}{2}\int_\Omega\int_\Omega J(x-y)[\varphi(y) - \varphi(x)]^2 dydx - \int_\Omega r(x)\varphi^2(x)dx}{\|\varphi\|^2_{L^2(\Omega)}}. \tag{3}$$

In particular, suppose that $r(x) \neq constant$, then $\mu_r > 0$ provided that $\overline{r} \geq 0$, where $\overline{r} = \frac{1}{|\Omega|}\int_\Omega r(x)dx$.

Proof. The existence of a principal eigenpair (μ_r, ϕ_r) was proved in Coville [10] where the existence of a principal eigenpair was established for a more general nonlocal operator and Ω is allowed to be unbounded. In particular, it was shown in Theorem 1.1 of Coville [10] that $\mu_r > \sup_{x\in\Omega} b(x)$. Recall that $b(x) = r(x) - d\int_\Omega J(x-y)dy$. This implies that $(\lambda - b(x))^{-1}$ is a bounded and continuous function for all $x \in \overline{\Omega}$ whenever $\lambda \geq \mu_r$. Let $\mathcal{K} : L^2(\Omega) \to L^2(\Omega)$ and $\mathcal{B} : L^2(\Omega) \to L^2(\Omega)$ be defined by

$$(\mathcal{K}\varphi)(x) = -d\int_\Omega J(x-y)\varphi(y)dy \quad \text{and} \quad (\mathcal{B}\varphi)(x) = -b(x)\varphi(x), \quad \varphi \in L^2(\Omega), \tag{4}$$

respectively. Clearly, $-L_r = \mathcal{K} + \mathcal{B}$ on $L^2(\Omega)$ and both $\mathcal{K}$ and $\mathcal{B}$ are self-adjoint. Moreover, due to the facts that $\mathcal{K}$ is compact and that $\lambda \in \rho(\mathcal{B})$ if $\lambda \leq -\mu_r$, it follows from Theorem 8.15 of Schmüdgen [29] that $(-\infty, -\mu_r] \subset [\sigma_d(-L_r) \bigcup \rho(-L_r)]$. Since $\phi_r \in L^2(\Omega)$, as a result, $-\mu_r \in \sigma_d(-L_r)$ with $D(-L_r) = L^2(\Omega)$. Note that $-L_r$ is a lower semi-bounded self-adjoint operator on $L^2(\Omega)$. In fact, let $\langle \cdot, \cdot \rangle$ be the inner product for $L^2(\Omega)$, then we have $\langle -L_r\varphi, \varphi\rangle \geq -m\|\varphi\|_{L^2(\Omega)}$ as long as $m \geq |r(x)|_{L^\infty(\Omega)}$. In addition, as $-L_r$ is bounded, we have $(-\infty, -\|L_r\| - 1] \subset \rho(-L_r)$. Let $\omega_r = \inf\{\mu \in \mathbb{R} \mid \mu \in \sigma_{\rm ess}(-L_r)\}$, it follows that $-\mu_r < \omega_r$. Apparently, $(-\|L_r\| - 1, \omega_r) \bigcap \sigma_d(-L_r) \neq \varnothing$ as $-\mu_r \in (-\|L_r\| - 1, \omega_r)$.

Let $\lambda_1 = \inf_{\varphi \in L^2(\Omega), \varphi \neq 0} \|\varphi\|^{-2}_{L^2(\Omega)} \langle -L_r\varphi, \varphi\rangle$. Clearly, $\lambda_1 \leq -\mu_r < \omega_r$. It then follows from Theorem XIII.1 of Reed and Simon [27] that $\lambda_1 \in \sigma_d(-L_r)$. Indeed, we have $\lambda_1 = -\mu_r$. If otherwise, let ϕ_1 be an eigenfunction associated with λ_1. Note that

$|\phi_1|$ is also an eigenfunction for λ_1 since $\langle -L_r|\varphi|, |\varphi|\rangle \leq \langle -L_r\varphi, \varphi\rangle$ for all $\varphi \in L^2(\Omega)$. Then we find that $\langle |\phi_1|, \phi_r\rangle = 0$ since $-L_r$ is self-adjoint. This is impossible as $\phi_r > 0$. Thus, $\lambda_1 = -\mu_r$. Namely, (3) holds, and

$$-\mu_r\|\varphi\|_{L^2(\Omega)} \leq \langle -L_r\varphi, \varphi\rangle = \frac{1}{2}\int_\Omega\int_\Omega J(x-y)[\varphi(y)-\varphi(x)]^2dydx - \int_\Omega r(x)\varphi^2(x)dx$$

for all $\varphi \in L^2(\Omega)$.

It remains to prove the last part of the lemma. Let $\phi > 0$ be an eigenfunction associated with μ_r, that is,

$$\int_\Omega J(x-y)[\phi(y)-\phi(x)]dy + r(x)\phi(x) = \mu_r\phi(x).$$

Multiplying both sides of the above equation by $1/\phi$ and integrating the resulting equation over Ω yield that

$$\frac{md}{2}\int_\Omega\int_\Omega J(x-y)[\phi(y)-\phi(x)]^2dydx + \int_\Omega r(x)dx \leq |\Omega|\mu_r.$$

Here $m = 1/|\phi|^2_{L^\infty(\Omega)}$ and we used the fact that

$$\begin{aligned}
&\int_\Omega\int_\Omega J(x-y)[\phi(y)-\phi(x)]dy\frac{1}{\phi(x)}dx \\
&\quad = -\frac{1}{2}\int_\Omega\int_\Omega J(x-y)[\phi(y)-\phi(x)]\left[\frac{1}{\phi(y)}-\frac{1}{\phi(x)}\right]dydx \\
&\quad \geq \frac{m}{2}\int_\Omega\int_\Omega J(x-y)[\phi(y)-\phi(x)]^2
\end{aligned}$$

since

$$-[\phi(y)-\phi(x)]\left[\frac{1}{\phi(y)}-\frac{1}{\phi(x)}\right] \geq \frac{1}{|\phi|^2_{L^\infty(\Omega)}}[\phi(y)-\phi(x)]^2$$

for all $x, y \in \Omega$. Moreover, it follows from the Poincaré type inequality of Andreu et al. [1] that

$$\int_\Omega\int_\Omega J(x-y)[\phi(y)-\phi(x)]^2dydx \geq \beta\int_\Omega\left|\phi(x)-\frac{1}{|\Omega|}\int_\Omega\phi(z)dz\right|^2dx,$$

where $\beta > 0$ is a constant depending only upon J and Ω. Since $\phi \neq$ constant and $\overline{r} \geq 0$. The desired conclusion follows. ∎

PROPOSITION 2.2. *Assume that $r_1, r_2 \in C(\overline{\Omega})$. Let $b_i(x) = r_i(x) - \int_\Omega J(x-y)dy$ $(i=1,2)$. Suppose that there exists sub-domains $\Omega_i \subset \Omega$ such that $[\kappa_i - b_i(x)]^{-1} \notin L^1(\Omega_i)$, where $\kappa_i = \sup_{x\in\overline{\Omega}} b_i$. Let $L_{r_i} : C(\overline{\Omega}) \to C(\overline{\Omega})$ be defined by (2). Assume that $r_1 \gneq r_2$ for all $x \in \overline{\Omega}$. Then $\mu_1 > \mu_2$, where μ_i is the principal eigenvalue of L_{r_i} $(i=1,2)$.*

Proof. Let ϕ_i be an eigenfunction associated with μ_i $(i=1,2)$. Then we have

$$\int_\Omega J(x-y)[\phi_1(y)-\phi_1(x)]dy + r_1(x)\phi_1(x) = \mu_1\phi_1(x),$$

$$\int_\Omega J(x-y)[\phi_2(y)-\phi_2(x)]dy + r_2(x)\phi_2(x) = \mu_2\phi_2(x).$$

Multiplying both sides of the first equation by ϕ_2, both sides of the second equation by ϕ_2, and integrating the resulting equations over Ω, we have $(i = 1, 2)$

$$\int_\Omega\int_\Omega J(x-y)[\phi_1(y)-\phi_1(x)][\phi_2(y)-\phi_2(x)]dydx + \int_\Omega r_i(x)\phi_1\phi_2 dx = \mu_i \int_\Omega \phi_1\phi_2 dx.$$

Note that $\phi_i > 0$ for all $x \in \overline{\Omega}$. Subtracting these two equalities yields that

$$0 < \int_\Omega [r_1(x) - r_2(x)]\phi_1(x)\phi_2(x)dx = (\mu_1 - \mu_2)\int_\Omega \phi_1(x)\phi_2(x)dx.$$

Since the right side of the above equation is strictly positive, it follows that $\mu_1 > \mu_2$.□

3. Existence and stability of stationary solutions. We now proceed to study the steady states of (1) and their stabilities. Note that (1) always has an infection-free steady state given by $(w^0, u^0, v^0) = \left(\frac{s(x)}{b}, 0, 0\right)$. A positive steady state of (1) is particularly of interest as it represents an infection state, we hence are led to study the solution(s) to

$$d\int_\Omega J(x-y)[v(y)-v(x)]dy + v(x)\left[\frac{pc(x)s(x)}{a[b+c(x)v(x)]} - q\right] = 0, \quad x \in \overline{\Omega}. \tag{5}$$

Unless otherwise stated, the following assumptions will be needed throughout the rest of paper.

(H1) $J \in C_c^1(\mathbb{R}^n)$ $(n = 1$ or $2)$, $J \gneq 0$, and $J(0) > 0$;

(H2) a, b, d, p, q are positive constants, $s \in C^2(\overline{\Omega})$ and $s \gneq 0$ for all $x \in \overline{\Omega}$, $c \in C^2(\overline{\Omega})$ and $c > 0$ for all $x \in \overline{\Omega}$, where $\Omega \subset \mathbb{R}^n$ $(n = 1$ or $2)$ is a bounded and connected domain.

Set

$$\mathcal{S}_0 = -\inf_{\varphi\in L^2(\Omega), \|\varphi\|_{L^2(\Omega)}=1}\left\{\frac{d}{2}\int_\Omega\int_\Omega J(x-y)[\varphi(y)-\varphi(x)]^2 dydx - \int_\Omega\left[\frac{pc(x)w^0(x)}{a} - q\right]\varphi^2(x)dx\right\},$$

$$\hat{S}_0 = \frac{1}{|\Omega|}\int_\Omega\left[\frac{pc(x)w^0(x)}{a} - q\right]dx,$$

$$S(\lambda, x) = \frac{pc(x)w^0(x)}{\lambda + a} - (\lambda + q), \quad \mathrm{Re}\lambda > -a.$$

Also define an operator $L_{S,\lambda} : C(\overline{\Omega}) \to C(\overline{\Omega})$ by

$$L_{S,\lambda}\varphi(x) = \int_\Omega J(x-y)[\varphi(y)-\varphi(x)]dy + S(\lambda, x)\varphi(x), \ \varphi \in C(\overline{\Omega}), \quad \mathrm{Re}\lambda > -a. \tag{6}$$

Remark 3.1. Thanks to (H1) and (H2), for each $\lambda > -a$, $S(\lambda, x) - \int_\Omega J(x - y)dy \in C^2(\overline{\Omega})$, which, as shown in Coville [10], guarantees the existence a principal eigenvalue of $L_{S,\lambda}$. Denote the principal eigenvalue of $L_{S,\lambda}$ in $C(\overline{\Omega})$ by $\mu(\lambda)$. Note that $\mu(\lambda)$ is analytic in λ and $\mu(0) = \mathcal{S}_0$. In particular, when λ takes on real values, simple calculation shows that $\mu'(\lambda) < 0$. In light of Lemma 2.1, $\mathcal{S}_0 > 0$ provided that $\hat{S}_0 \geq 0$. In case that s and c are independent of x, we have

$$\hat{S}_0 = \frac{pcs}{ab} - q = q(R_0 - 1),$$

where $R_0 = \frac{pcs}{qab}$ is the basic reproduction number of the virus (Nowak and May [24]). Thus, $\mathcal{S}_0$ has the same sign as the basic reproduction number minus unity $(R_0 - 1)$. In what follows, we will see that $\mathcal{S}_0$ plays a role in determining the stabilities of stationary solutions to (5).

THEOREM 3.2. *Assume that (H1) and (H2) are satisfied. Suppose that $\mathcal{S}_0 \leq 0$. Then (5) has no positive solutions. Namely, model (1) has no non-negative steady states other than $(w^0, u^0, v^0) = (\frac{s(x)}{b}, 0, 0)$. Moreover, (w^0, u^0, v^0) is uniformly asymptotically stable in X provided that $\mathcal{S}_0 < 0$, where $X = C(\overline{\Omega}) \times C(\overline{\Omega}) \times C(\overline{\Omega})$.*

Proof. We first show that (5) has no positive solutions by contradiction. Assume to the contrary that (5) has a positive solution $v^* \in C(\overline{\Omega})$. Let $v^*(x_*) = \inf_{x\in\Omega} v^*(x)$ for some $x_* \in \overline{\Omega}$ and $v^*(x^*) = \sup_{x\in\Omega} v^*(x)$ for some $x^* \in \overline{\Omega}$. Clearly, $v^*(x_*) \neq v^*(x^*)$ as $v^* \neq$ constant. It is easy to see that $v^*(x) > 0$ for all $x \in \overline{\Omega}$. Note that

$$\int_\Omega J(x-y)[v^*(y) - v^*(x_*)]dy \geq 0 \quad \text{for all } \ x \in \overline{\Omega}.$$

As a result, we have that $\frac{pc(x_*)s(x_*)}{a[b+c(x_*)v(x_*)]} - q \leq 0$. Hence, $v^*(x_*) \geq \frac{ps(x_*)}{a} - \frac{bq}{c(x_*)}$. Likewise, we have $v^*(x^*) \leq \frac{ps(x^*)}{a} - \frac{bq}{c(x^*)}$. That is,

$$\frac{p\inf_{x\in\Omega} s(x)}{a} - \frac{bq}{\inf_{x\in\Omega} c(x)} \leq v^*(x) \leq \frac{p\sup_{x\in\Omega} s(x)}{a} - \frac{bq}{\sup_{x\in\Omega} c(x)} \quad \text{for all } \ x \in \overline{\Omega}.$$

Now let ψ be a positive eigenfunction corresponding to $\mathcal{S}_0$. Namely,

$$d\int_\Omega J(x-y)[\psi(y) - \psi(x)]dy + \left[\frac{pc(x)w^0(x)}{a} - q\right]\psi(x) = \mathcal{S}_0\psi(x).$$

By multiplying this equation by v^* and (5) by ψ, respectively, and integrating the resulting equations over Ω, we find that

$$\begin{aligned}
&-\frac{d}{2}\int_\Omega\int_\Omega J(x-y)[\psi(y)-\psi(x)][v^*(y)-v^*(x)]dydx \\
&\qquad + \int_\Omega \left[\frac{pc(x)w^0(x)}{a} - q\right]\psi(x)v^*(x)dx = \mathcal{S}_0\int_\Omega \psi(x)v^*(x)dx, \\
&-\frac{d}{2}\int_\Omega\int_\Omega J(x-y)[v^*(y)-v^*(x)][\psi(y)-\psi(x)]dydx \\
&\qquad + \int_\Omega \left[\frac{pc(x)w^0(x)}{a[1+(c(x)v^*(x))/b]} - q\right]\psi(x)v^*(x)dx = 0.
\end{aligned}$$

Subtracting these equations yields that

$$\int_\Omega \left[\frac{pc(x)w^0(x)}{a} - \frac{pc(x)w^0(x)}{a[1+(c(x)v^*(x))/b]}\right]\psi(x)v^*(x)dx = \mathcal{S}_0\int_\Omega \psi(x)v^*(x)dx \leq 0.$$

As $\psi, v^* > 0$ for all $x \in \overline{\Omega}$, and $pc(x)w^0(x)/a - pc(x)w^0(x)/a[1+(c(x)v^*(x))/b] \gneq 0$ for $x \in \overline{\Omega}$, the integral of the right hand side of the above equation is strictly greater than zero, which obviously is a contradiction. This contradiction confirms that (5) has no positive solutions if $\mathcal{S}_0 \leq 0$. It is easy to see that (1) has no non-negative steady state other than (w^0, u^0, v^0).

It remains to show that (w^0,u^0,v^0) is stable in X if $\mathcal{S}_0<0$. The linearization of (1) around (w^0,u^0,v^0) for perturbation of functions $(w,u,v)\in C([0,T),X)$ is given by the system

$$\frac{\partial}{\partial t}\begin{pmatrix} w\\ u\\ v\end{pmatrix}=\begin{pmatrix} -b & 0 & -cw^0\\ 0 & -a & cw^0\\ 0 & p & L_q\end{pmatrix}\begin{pmatrix} w\\ u\\ v\end{pmatrix},$$

where $L_q: C(\overline{\Omega})\to C(\overline{\Omega})$ is defined by $L_q\varphi(x)=\int_\Omega J(x-y)[\varphi(y)-\varphi(x)]dy-q\varphi(x)$.

Now let

$$\mathcal{L}_0=\begin{pmatrix} -b & 0 & -cw^0\\ 0 & -a & cw^0\\ 0 & p & L_q\end{pmatrix}.$$

Obviously, $\mathcal{L}_0$ is a bounded linear operator on X and is the generator of the strongly (actually uniformly) continuous semigroup $\{e^{\mathcal{L}_0 t}\}_{t\geq 0}$ given by

$$e^{\mathcal{L}_0 t}=\sum_{n=0}^{\infty}\frac{t^n\mathcal{L}_0^n}{n!},\ t\geq 0.$$

Denote the spectral bound of $\mathcal{L}_0$ by

$$\mathfrak{s}(\mathcal{L}_0)=\sup\{\mathrm{Re}\lambda\mid\lambda\in\sigma(\mathcal{L}_0)\}.$$

Given $\epsilon>0$, it follows from Engel and Nagel [14] that

$$\|e^{\mathcal{L}_0 t}\|\leq M_\epsilon e^{(\mathfrak{s}(\mathcal{L}_0)+\epsilon)t},\ t\geq 0$$

for some positive constant M_ϵ. Therefore, to complete the proof, it is sufficient to show that $\mathfrak{s}(\mathcal{L}_0)<0$. To this end, we proceed to show that there exists $\delta>0$ for which $\{\lambda\in\mathbb{C}\mid\mathrm{Re}\lambda\geq-\delta\}\subset\rho(\mathcal{L}_0)$. Let $L_{S,\lambda}$ be the operator defined by (6). Again, let $\mu(\lambda)$ be the principal eigenvalue of $L_{S,\lambda}$ in $C(\overline{\Omega})$. Clearly, $\mu(0)=\mathcal{S}_0$. As $\mathcal{S}_0<0$, from the monotonicity of $S(\lambda,x)$ in λ, it follows that $\mu(\lambda)<0$ for all $\lambda>0$, which implies that $0\in\rho(L_{S,\lambda})$ for all $\lambda\geq 0$. In addition, by virtue of the continuity of $S(\lambda,x)$ with respect to λ, there exists $\delta>0$ with $\delta\leq\frac{1}{2}\min\{b,a,q\}$ such that $\mu(\lambda)<0$ for all $\lambda\in[-\delta,0)$. Consequently, $0\in\rho(L_{S,\lambda})$ for all $\lambda\geq-\delta$.

Given that $\lambda\geq-\delta$, to show $\lambda\in\rho(\mathcal{L}_0)$, we consider the resolvent equation $(\lambda I-\mathcal{L}_0)(w,u,v)^T=(h_1,h_2,h_3)^T$, where $(h_1,h_2,h_3)^T\in X$. Namely,

$$\begin{cases}(\lambda+b)w+cw^0v=h_1,\\ (\lambda+a)u-cw^0v=h_2,\\ -pu+\lambda v-L_qv=h_3.\end{cases}\tag{7}$$

As $\lambda+a\neq 0$ and $\lambda+b\neq 0$, it is easy to see that

$$(w,u,v)=\left(\frac{h_1+cw^0L_{S,\lambda}^{-1}(h_3+\frac{ph_2}{\lambda+a})}{\lambda+b},\frac{h_2-cw^0L_{S,\lambda}^{-1}(h_3+\frac{ph_2}{\lambda+a})}{\lambda+a},-L_{S,\lambda}^{-1}(h_3+\frac{ph_2}{\lambda+a})\right)$$

is the unique solution to (7). Hence $\lambda\in\rho(\mathcal{L}_0)$ if $\lambda\geq-\delta$.

In case that $\lambda\in\mathbb{C}$ and $\mathrm{Im}\lambda\neq 0$, we write $\lambda=\lambda_1+i\lambda_2$ with $\lambda_1,\lambda_2\in\mathbb{R}$, and $v=v_1+iv_2$, where v_1,v_2 take real values. In view of the above argument, in order to prove that $\lambda\in\rho(\mathcal{L}_0)$ whenever $\mathrm{Re}\lambda\geq-\delta$, it suffices to show that $0\in\rho(L_{S,\lambda})$ if $\mathrm{Re}\lambda\geq-\delta$. First notice that $L_{S,\lambda}$ is also a bounded linear operator on $L^2(\Omega)$.

Moreover, it is not difficult to show that $\ker(L_{S,\lambda}) = \{0\}$ for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda \geq -\delta$. In fact, consider

$$\int_\Omega J(x-y)[v(y)-v(x)]dy - (\lambda+q)v + \frac{pc(x)w^0(x)v}{\lambda+a} = 0, \quad v \in L^2(\Omega).$$

By multiplying both sides of this equation by $-\overline{v}$, we have that

$$\frac{1}{2}\int_\Omega\int_\Omega \big\{[v_1(y)-v_1(x)]^2 + [v_2(y)-v_2(x)]^2\big\}dydx$$
$$-\int_\Omega \left[\frac{pc(x)w^0(x)(\lambda_1+a)}{(\lambda_1+a)^2+\lambda_2^2} - (\lambda_1+q)\right]v\overline{v}dx = 0.$$

Notice that

$$\frac{pc(x)w^0(x)(\lambda_1+a)}{(\lambda_1+a)^2+\lambda_2^2} - (\lambda_1+q) \leq S(x,\lambda_1)$$

if $\lambda_1 \geq -\delta$ and $\lambda_2 \neq 0$. Then Lemma 2.1 and Remark 2.2 imply that

$$-\mu(\lambda_1)\|v\|^2_{L^2(\Omega)} \leq \frac{1}{2}\int_\Omega\int_\Omega \big\{[v_1(y)-v_1(x)]^2 + [v_2(y)-v_2(x)]^2\big\}dydx$$
$$-\int_\Omega \left[\frac{pcw^0(\lambda_1+a)}{(\lambda_1+a)^2+\lambda_2^2} - (\lambda_1+q)\right]v\overline{v}dx.$$

As $\mu(\lambda_1) < 0$ if $\lambda_1 \geq -\delta$, this implies that $v = 0$. Namely, $\ker(L_{S,\lambda}) = \{0\}$ if $\mathrm{Re}\lambda \geq -\delta$. Let $L^*_{S,\lambda}$ be the adjoint operator of $L_{S,\lambda}$ on $L^2(\Omega)$. Then we have

$$L^*_{S,\lambda}v(x) = \int_\Omega J(x-y)[v(y)-v(x)]dy - \overline{(\lambda+q)}v + \frac{pc(x)w^0(x)v}{\overline{\lambda+a}}.$$

The same reasoning shows that $\ker(L^*_{S,\lambda}) = \{0\}$. Thus, $\overline{\mathcal{R}(L_{S,\lambda})} = L^2(\Omega)$. Clearly, $0 \in \mathbb{C}\setminus\sigma_{co}(L_{S,\lambda})$ if $\mathrm{Re}\lambda \geq -\delta$. Furthermore, we have $0 \in \mathbb{C}\setminus\sigma_q(L_{S,\lambda})$. In fact, if $0 \in \sigma_q(L_{S,\lambda})$, there would be a Weyl sequence $\{v_n\}$ such that $\langle -L_{S,\lambda}v_n, v_n\rangle \to 0$ as $n \to \infty$, which as above implies that $-\mu(\lambda_1)\|v_n\|_{L^2(\Omega)} \to 0$ as $n \to \infty$. This is a contradiction. Thus, we must have that $0 \in \mathbb{C}\setminus[\sigma_q(L_{S,\lambda})\bigcup\sigma_{co}(L_{S,\lambda})]$. Then, from the fact that $\rho(L_{S,\lambda}) = \mathbb{C}\setminus[\sigma_q(L_{S,\lambda})\bigcup\sigma_{co}(L_{S,\lambda})]$, we infer that $0 \in \rho(L_{S,\lambda})$ for all $\mathrm{Re}\lambda \geq -\delta$ with $D(L_{S,\lambda}) = L^2(\Omega)$.

Now fix $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda \geq -\delta$. Let $\mathcal{P}: L^2(\Omega) \to L^2(\Omega)$ be defined by

$$(8)\qquad \begin{aligned}(\mathcal{P}v)(x) &= P(x)v(x) = [-\int_\Omega J(x-y)dy + S(x,\lambda)]v(x),\\ P(x) &= -\int_\Omega J(x-y)dy + S(x,\lambda).\end{aligned}$$

Note that $P \in C(\overline{\Omega})$. We next show that $0 \in \Lambda^c$, where $\Lambda = \{z \in \mathbb{C} \mid z = P(x), x \in \overline{\Omega}\}$. Assume to the contrary this is not true, then in view of Schmüdgen [29], there holds that $0 \in \Lambda \subseteq \sigma(\mathcal{P})$. Since $\mathcal{P}$ is a normal operator on $L^2(\Omega)$, we have $\sigma(\mathcal{P}) = \sigma_p(\mathcal{P}) \cup \sigma_c(\mathcal{P})$. It is easy to see that $\sigma_p(\mathcal{P}) \subseteq \sigma_q(\mathcal{P})$. In fact, if $\lambda \in \sigma_p(\mathcal{P})$, let $\psi \in L^2(\Omega)$ be an eigenfunction corresponding to λ, then

$$[\lambda - P(x)]\psi\overline{\psi} = \mathrm{Re}[\lambda - P(x)]\psi\overline{\psi} + i\mathrm{Im}[\lambda - P(x)]\psi\overline{\psi} = 0.$$

Write $\Xi = \{x \in \Omega | \psi\overline{\psi} \neq 0\}$. Obviously, the measure of Ξ is positive. Hence, $[\lambda - P(x)] = 0$ in Ξ. This implies that any L^2 function with support in Ξ belongs to $\ker(\lambda I - \mathcal{P})$ and $\dim\ker(\lambda I - \mathcal{P}) = \infty$. Thus, $\sigma_p(\mathcal{P}) \subseteq \sigma_q(\mathcal{P})$ and $\sigma(\mathcal{P}) = \sigma_q(\mathcal{P})$. On the other hand, note that $L_{S,\lambda} = -\mathcal{K} + \mathcal{P}$, where $\mathcal{K}$ is given by (4), hence it follows from Proposition 1.5 of Appell et al. [2] that $\sigma_q(L_{S,\lambda}) = \sigma_q(\mathcal{P})$ and $0 \in \sigma_q(L_{S,\lambda})$, which however contradicts the fact that $0 \in \rho(L_{S,\lambda})$. Thus, we must have $0 \in \mathbb{C} \setminus \Lambda$. As Λ is a compact subset of $\mathbb{R}^2$ for fixed λ, there exists a $\omega_\lambda > 0$ for which $\mathrm{dist}(0, \Lambda) \geq \omega_\lambda$. In other words, $|P(x)| \geq \omega_\lambda$ or $|P(x)|^{-1} \leq 1/\omega_\lambda$ for all $x \in \overline{\Omega}$. Clearly, $P^{-1} \in C(\overline{\Omega})$. Given $f \in C(\overline{\Omega})$, as $f \in L^2(\Omega)$, there is a unique $v_f \in L^2(\Omega)$ such that $L_{S,\lambda} v_f = f$ and $\|v_f\|_{L^2(\Omega)} \leq K\|f\|_{L^2(\Omega)} \leq K\sqrt{|\Omega|}\|f\|_X$ for some $K > 0$, that is independent of f. Moreover, we have that

$$v_f(x) = -\frac{1}{P(x)}\int_\Omega J(x-y)v_f(y)dy + \frac{f(x)}{P(x)}.$$

It is clear that $v_f \in C(\overline{\Omega})$ and $\|v_f\|_X \leq K'\|f\|_X$ for some $K' > 0$. Consequently, for any $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda \geq -\delta$, $0 \in \rho(L_{S,\lambda})$ with $D(L_{S,\lambda}) = C(\overline{\Omega})$. Therefore, we infer that $\{\lambda \in \mathbb{C} \mid \mathrm{Re}\lambda \geq -\delta\} \subset \rho(\mathcal{L}_0)$, which implies that $\mathfrak{s}(\mathcal{L}_0) < 0$ as desired.

Now set

$$F(w,u,v) = \begin{pmatrix} -cw(x)v(x) \\ cw(x)v(x) \\ 0 \end{pmatrix}.$$

Then $F \in C^1(X)$. Note that $(w + w^0, u, v)$ is a solution of (1) with initial data $(w(0,x) + w^0(x), u(0,x), v(0,x))$ if and only if (w,u,v) is a solution to

$$\frac{\partial}{\partial t}\begin{pmatrix} w \\ u \\ v \end{pmatrix} = \mathcal{L}_0 \begin{pmatrix} w \\ u \\ v \end{pmatrix} + F(w,u,v)$$

with initial data $(w(0,x), u(0,x), v(0,x))^T$. Obviously, $(0,0,0)^T$ is a stationary solution of the above equation and $\|F(w,u,v)\|_X = o(\|(w,u,v)^T\|_X)$ as $\|(w,u,v)^T\|_X \to 0$. By using Theorem 5.1.1 of Henry [19], we finally conclude that (w^0, u^0, v^0) is uniformly asymptotically stable in X. The proof is completed. ◻

THEOREM 3.3. *Assume that (H1) and (H2) are satisfied. Suppose that $\mathcal{S}_0 > 0$. Then (w^0, u^0, v^0) is unstable in X.*

Proof. We shall prove that $\mathfrak{s}(\mathcal{L}_0) \in \sigma_p(\mathcal{L}_0)$ and $\mathfrak{s}(\mathcal{L}_0) > 0$, where $\mathcal{L}_0$ is given in the proof of Theorem 3.2, and $\mathfrak{s}(\mathcal{L}_0) = \sup\{\mathrm{Re}\lambda \mid \lambda \in \sigma(\mathcal{L}_0)\}$. Let $\mu(\lambda)$ be the principal eigenvalue of $L_{S,\lambda}$. By the assumption, we have $\mu(0) = \mathcal{S}_0 > 0$. Since $S(\lambda, x) \to -\infty$ uniformly as $\lambda \to \infty$, by the monotonicity of $\mu(\lambda)$ $(\mu'(\lambda) < 0)$, there exists a $\lambda_m > 0$ such that $\mu(\lambda_m) < 0$ for all $\lambda \geq \lambda_m$. It then follows from the mean value theorem that $\mu(\lambda^*) = 0$ for some $\lambda^* \in (0, \lambda_m)$. In addition, λ^* is the only zero of $\mu(\lambda)$ in $[0,\infty)$ since $\mu'(\lambda) < 0$. This also implies that $\mu(\lambda) < 0$ for all $\lambda > \lambda^*$. In other words, $0 \in \rho(L_{S,\lambda})$ if $\lambda > \lambda^*$. With the same reasoning as that used in the proof of Theorem 3.2, we can infer that $\lambda \in \rho(\mathcal{L}_0)$ provided that $\mathrm{Re}\lambda > \lambda^*$. Now let $\varphi^* \in \ker(\mu(\lambda^*)I - L_{S,\lambda^*})$. It is easy to see that

$$\ker(\lambda^* I - \mathcal{L}_0) = \mathrm{span}\left(\frac{cw^0\varphi^*}{\lambda^* + b}, \frac{cw^0\varphi^*}{\lambda^* + a}, \varphi^*\right).$$

Namely, $\lambda^* \in \sigma_p(\mathcal{L}_0)$ and $\mathfrak{s}(\mathcal{L}_0) = \lambda^* > 0$. It then follows from Theorem 5.1.3 of Henry [19] that (w^0, u^0, v^0) is unstable in X. The proof is completed. ◻

PROPOSITION 3.4 (Coville [10]). *Assume that $g(x,\tau) \in C^{0,1}(\overline{\Omega} \times \mathbb{R}^+)$ and $\theta g(x,\tau) \lneq g(x,\theta\tau)$ for $\theta > 1$. Let $v_1, v_2 \in X$ satisfy*

$$\int_\Omega J(x-y)[v_1(y)-v_1(x)]dy + g(x,v_1) \le 0 \le \int_\Omega J(x-y)[v_2(y)-v_2(x)]dy + g(x,v_2).$$

Assume further that $v_1(x) > 0$ for all $x \in \overline{\Omega}$. Then $v_1 \ge v_2$.

Proof. See section 6.3 of Coville [10] for detail. ∎

THEOREM 3.5. *Assume that (H1) and (H2) are satisfied. Suppose that $\mathcal{S}_0 > 0$. Then (1) has a unique positive steady state (w^*, u^*, v^*) which is uniformly asymptotically stable in X.*

Proof. Note that (1) has a positive steady state if and only if there exists a positive solution to equation (5). We next show that $\underline{v} = \epsilon\phi$ is a sub-solution of (5), where $\epsilon > 0$ is a sufficiently small constant and $\phi > 0$ is a eigenfunction associated with $\mathcal{S}_0$. Namely,

$$d\int_\Omega J(x-y)[\phi(y)-\phi(x)]dy + \left[\frac{pc(x)s(x)}{ab} - q\right]\phi(x) = \mathcal{S}_0\phi(x).$$

Thus, whenever ϵ is sufficiently small, we find

$$\begin{aligned} &d\int_\Omega J(x-y)\epsilon[\phi(y)-\phi(x)]dy + \left[\frac{pc(x)s(x)}{a[b+\epsilon c(x)\phi]} - q\right]\epsilon\phi \\ &= \left[\mathcal{S}_0 + \frac{pc(x)s(x)}{a[b+\epsilon c(x)\phi]} - \frac{pc(x)s(x)}{ab}\right]\epsilon\phi > 0. \end{aligned}$$

Meanwhile, it is easy to see that $[\frac{pc(x)s(x)}{a[b+c(x)M]} - q] \le 0$, where $M > 0$ is a constant and is sufficiently large. Now fix M and let $\overline{v} \equiv M$. Clearly, we have

$$d\int_\Omega J(x-y)\epsilon[\overline{v}(y)-\overline{v}(x)]dy + \left[\frac{pc(x)s(x)}{a[b+c(x)\overline{v}]} - q\right]\overline{v} \le 0.$$

Set $f(x,\tau) = \tau[\frac{pc(x)s(x)}{a[b+c(x)\tau]} - q]$ and let $\nu > \max_{(x,\tau)\in\overline{\Omega}\times[0,2M]} |f_\tau(x,\tau)|$.

Now define $\mathcal{F} : X \to X$ by

$$(\mathcal{F}v)(x) = (\nu I - L_0)^{-1}[\nu v + f(x,v)], \;\; v \in X,$$

where $(L_0 v)(x) = d\int_\Omega J(x-y)[v(y)-v(x)]dy$. As $\mathfrak{s}(L_0) = 0$, due to Bates and Zhao [5], $(\nu I - L_0)^{-1}$ is well defined and is a positive operator on X; that is, $(\nu I - L_0)^{-1}v \ge 0$ if $v \ge 0$. Consequently, $\mathcal{F}v_1 \ge \mathcal{F}v_2$ provided that $0 \le v_2 \le v_1 \le M$. On the other hand, simple calculation shows that $f_{\tau\tau} \le 0$. Hence, $f(x, t\theta_1 + (1-t)\theta_2) \ge tf(x,\theta_1) + (1-t)f(x,\theta_1)$ for $t \in [0,1]$ and $\theta_1, \theta_2 \in \mathbb{R}$. This implies that $\mathcal{F}(tu + (1-t)w) \ge t\mathcal{F}u + (1-t)\mathcal{F}w$ for $u, w \in X$ with $u, w \ge 0$. Notice that (5) is equivalent to $\mathcal{F}v = v$. In addition, as $(\nu I - L_0)^{-1}$ is a positive operator, it is easy to see that $\mathcal{F}\underline{v} \ge \underline{v}$ and $\mathcal{F}\overline{v} \le \overline{v}$. Therefore, it follows from Du [13] that $\mathcal{F}$ has a unique fixed point v^* in Θ, where $\Theta = \{v \in X \mid \underline{v} \le v \le \overline{v}\}$. Thus, v^* is a positive solution of (5). To prove the uniqueness of v^*, let w^* be a positive solution of (5). Then Proposition 3.4 implies that $v^* \ge w^*$ and $v^* \le w^*$. Therefore, v^* is the unique positive solution of (5). Now clearly, (1) has a unique positive steady state whose w, u components are given by

$$w^*(x) = \frac{s(x)}{b + c(x)v^*(x)}, \; u^*(x) = \frac{s(x)v^*(x)}{a[b+c(x)v^*(x)]}.$$

To consider the stability of (w^*, u^*, v^*), we linearize (1) around (w^*, u^*, v^*) for perturbation of functions $(w, u, v) \in C([0, T), X)$ and obtain the following system

$$\frac{\partial}{\partial t}\begin{pmatrix} w \\ u \\ v \end{pmatrix} = \begin{pmatrix} -b - cv^* & 0 & -cw^* \\ cv^* & -a & cw^* \\ 0 & p & L_q \end{pmatrix}\begin{pmatrix} w \\ u \\ v \end{pmatrix}.$$

Let $\mathcal{L}_* : C(\overline{\Omega}) \to C(\overline{\Omega})$ be defined by

$$\mathcal{L}_* = \begin{pmatrix} -b - cv^* & 0 & -cw^* \\ cv^* & -a & cw^* \\ 0 & p & L_q \end{pmatrix}.$$

In light of the proof of Theorem 3.2, to establish the stability of (w^*, u^*, v^*), it is sufficient to show that there exists $\delta > 0$ for which $\{\lambda \in \mathbb{C} \mid \mathrm{Re}\lambda \geq -\delta\} \subset \rho(\mathcal{L}_*)$. To this end, we first prove that $\lambda \in \rho(\mathcal{L}_*)$ if $0 \in \rho(L_{S_*,\lambda})$. Here $L_{S_*,\lambda}$ is given by $L_0 + S_*(\lambda, x)$ and

$$\begin{aligned} S_*(\lambda, x)v(x) = \bigg[&\frac{pc(x)s(x)}{(\lambda + a)[b + c(x)v^*(x)]} - (\lambda + q) \\ &- \frac{pc^2(x)s(x)v^*(x)}{(\lambda + a)(\lambda + b + cv^*)[b + c(x)v^*(x)]}\bigg]v(x). \end{aligned}$$

Set $m(x) = \frac{pc(x)s(x)}{a[b+c(x)v^*(x)]} - q$ and let $L_m : C(\overline{\Omega}) \to C(\overline{\Omega})$ be defined by $L_m = L_0 + m(x)$. As v^* is the unique positive solution of (5), that is, $L_m v^* = 0$, it follows from [5] that the principal eigenvalue of L_m is zero. Denote the principal eigenvalue of $L_{S_*,\lambda}$ by $\mu_*(\lambda)$. When $\lambda \in \mathbb{R}$ and $\lambda \geq 0$, it is obvious that $S_*(\lambda, x) \lneq m(x)$ for all $x \in \overline{\Omega}$. Hence, it follows from Remark 2.2 that $\mu_*(\lambda) < 0$ provided that $\lambda \geq 0$. In addition, $\mu_*(\lambda)$ is analytic in λ whenever $\mathrm{Re}\lambda > \max\{-a, -b\}$ since $S_*(\lambda, x)$ is analytic in λ. Thus, there exists $\delta > 0$ sufficiently small such that $\mu^*(\lambda) < 0$ for all $\lambda \geq -\delta$. Consequently, $0 \in \rho(L_{S_*,\lambda})$ as long as $\lambda \geq -\delta$. Given $(h_1, h_2, h_3) \in X$, the system

$$\begin{cases} (\lambda + b + cv^*)w + cw^*v = h_1, \\ -cv^*w + (\lambda + a)u - cw^*v = h_2, \\ -pu + \lambda v - L_q v = h_3 \end{cases} \tag{9}$$

has a unique solution given by

$$\begin{aligned} w &= -\frac{cw^*v}{\lambda + b + cv^*} + \frac{h_1}{\lambda + b}, \\ u &= -\frac{c^2w^*v^*v}{(\lambda + a)(\lambda + b + cv^*)} + \frac{cw^*v}{\lambda + a} + \frac{cv^*h_1}{(\lambda + a)(\lambda + b)} + \frac{h_2}{\lambda + a}, \\ v &= L^{-1}_{S_*,\lambda}\left[\frac{-pcv^*h_1}{(\lambda + a)(\lambda + b)} + \frac{-ph_2}{\lambda + a} - h_3\right]. \end{aligned}$$

Namely, $\lambda \in \rho(\mathcal{L}_*)$ if $\lambda \geq -\delta$. In case that $\lambda \in \mathbb{C}$ with $\mathrm{Im}\lambda \neq 0$, by utilizing the same argument given in the proof of Theorem 3.2, we can show that $\lambda \in \rho(\mathcal{L}_*)$ if $\mathrm{Re}\lambda \geq -\delta$. Therefore, $\{\lambda \in \mathbb{C} \mid \mathrm{Re}\lambda \geq -\delta\} \subset \rho(\mathcal{L}_*)$. The proof is completed. ∎

4. Impacts of dispersal rate. In this section, we discuss the impacts of dispersal rate on solutions of (5). The discussion is motivated by an observation made in

Funk et al. [15] that the increased transport rate d_v for viruses between the different sites may give rise to a smoothed viral load between different sites. As argued in Graw and Perelson [16], this may indicate that "the average virus load in the neighborhood of a grid site has a higher influence on the equilibrium viral load at this site than more distant sites". Thus, it is a natural question to ask if similar phenomena can be observed for the spatial dynamics of (5). As matter of the fact, under suitable conditions, it can be shown that solutions of (5) tend to be more spatially homogeneous as d goes to infinity while the solutions of (5) display spatial heterogeneity as d goes to zero.

Let $\zeta(x) \in C(\overline{\Omega})$ be the function satisfying $f(x, \zeta(x)) \equiv 0$. Namely,

$$\zeta(x) = \frac{ps(x)}{aq} - \frac{b}{c} \tag{10}$$

THEOREM 4.1. *Let $\zeta(x)$ be defined by (10). Assume that $\zeta(x) > 0$ for all $x \in \overline{\Omega}$. Then (5) possesses a unique positive solution v_d for each $d > 0$. In particular, v_d converges uniformly to $\zeta(x)$ in $\overline{\Omega}$ as d goes to zero.*

Proof. Since $\zeta > 0$, we have $\overline{pcs/ab - q} > 0$. Hence $\mathcal{S}_0 > 0$. It then follows from Theorem 3.5 that (5) has a unique positive solution v_d for each $d > 0$. Now set

$$\underline{v}_d = \zeta(x) - \sqrt{d}, \quad \overline{v}_d = \zeta(x) + \sqrt{d}.$$

Write $f(x, \tau) = \tau h(x, \tau)$; that is, $h(x, \tau) = \frac{pcs}{a[b+c\tau]} - q$. Using the fact that $h(x, \zeta) = 0$ and the mean value theorem, we have that

$$f(x, \underline{v}_d) = -\sqrt{d} \int_0^1 h_\tau(x, \zeta - t\sqrt{d}) dt \underline{v}_d, \; f(x, \overline{v}_d) = \sqrt{d} \int_0^1 h_\tau(x, \zeta + t\sqrt{d}) dt \overline{v}_d.$$

Notice that

$$\int_0^1 h_\tau(x, \zeta - t\sqrt{d}) dt \underline{v}_d \to h_\tau(x, \zeta)\zeta, \quad \int_0^1 h_\tau(x, \zeta + t\sqrt{d}) dt \overline{v}_d \to h_\tau(x, \zeta)\zeta$$

uniformly in $\overline{\Omega}$ as $d \to 0$. On the other hand, we have

$$L_0 \underline{v}_d = L_0 \overline{v}_d = \sqrt{d} \int_\Omega \sqrt{d} J(x - y)[\zeta(y) - \zeta(x)] dy.$$

As $h_\tau(x, \zeta) < 0$ for all $x \in \overline{\Omega}$, there exists $D > 0$ such that $\underline{v}_d$ and $\overline{v}_d$ are the sub-solution and super-solution of (5), respectively if $d \leq D$. Hence, Proposition 3.4 implies that $\underline{v}_d \leq v_d \leq \overline{v}_d$ provided that $d \leq D$. Then desired conclusion follows. The proof is completed. ∎

PROPOSITION 4.2. *Let v_d be the unique positive solution of (5). Then $v_d \in C^\alpha(\overline{\Omega})$ provided that d is sufficiently large and v_d satisfying $\|v_d\|_{C^\alpha} \leq K$ with some positive constants $\alpha \in (0, 1)$ and $K > 0$ for all $d \geq D$.*

Proof. We first note that there exists $M > 0$ such that $f(x, M) \leq 0$. It is obvious that $\overline{v}_d = M$ is a sub-solution of (5) for all $d > 0$. Hence, it follows from Proposition 3.4 that $|v_d|_{L^\infty(\Omega)} \leq M$. Given $x \in \overline{\Omega}$, let $h > 0$ be chosen so that $B_h(x) \cap \overline{\Omega} \neq \varnothing$, where $B_h(x) := \{y \in \mathbb{R}^n \mid |y-x| < h\}$. Set $v_d^h = v_d(x+h) - v_d(x)$. Then we find

$$\begin{aligned}
&\left[\int_\Omega J(x-y)dy - d^{-1}\int_0^1 f_s(x, tv_d(x+h) + (1-t)v_d(x))dt\right] v_d^h \\
&\quad = \int_\Omega [J(x+h-y) - J(x-y)]v_d(y)dy - \int_\Omega [J(x+h-y) - J(x-y)]dy v_d(x) \\
&\qquad + f(x+h, v_d(x+h)) - f(x, v_d(x+h)).
\end{aligned}$$

Write

$$R_h(x) = \int_\Omega J(x-y)dy - d^{-1}\int_0^1 f_s(x, tv_d(x+h) + (1-t)v_d(x))dt.$$

As $\int_\Omega J(x-y)dy > 0$ for all $x \in \overline{\Omega}$, it is easy to see that $R_h(x) \geq \theta > 0$ for some positive constant θ for all $(x,h) \in \overline{\Omega} \times (0,1)$ as long as d is sufficiently large. In view of (H1) and (H2), we see that $f \in C^{\alpha,1}(\overline{\Omega} \times \mathbb{R}^+)$ for some $\alpha \in (0,1)$. Then notice that

$$\begin{aligned}
\frac{v_d^h}{h^\alpha} = {}& \frac{1}{R_h(x)}\left\{\int_\Omega \left[\frac{J(x+h-y) - J(x-y)}{h^\alpha}\right] v_d(y)dy\right. \\
& \left. - \int_\Omega \left[\frac{J(x+h-y) - J(x-y)}{h^\alpha}\right] dy v_d(x)\right\} \\
& + \frac{1}{R_h(x)}\left\{\frac{f(x+h, v_d(x+h)) - f(x, v_d(x+h))}{h^\alpha}\right\}.
\end{aligned}$$

Due to the assumptions on J and f, there exists $K > 0$ independent of x and h, such that $|h^{-\alpha}v_d^h|_{L^\infty} \leq K$ provided that d is sufficiently small. Thus, the desired conclusion follows. The proof is completed. ∎

Owing to Proposition 4.2 and the Arzelà-Ascoli lemma, $\{v_d\}$ converges to some function $v^* \in C(\overline{\Omega})$ uniformly in $\overline{\Omega}$ as $d \to \infty$. By taking limit in (5), that is

$$\lim_{d\to\infty}\int_\Omega J(x-y)[v_d(y) - v_d(x)]dy = -\lim_{d\to\infty} d^{-1}f(x, v_d),$$

we immediately find that $L_0 v^* = 0$. Since $\ker(L_0) = \operatorname{span}\{1\}$, v^* must be a constant. We have the next theorem.

THEOREM 4.3. *Assume that $\overline{pcs(x)} - abq \geq 0$. Let all the assumptions of Proposition 4.2 are satisfied. Assume that $c(x)$ is independent of $x \in \overline{\Omega}$. Then (5) possesses a unique positive solution v_d for each $d > 0$. In particular, $\{v_d\}$ converges to $v^* = \frac{\overline{pcs(x)} - abq}{acq}$ uniformly in $\overline{\Omega}$ as $d \to \infty$.*

Proof. The existence of a unique positive solution v_d of (5) follows from the same argument as that of Theorem 3.5. The rest of the proof relies on the Crandall-Rabinowitz bifurcation theorem and is similar to that of Theorem A.2 of Cantrell et al. [8]. Let $V = \{u \in C(\overline{\Omega}) \mid \int_\Omega u dx = 0\}$. Write $\mu = d^{-1}$. Let $\Psi : \mathbb{R} \times V \times \mathbb{R}^+ \to X$ be defined by

$$\Psi(k, u, \mu) = \int_\Omega J(x-y)[u(y) - u(x)]dy + \mu(u+k)\left(\frac{pcs(x)}{a[b + c(u+k)]} - q\right),$$

where k is an arbitrary constant. Clearly, $\Psi(k, u, \mu) = 0$ is equivalent to (5) when $\mu > 0$. If $\mu = 0$, then $\Psi(k, u, 0) = 0$ implies that $u = 0$. Let $D\Psi(k, u, \mu)$ denote the

Fréchet derivative of Ψ at (u, μ). Then we have

$$
\begin{aligned}
D\Psi(k,u,\mu)(v,\eta) = &\int_\Omega J(x-y)[v(y)-v(x)]dy + \mu\left(\frac{abpcs(x)}{[ab+ac(u+k)]^2} - q\right)v \\
&+\eta(u+k)\left(\frac{pcs(x)}{a[b+c(u+k)]} - q\right).
\end{aligned}
$$

Thus

$$
D\Psi(k,0,0)(v,\eta) = \int_\Omega J(x-y)[v(y)-v(x)]dy + \eta k\left(\frac{pcs(x)}{a[b+ck]} - q\right).
$$

If $\overline{pcs(x)} - abq > 0$, then there exist two solutions to $\overline{k(pcs(x)/[ab+ack]-q)} = 0$, which are $k_1 = (\overline{pcs(x)} - abq)/acq$ and $k_2 = 0$. If k is equal to neither k_1 nor k_2, that is, $\overline{k(pcs(x)/[ab+ack]-q)} \neq 0$, following Cantrell et al. [8], we can show that $D\Psi(k,0,0) \in B(V\times\mathbb{R}, X)$ is invertible. In fact, assume to the contrary that $\ker D\Psi(k,0,0)\setminus\{\mathbf{0}\} \neq \varnothing$. Let $(u^\star,\eta^\star)\neq 0$ and $(u^\star,\eta^\star)\in \ker D\Psi(k,0,0)$. Then, it is easy to see that

$$
\eta^\star\left[\overline{k(pcs(x)/[ab+ack]-q)}\right] = \int_\Omega\int_\Omega J(x-y)[u^\star(y)-u^\star(x)]dydx = 0.
$$

This implies that $\eta^\star = 0$, and consequently, $u^\star = 0$ as $u^\star \in V$, which is a contradiction. Hence, $\ker D\Psi(k,0,0)\setminus\{\mathbf{0}\} = \varnothing$. Now let $g \in X$, as $\overline{k(pcs(x)/[ab+ack]-q)} \neq 0$, we write $\eta_g = \overline{g}/\overline{k(pcs(x)/[ab+ack]-q)}$. In other words, $\overline{g} = \eta_g \overline{k(pcs(x)/[ab+ack]-q)}$. In view of the Poincaré-type inequality of Andreu et al. [1] and Lemma 2.2 of Bates and Zhao [6], there exists a unique $u_g \in L^2(\Omega)$ such that

$$
\int_\Omega J(x-y)[u_g(y)-u_g(x)]dy = g - \eta_g\left[\frac{kpcs(x)}{ab+ack} - q\right].
$$

In particular, we have

$$
\int_\Omega u_g dx = 0, \quad u_g = \frac{1}{\int_\Omega J(x-y)dy}\left[\int_\Omega J(x-y)u_g(y)dy + \eta_g k\left(\frac{pcs(x)}{a[b+ck]} - q\right) - g\right].
$$

With the same argument as that given in the proof for Theorem 3.2, we infer that $u_g \in C(\overline{\Omega})$. Namely, $\mathrm{Range}(D\Psi(k,0,0)) = X$. Thus, $D\Psi(k,0,0)$ has a bounded inverse. This implies that the line of constants $\{(k,0,0) \mid k\in\mathbb{R}\}$ is the only branch of solutions to $\Psi(k,u,\mu) = 0$ in a neighborhood of $(k,0,0)$.

Now let $k = k_1 = (\overline{pcs(x)} - abq)/acq$, then the same reasoning implies that there exists a unique $v^\circ \in V$ such that

$$
\int_\Omega J(x-y)[v^\circ(y)-v^\circ(x)]dy + k_1\left(\frac{pcs(x)}{a[b+ck_1]} - q\right) = 0.
$$

Therefore, $\ker D\Psi(k_1,0,0) = \{\tau(v^\circ,1), \tau\in\mathbb{R}\}$. In addition, note that

$$
D\Psi(k_1,0,0)(u,\eta) = [D\Psi(k_1,0,0)+\mathcal{H}](u,\eta) - \mathcal{H}(u,\eta),
$$

where $\mathcal{H} : V\times\mathbb{R}\to X$ is given by $\mathcal{H}(u,\eta) = \theta\eta$, $\theta\neq 0$ is a fixed constant, and so $D\Psi(k_1,0,0)(u,\eta)$ is Fredholm of index 0 since $[D\Psi(k_1,0,0)+\mathcal{H}]$ is invertible and $\mathcal{H}$ is compact. Moreover, we have

$$
D_k D\Psi(k_1,0,0)(u,\eta) = \eta\left[\frac{abpcs(x)}{(ab+ck_1)^2} - q\right].
$$

Since $\overline{abpcs(x)/(ab+ck_1)^2 - q} \neq 0$, $D_k D\Psi(k_1,0,0)(u^\circ,1) \notin \text{Range}(D\Psi(k_1,0,0))$. Hence it follows from the Crandall-Rabinowitz bifurcation theorem that there is a nontrivial continuously differentiable curve through $(k_1,0,0)$,

$$\{(k(\tau),v(\tau),\mu(\tau)) \in \mathbb{R}\times V\times\mathbb{R} \mid \tau\in(-\delta,\delta),(k(0),v(0),\mu(0))=(k_1,0,0)\}$$

such that $\Psi(k(\tau),v(\tau),\mu(\tau))=0$ for $\tau\in(-\delta,\delta)$, and $(u,\mu)=\tau(v^\circ,1)+o(\tau)$. Moreover, as $\mu'(0)>0$, it follows from the Inverse Function Theorem that $\mu(\cdot)$ is a differmorphism for $\tau\in(-\epsilon,\epsilon)$ with $\epsilon>0$ being sufficiently small and $\tau=\tau^*(\mu)$ for some $\tau^*\in C^1(\mathbb{R})$. Recall that $\mu=1/d$ if $\mu>0$. Since $k_1>0$ and $k(\tau^*(\mu))+\tau^*(\mu)v^\circ>0$ provided that μ is sufficiently small, thanks to the uniqueness of v_d, there holds $v_d=k(\tau^*(\mu))+u(\tau^*(\mu))$. On the other hand, Proposition 4.2 shows that $v_d\to v^*$ for some $v^*\in C(\overline{\Omega})$ as $d\to\infty$. Thus, $v^*=(\overline{pcs(x)}-abq)/acq$. In addition, the same argument as that given for Theorem A.2 of [8] shows that $k\neq 0$ under the condition that $\overline{pcs(x)}-abq>0$. Hence, we must have $v_d\to\overline{pcs(x)}-abq)/acq$ as $d\to\infty$. In case that $\overline{pcs(x)}-abq=0$, by employing the argument given in Theorem A.2 of [8], we infer that $v_d\to 0$ as $d\to\infty$. Namely, $v_d\to\overline{pcs(x)}-abq$ as $d\to\infty$. The proof is completed. ∎

It is also interesting to ask if $\overline{v}_d$ as a function of d possesses extreme values, and if so, where the extreme values are attained. A study of the differentiability of $\overline{v}_d$ with respect to d may offer useful clues. It can be shown that $v_d: d\to C(\overline{\Omega})$ is differentiable if d is sufficiently small. Suppose that all assumptions of Theorem 4.1 are satisfied. Notice that $f_\tau(x,\zeta(x))=\zeta(x)h_\tau(x,\zeta(x))<0$ for all $x\in\overline{\Omega}$. Let $L^d_\zeta=dL_0+f_\tau(x,\zeta(x))$ and denote its principal eigenvalue by μ_ζ. Due to Lemma 2.1, we have $-\mu_\zeta=\langle -L_\zeta u,u\rangle\geq\inf_{x\in\overline{\Omega}}-f_\tau(x,\zeta)>0$, which implies that $0\in\rho(L_\zeta)$ if L^d_ζ is considered as an operator in $L^2(\Omega)$. Let $f\in L^2(\Omega)$. As L^d_ζ is self-adjoint in $L^2(\Omega)$, $\|u_f\|_{L^2(\Omega)}\leq\theta^{-1}\|f\|_{L^2(\Omega)}$, where $\theta=\inf_{x\in\overline{\Omega}}|f_\tau(x,\zeta|$ and u_f solves $L^d_\zeta w=g$. In particular, if $g\in X:=C(\overline{\Omega})$, then simple calculation yields that

$$u_g=[d\int_\Omega J(x-y)dy-f_\tau(x,\zeta(x))]^{-1}\left\{d\int_\Omega J(x-y)u_g(y)dy+g\right\}.$$

Thus, $u_g\in X$. Moreover, given that $d<1$, then

$$\|u_g\|_X\leq\sup_{x\in\overline{\Omega}}\theta^{-1}\int_\Omega|J(x-y)|^2dy\|u_g\|_{L^2(\Omega)}+\theta^{-1}\|g\|_X\leq C\|g\|_X.$$

Here $C>0$ is a constant depending only on $J,|\Omega|$, and θ. Due to the continuity of f_τ, there exists $\epsilon>0$ sufficiently small such that $\epsilon<\zeta$ and $f_\tau(x,\xi(x))<0$ as long as $\zeta-\epsilon\leq\xi(x)\leq\zeta-\epsilon$. Given that $\xi\in X$. Let $L^d_\xi:=dL_0+f_s(x,\xi)$. Then L^d_ξ is also invertible. In addition, it follows that $\|(L^d_\xi)^{-1}\|\leq\vartheta$ for some $\vartheta>0$ provided that $\|\xi-\varsigma\|\leq\epsilon$ and ϵ is sufficiently small. Hence, by following the same reasoning, $L^d_\xi u=g$ has a unique solution $u_g\in X$ for $g\in X$. In particular, $\|u_g\|\leq C'\|g\|_X$ for some positive constant C'. Given that $h>0$, since

$$(d+h)L_0v_{d+h}+f(x,v_{d+h})=0,\quad dL_0v_d+f(x,v_d)=0,$$

we have

$$dL_0[v_{d+h}-v_d]+\int_0^1 f_\tau(x,tv_{d+h}+(1-t)v_d)dt[v_{d+h}-v_d]=-hL_0v_{d+h}.$$

It follows that

$$\|u_{d+h} - u_d + h(dL_0 + f_\tau(x, v_d))^{-1} L_0 v_d\|_X = o|h|,$$

which apparently shows that v_d is differentiable with respect to d. Notice that $L_0 v_d = d^{-1} f(x, v_d)$. Hence, $\frac{\partial v_d}{\partial d} = (dL_0 + f_\tau(x, v_d))^{-1} f(x, v_d)$. In addition, a straightforward calculation yields that

$$\int_\Omega f(x, v_d) \frac{\partial v_d}{\partial d} dx = 0.$$

5. Asymptotic stability of steady states. In this section, we study the asymptotic behavior of the positive solutions of (1). Similar to the evolution systems studied in Cantrell et al. [7], bounded forward orbits of (1) are generally not pre-compact in the phase space, and so the LaSalle invariance principle is seemingly inapplicable. To cope with this difficulty, we adopt a super- and sub-solution technique to investigate the asymptotic behavior of the bounded positive solutions of (1). Under certain conditions, this technique helps to show that bounded positive solutions of (1) in an invariant manifold (region) converge exponentially to the infection-free steady state $(w^0(x), 0, 0)$ provided that $\mathcal{S}_0 < 0$.

PROPOSITION 5.1. *Assume that $(w, u, v) \in C^1([0, \infty), Y)$ satisfies*

$$\|(w, u, v)\|_{C([0,\infty),Y)} < \infty$$

and

$$\begin{aligned} w_t &\le a_{11} w + a_{12} u + a_{13} v, \\ u_t &\le a_{21} w + a_{22} u + a_{23} v, \\ v_t &\le \int_\Omega J(x-y)[v(y) - v(x)]dy + a_{31} w + a_{32} u + a_{33} v \end{aligned}$$

for $(t, x) \in [0, \infty) \times \overline{\Omega}$, where $a_{i,j} \in C([0, T), X)$ and $a_{i,j} \ge 0$ if $i \ne j$. Furthermore, suppose that $(w(0, x), u(0, x), v(0, x)) \le (0, 0, 0)$ for all $x \in \overline{\Omega}$. Then $(w, u, v) \le (0, 0, 0)$ a.e. in $[0, T) \times \overline{\Omega}$.

Proof. The proof is similar to that for parabolic systems. We only give a sketch. Write $(\breve{w}, \breve{u}, \breve{v}) = (w \vee 0, u \vee 0, v \vee 0)$ and $(\hat{w}, \hat{u}, \hat{v}) = (-w \vee 0, -u \vee 0, -v \vee 0)$. Note that

$$\begin{aligned} w_t &\le a_{11} w + a_{12} \breve{u} + a_{13} \breve{v}, \\ u_t &\le a_{21} \breve{w} + a_{22} u + a_{23} \breve{v}, \\ v_t &\le \int_\Omega J(x-y)[\breve{v}(t, y) - \breve{v}(t, x)]dy + \int_\Omega J(x-y)dy \hat{v} + a_{31} \breve{w} + a_{32} \breve{u} + a_{33} v. \end{aligned}$$

Then we find that

$$\begin{aligned} \frac{d}{dt} \int_\Omega \breve{w}^2 dx &\le 2 \int_\Omega [a_{11} \breve{w}^2 + a_{12} \breve{u}\breve{w} + a_{13} \breve{v}\breve{w}] dx, \\ \frac{d}{dt} \int_\Omega \breve{u}^2 dx &\le 2 \int_\Omega [a_{21} \breve{w}\breve{u} + a_{22} \breve{u}^2 + a_{23} \breve{v}\breve{u}] dx, \\ \frac{d}{dt} \int_\Omega \breve{v}^2 dx &\le 2 \int_\Omega [a_{31} \breve{w}\breve{v} + a_{32} \breve{u}\breve{v} + a_{33} \breve{v}^2] dx. \end{aligned}$$

Thus, Hölder inequality implies that

$$\frac{d}{dt}\int_\Omega [\breve{w}^2+\breve{u}^2+\breve{v}^2]dx \le K\int_\Omega [\breve{w}^2+\breve{u}^2+\breve{v}^2]dx$$

for some positive constant K. As $(\breve{w}_0,\breve{u}_0,\breve{v}_0)=(0,0,0)$, it follows from the comparison principle that $(\breve{w},\breve{u},\breve{v})=(0,0,0)$. $\square$

Definition 5.2. A pair of functions $(w^\pm,u^\pm,v^\pm)\in C^1([0,T),X)$ is said to be a pair of *coupled non-negative super- and sub-solutions* of (1) provided that $(0,0,0)\le (w^-,u^-,v^-)\le (w^+,u^+,v^+)$, and

$$\begin{aligned}
&s(x)-bw^+-c(x)w^+v^- - \frac{\partial w^+}{\partial t}\le 0\le s(x)-bw^- - c(x)w^-v^+ - \frac{\partial w^-}{\partial t},\\
&-au^+ + c(x)w^+v^+ - \frac{\partial u^+}{\partial t}\le 0\le -au^- + c(x)w^-v^- - \frac{\partial u^-}{\partial t},\\
&d\int_\Omega J(x-y)[v^+(t,y)-v^+(t,x)]dy - qv^+ + pu^+ - \frac{\partial v^+}{\partial t}\le 0,\\
&d\int_\Omega J(x-y)[v^-(t,y)-v^-(t,x)]dy - qv^- + pu^- - \frac{\partial v^-}{\partial t}\ge 0,
\end{aligned}$$

where $0<T\le\infty$ is a constant. In this pair, (w^+,u^+,v^+) is called the *super-solution* and (w^-,u^-,v^-) is called the *sub-solution.*

Proposition 5.3. *Assume that there exists a pair of coupled non-negative super- and sub-solutions of (1)* $(w^\pm,u^\pm,v^\pm)$ *in* $[0,\infty)\times\overline{\Omega}$. *In addition, assume that*

$$\|(w^\pm,u^\pm,v^\pm)\|_{C([0,\infty),X)}<\infty.$$

Then given $(w_0,u_0,v_0)\in X$ *with* $(w^-,u^-,v^-)\le (w_0,u_0,v_0)\le (w^+,u^+,v^+)$, *there is a unique solution* (w,u,v) *to (1) satisfying*

$$(w(0,x),u(0,x),v(0,x))=(w_0(x),u_0(x),v_0(x)) \ \ \textit{and}\ \ (w_0,u_0,v_0)\in C^1([0,\infty),X).$$

Moreover,

$$(w^-,u^-,v^-)\le (w,u,v)\le (w^+,u^+,v^+)\quad \textit{for all}\ \ (t,x)\in[0,\infty)\times\overline{\Omega}.$$

Proof. Write $(\overline{w}^0,\overline{u}^0,\overline{v}^0)=(w^+,u^+,v^+)$, $(\underline{w}^0,\underline{u}^0,\underline{v}^0)=(w^-,u^-,v^-)$, let $\alpha>0$ be a constant sufficiently large so as that $\alpha>\|cv^+\|_{C([0,\infty),X)}$. Set

$$\begin{aligned}
\overline{w}^{n+1} &= e^{-(b+\alpha)t}w_0 + \int_0^t e^{-(b+\alpha)(t-\tau)}[s(x)+\alpha\overline{w}^n(\tau,x)-c(x)\overline{w}^n(\tau,x)\underline{v}^n(\tau,x)]d\tau,\\
\overline{u}^{n+1} &= e^{-(a+\alpha)t}u_0 + \int_0^t e^{-(a+\alpha)(t-\tau)}\alpha\overline{u}^{n+1}+\alpha\overline{u}^n(\tau,x)+c(x)\overline{w}^n(\tau,x)\overline{v}^n(\tau,x)d\tau,\\
\overline{v}^{n+1} &= e^{-(q+\alpha)t}v_0\\
&\quad + \int_0^t e^{-(b+\alpha)(t-\tau)}\left[\int_\Omega J(x-y)[\overline{v}^n(\tau,y)-\overline{v}^n(\tau,x)]dy+\alpha\overline{v}^n(\tau,x)+p\overline{u}^n(\tau,x)\right]d\tau,
\end{aligned}$$

and

$$\underline{w}^{n+1} = e^{-(b+\alpha)t}w_0 + \int_0^t e^{-(b+\alpha)(t-\tau)}[s(x) + \alpha \underline{w}^n(\tau,x) - c(x)\underline{w}^n(\tau,x)\overline{v}^n(\tau,x)]d\tau,$$

$$\underline{u}^{n+1} = e^{-(a+\alpha)t}u_0 + \int_0^t e^{-(a+\alpha)(t-\tau)}\alpha \underline{u}^n + c(x)\underline{w}^n(\tau,x)\underline{v}^n(\tau,x)d\tau,$$

$$\underline{v}^{n+1} = e^{-(q+\alpha)t}v_0 + \int_0^t e^{-(b+\alpha)(t-\tau)}\left[\int_\Omega J(x-y)[\underline{v}^n(\tau,y) - \underline{v}^n(\tau,x)]dy + \alpha\underline{v}^n(\tau,x) + p\underline{u}^n(\tau,x)\right]d\tau.$$

First it is straightforward to verify that $(\underline{w}^1, \underline{u}^1, \underline{v}^1)$, $(\overline{w}^1, \overline{u}^1, \overline{v}^1) \in C^1([0,\infty), X)$. Notice that $\alpha w^+ - cw^+v^- \geq \alpha w^+ - cw^+v^+ \geq \alpha w^- - cw^-v^+$ for all $(t,x) \in [0,\infty)\times\overline{\Omega}$. Hence, the comparison principle implies that

$$(w^-, u^-, v^-) \leq (\underline{w}^1, \underline{u}^1, \underline{v}^1) \leq (\overline{w}^1, \overline{u}^1, \overline{v}^1) \leq (w^+, u^+, v^+).$$

By induction, we see that

$$(w^-, u^-, v^-) \leq (\underline{w}^n, \underline{u}^n, \underline{v}^n) \leq (\overline{w}^n, \overline{u}^n, \overline{v}^n) \leq (w^+, u^+, v^+), \quad n \geq 1,$$

and

$$(\underline{w}^n, \underline{u}^n, \underline{v}^n) \leq (\underline{w}^{n+1}, \underline{u}^{n+1}, \underline{v}^{n+1}) \leq (\overline{w}^{n+1}, \overline{u}^{n+1}, \overline{v}^{n+1}) \leq (\overline{w}^n, \overline{u}^n, \overline{v}^n).$$

Clearly, $(\underline{w}^n, \underline{u}^n, \underline{v}^n)$ and $(\overline{w}^n, \overline{u}^n, \overline{v}^n) \in C^1([0,\infty), X)$. In particular, for each $(t,x) \in [0,\infty)\times\overline{\Omega}$, both $(\underline{w}^n, \underline{u}^n, \underline{v}^n)$ and $(\overline{w}^n, \overline{u}^n, \overline{v}^n)$ are monotone and bounded in their components. For fixed $(t,x) \in [0,\infty)\times\overline{\Omega}$, let

$$(w_*(t,x), u_*(t,x), v_*(t,x)) = \lim_{n\to\infty}(\underline{w}^n(t,x), \underline{u}^n(t,x), \underline{v}^n(t,x))$$

and

$$(w^*(t,x), u^*(t,x), v^*(t,x)) = \lim_{n\to\infty}(\overline{w}^n(t,x), \overline{u}^n(t,x), \overline{v}^n(t,x)).$$

Apparently, we have

$$(w^-, u^-, v^-) \leq (w_*, u_*, v_*) \leq (w^*, u^*, v^*) \leq (w^+, u^+, v^+) \tag{11}$$

for all $(t,x) \in [0,\infty)\times\overline{\Omega}$. By using Lebesgue dominated convergence theorem and passing the limits in the above equations, we find that

$$w^* = e^{-(b+\alpha)t}w_0 + \int_0^t e^{-(b+\alpha)(t-\tau)}[s(x) + \alpha w^*(\tau,x) - c(x)w^*(\tau,x)v_*(\tau,x)]d\tau,$$

$$u^* = e^{-(a+\alpha)t}u_0 + \int_0^t e^{-(a+\alpha)(t-\tau)}\alpha u^*(\tau,x) + c(x)w^*(\tau,x)v^*(\tau,x)d\tau,$$

$$v^* = e^{-(q+\alpha)t}v_0 + \int_0^t e^{-(b+\alpha)(t-\tau)}\left[\int_\Omega J(x-y)[v^*(\tau,y) - v^*(\tau,x)]dy + \alpha v^*(\tau,x) + pu^*(\tau,x)\right]d\tau,$$

and

$$w_* = e^{-(b+\alpha)t}w_0 + \int_0^t e^{-(b+\alpha)(t-\tau)}[s(x) + \alpha w_*(\tau,x) - c(x)w_*(\tau,x)v^*(\tau,x)]d\tau,$$

$$u_* = e^{-(a+\alpha)t}u_0 + \int_0^t e^{-(a+\alpha)(t-\tau)}\alpha u_*(\tau,x) + c(x)w_*(\tau,x)v_*(\tau,x)d\tau,$$

$$v_* = e^{-(q+\alpha)t}v_0 + \int_0^t e^{-(b+\alpha)(t-\tau)}\left[\int_\Omega J(x-y)[v_*(\tau,y) - v_*(\tau,x)]dy + \alpha v_*(\tau,x) + pu_*(\tau,x)\right]d\tau.$$

Let $Y = L_\infty(\Omega) \times L_\infty(\Omega) \times L_\infty(\Omega)$. Thanks to the fact that both (w^*, u^*, v^*) and (w_*, u_*, v_*) are bounded, we have that (w^*, u^*, v^*) and $(w_*, u_*, v_*) \in C([0,\infty), Y)$. This implies that (w^*, u^*, v^*) and $(w_*, u_*, v_*) \in C^1([0,\infty), Y)$. Now set $(\widehat{w}, \widehat{u}, \widehat{v}) = (w^*-w_*, u^*-u_*, v^*-v_*)$. Clearly, $(\widehat{w}, \widehat{u}, \widehat{v}) \in C^1([0,\infty), Y)$ and $\|(\widehat{w}, \widehat{u}, \widehat{v})\|_{C([0,\infty),Y)} < \infty$. In addition, by the mean value theorem, we have

$$\begin{aligned} \widehat{w}_t &\le M(\widehat{w} + \widehat{v}), \\ \widehat{u}_t &\le M(\widehat{u} + \widehat{w} + \widehat{v}), \\ \widehat{v}_t &\le \int_\Omega J(x-y)[\widehat{v}(t,y) - \widehat{v}(t,x)]dy + M(\widehat{u} + \widehat{w} + \widehat{v}) \end{aligned}$$

for some positive constant M. As $(\widehat{w}(0,x), \widehat{u}(t,x), \widehat{v}(t,x)) = (0,0,0)$, it follows from Proposition 5.1 that $(w^*(t,\cdot), u^*(t,\cdot), v^*(t,\cdot)) \le (w_*(t,\cdot), u_*(t,\cdot), v_*(t,\cdot))$ a.e. in $\overline{\Omega}$. By (11), we see that $(w^*(t,\cdot), u^*(t,\cdot), v^*(t,\cdot)) = (w_*(t,\cdot), u_*(t,\cdot), v_*(t,\cdot))$ a.e. in $\overline{\Omega}$ for each $t \in (0,\infty)$. Hence, (w^*, u^*, v^*) is a solution of (1) in Y with $(w^*(0), u^*(0), v^*(0)) = (w_0, u_0, v_0)$.

We next show that $(w^*, u^*, v^*) \in C^1([0,\infty), X)$. By virtue of Banach's fixed point theorem, for (w_0, u_0, v_0), there exists a unique solution $(\tilde{w}, \tilde{u}, \tilde{v}) \in C^1([0, T_{max}), X)$ to (1) satisfying $(\tilde{w}(0), \tilde{u}(0), \tilde{v}(0)) = (w_0, u_0, v_0)$ for some $T_{max} > 0$. Obviously, $(\tilde{w}, \tilde{u}, \tilde{v}) \in C^1([0, T_{max}), Y)$, therefore the uniqueness implies that $(w^*, u^*, v^*) = (\tilde{w}, \tilde{u}, \tilde{v})$. The standard argument shows that $T_{max} = \infty$. Namely, $(w^*, u^*, v^*) \in C^1([0,\infty), X)$ is the unique solution of (1). The proof is completed. ∎

To state and prove the next result, denote

$$X_1^+ = \{(w,u,v) \in X \mid 0 \le w \le w^0, u, v \ge 0\}.$$

THEOREM 5.4. *Assume that $\mathcal{S}_0 < 0$. Then (w^0, u^0, v^0) is asymptotically stable in X_1^+. More precisely, given that $(w_0, u_0, v_0) \in X_1^+$, then the solution $(w(t, w_0), u(t, u_0), v(t, v_0))$ of (1) satisfying $(w(0, w_0), u(0, u_0), v(0, v_0)) = (w_0, u_0, v_0)$ exists globally and $(w(t, w_0), u(t, u_0), v(t, v_0)) \in X_1^+$ for all $t > 0$. In particular, $(w(t, w_0), u(t, u_0), v(t, v_0))$ converges exponentially to $(w^0(x), 0, 0)$ as $t \to \infty$.*

Proof. We again let $\mu(\lambda)$ be the principal eigenvalue of $L_{S,\lambda}$ defined in (6). Note that $\mu(\lambda)$ is continuous in λ. Since $\mu(0) = \mathcal{S}_0 < 0$, there exists $\lambda^* < 0$ such that $\mu(\lambda^*) - \lambda^* < 0$. Let $\phi_1 > 0$ be an eigenfunction associated with $\mu(\lambda^*)$. Next let $k > 0$ be a positive constant and set

$$(w^+(t,x), u^+(t,x), v^+(t,x)) = \left(w^0(x), \frac{k}{\lambda^* + a}c(x)w^0(x)\phi_1(x)e^{\lambda^* t}, k\phi_1(x)e^{\lambda^* t}\right)$$

for $(t,x) \in \mathbb{R}^+ \times \overline{\Omega}$ and

$$(w^-(t,x), u^-(t,x), v^-(t,x)) = (0,0,0).$$

It is straightforward to verify that

$$s(x) - bw^+ - c(x)w^+v^- - \frac{\partial w^+}{\partial t} \leq 0,$$

$$-au^+ + c(x)w^+v^+ - \frac{\partial u^+}{\partial t} = c(x)w^0(x)\phi_1(x)e^{\lambda^* t}[-\frac{ak}{\lambda^*+a} + k - \frac{k\lambda^*}{\lambda^*+a}] \leq 0,$$

and

$$\int_\Omega J(x-y)[v^+(t,y) - v^+(t,x)]dy - qv^+(t,x) + pu^+(t,x) - \frac{\partial v^+}{\partial t}$$
$$= ke^{\lambda^* t}\left\{ \int_\Omega J(x-y)[\phi_1(y) - \phi_1(x)]dy + \left(\frac{pc(x)w^0(x)}{\lambda^*+a} - q\right)\phi_1(x) - \lambda^*\phi_1(x)\right\}$$
$$= ke^{\lambda^* t}\phi_1(x)[\mu(\lambda^*) - \lambda^*] \leq 0.$$

In addition, we have

$$s(x) - bw^- - c(x)w^-v^+ - \frac{\partial w^-}{\partial t} = s(x) \geq 0,$$

$$-au^- + c(x)w^-v^- - \frac{\partial u^-}{\partial t} = 0,$$

$$\int_\Omega J(x-y)[v^-(t,y) - v^-(t,x)]dy - qv^-(t,x) + pu^-(t,x) - \frac{\partial v^-}{\partial t} = 0.$$

By Definition 5.2, $(w^\pm, u^\pm, v^\pm)$ given above is a pair of coupled super-sub solutions. Given $(w_0, u_0, v_0) \in X_1^+$, as c, w^0, and ϕ_1 are strictly positive, there exists $k > 0$ such that $(w_0, u_0, v_0) \leq (w^+, u^+, v^+)$ for all $x \in \overline{\Omega}$. Hence, it follows from Proposition 5.3 that

$$(0,0,0) \leq (w(t,t_0,w_0), u(t,t_0,u_0), v(t,t_0,v_0))$$
$$\leq \left(w^0(x), \frac{k}{\lambda^*+a}c(x)w^0(x)\phi_1(x)e^{\lambda^* t}, k\phi_1(x)e^{\lambda^* t}\right) \text{ for all } (t,x) \in \mathbb{R}^+ \times \overline{\Omega}.$$

This immediately implies that $(w(t,t_0,w_0), u(t,t_0,u_0), v(t,t_0,v_0))$ exists for all $t > 0$ and $(u(t,t_0,u_0), v(t,t_0,v_0))$ converges exponentially to $(0,0)$ as $t \to \infty$. We next show that $w(t,t_0,w_0)$ also converges to 0 exponentially as $t \to \infty$.

Notice that

$$\frac{\partial (w-w^0)^2}{\partial t} = -2b(w-w^0)^2 - 2cwv(w-w^0).$$

This shows that

$$(w-w^0)^2 = e^{-2bt}[w(0,x) - w^0(x)]^2 - \int_0^t e^{-2b(t-\tau)}2cwv(w-w^0)vd\tau.$$

Assume without loss of generality that $|\lambda^*| < 2b$, let $K = 2\|cw\|$, then

$$\|w-w^0\|^2 \leq e^{-2bt}\|w-w^0\|^2 + K\int_0^t e^{-2b(t-\tau)}\|v(\tau)\|d\tau$$
$$\leq e^{-2bt}\|w-w^0\|^2 + Ke^{-2bt}\int_0^t e^{(\lambda^*+2b)\tau}d\tau$$
$$= e^{-2bt}\|w-w^0\|^2 + \frac{K}{\lambda^*+2b}[e^{\lambda^* t}(1 - e^{(-2b-\lambda^*)t})].$$

Namely, $w(t,t_0,w_0)$ converges to 0 exponentially as $t \to \infty$. The proof is completed. $\square$

6. Numerical simulations. In this section, we provide numerical approximations of solutions of (1) to illustrate stabilities of both the disease-free steady state and the infection steady state. For the sake of simplicity we assume that all coefficients are a constant. Take

$$s = 1.5,\ b = 2,\ c = 0.001,\ a = 1,\ d = 10, q = 5.5,\ p = 1.$$

One can verify that $\mathcal{S}_0 < 0$, so Theorem 3.5 implies that the disease-free steady state $(0.75, 0, 0)$ is the only non-negative steady state of (1). In addition, it is stable. Given that $\Omega \subset \mathbb{R}$ is a bounded domain, we assume $\Omega = (-1, 1)$ and consider initial data as follows:

$$\begin{aligned} w_0(x) &= 0.55 + 0.01 \sin(3\pi x + 0.1),\\ u_0(x) &= 0.2 + 0.01 \cos(2\pi x + 0.1),\\ v_0(x) &= 0.4 + 0.01 \sin(20\pi x + 0.1). \end{aligned}$$

The snapshots of the solution $(w(t, x), u(t, x), v(t, x))$ with $t = 0, 1.3, 1.6, 1.9$ are given in Fig. 1.

In case that $\Omega \subset \mathbb{R}^2$ is a bounded domain, we assume that $\Omega = (-1, 1) \times (-1, 1)$ and select initial data as follows:

$$\begin{aligned} w_0(x, y) &= 0.55 + 0.01 \sin(3\pi x + 0.1) \cos(3\pi y + 0.1),\\ u_0(x, y) &= 0.2 + 0.01 \cos(2\pi x + 0.1) \sin(2\pi y + 0.1),\\ v_0(x, y) &= 0.4 + 0.01 \sin(5\pi x + 0.1)(x^2 + y^2). \end{aligned}$$

The snapshots of the solution $(w(t, x, y), u(t, x, y), v(t, x, y))$ with $t = 0, 0.5, 0.75, 1.0$ are given in Fig. 2.

To demonstrate stability of the infection steady state, we assume that

$$s = 4,\ b = 2,\ c = 1,\ a = 1,\ d = 10,\ q = 0.5,\ p = 2.$$

Simple calculation shows that the infection steady state is given by $(0.25, 3.5, 14)$, which is the only positive steady state of (1) and is stable. Note that $\mathcal{S}_0 > 0$. When $\Omega \subset \mathbb{R}$, we again assume that $\Omega = (-1, 1)$ and adopt initial data as follows:

$$\begin{aligned} w_0(x) &= 0.3 + 0.01 \sin(3\pi x + 0.1),\\ u_0(x) &= 3 + 0.01 \cos(2\pi x + 0.1),\\ v_0(x) &= 12 + 0.001 \sin(2\pi x + 0.1)e^{-x^2}. \end{aligned}$$

The snapshots of the solution $(w(t, x), u(t, x), v(t, x))$ with $t = 1, 1.3, 1.6, 1.9$ are given in Fig. 3.

In case that $\Omega \subset \mathbb{R}^2$ is a bounded domain, we assume that $\Omega = (-1, 1) \times (-1, 1)$ and choose initial data as follows:

$$\begin{aligned} w_0(x, y) &= 0.3 + 0.01 \sin(3\pi x + 0.1) \cos(3\pi y + 0.1),\\ u_0(x, y) &= 3 + 0.01 \cos(2\pi x + 0.1) \sin(2\pi y + 0.1),\\ v_0(x, y) &= 12 + 0.01(x^2 + y^2) \cos(2\pi y + 0.1)xe^{-(x^2+y^2)}. \end{aligned}$$

The snapshots of the solution $(w(t, x, y), u(t, x, y), v(t, x, y))$ with $t = 0, 0.5, 0.75, 1.0$ are given in Fig. 4.

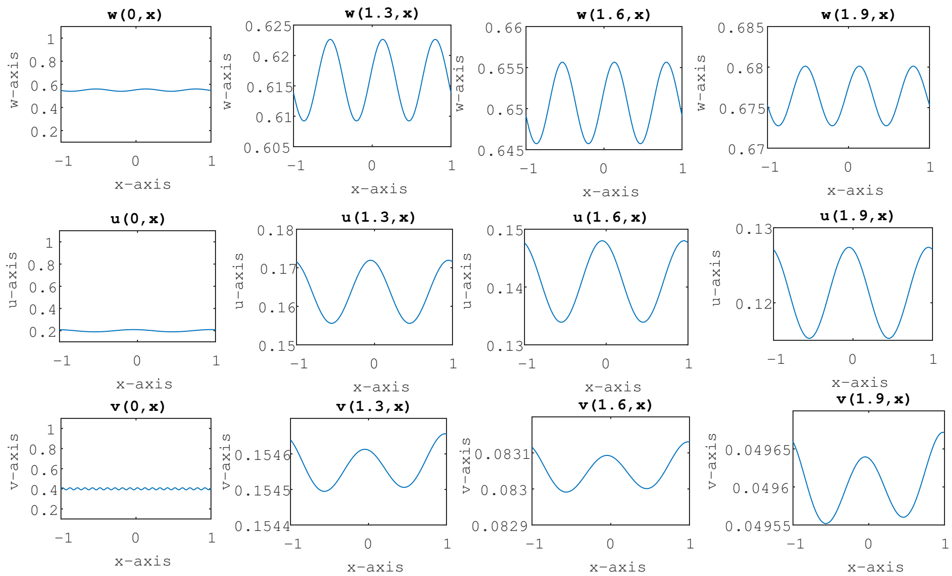

FIG. 1. *The snapshots of the solution* $(w(t,x),u(t,x),v(t,x))$ *of* (1) *in a one-dimensional spatial domain with* $t=0,1.3,1.6,1.9$*, which converges to the disease-free steady state* $(0.75,0,0)$.

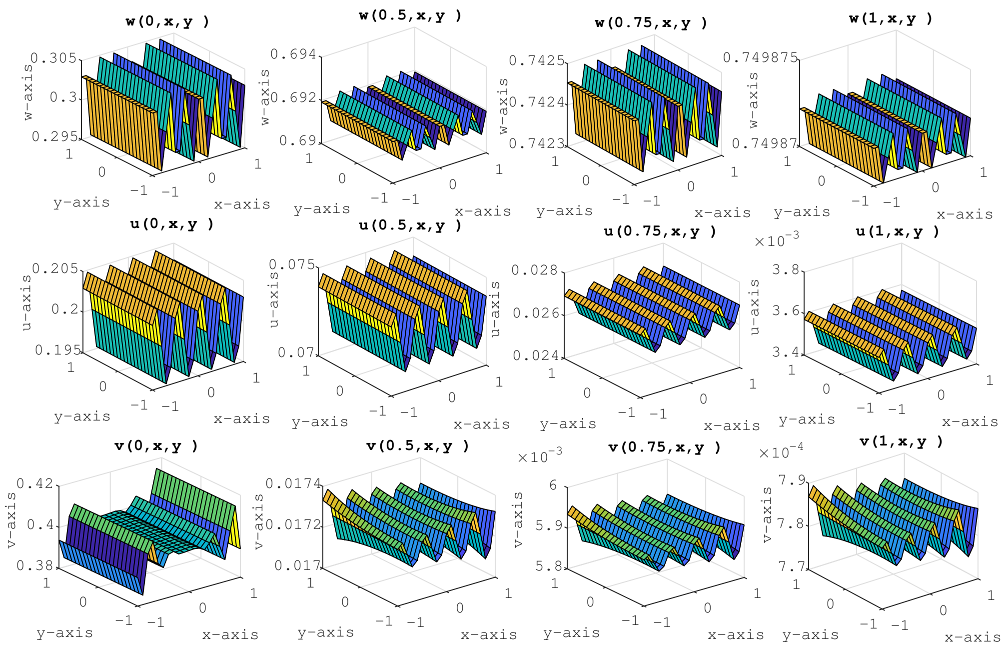

FIG. 2. *The snapshots of the solutions* $(w(t,x,y),u(t,x,y),v(t,x,y))$ *of* (1) *converging to the disease-free steady state* $(0.75,0,0)$ *in a two dimensional spatial domain with* $t = 0, 0.5, 0.75, 1.0$.

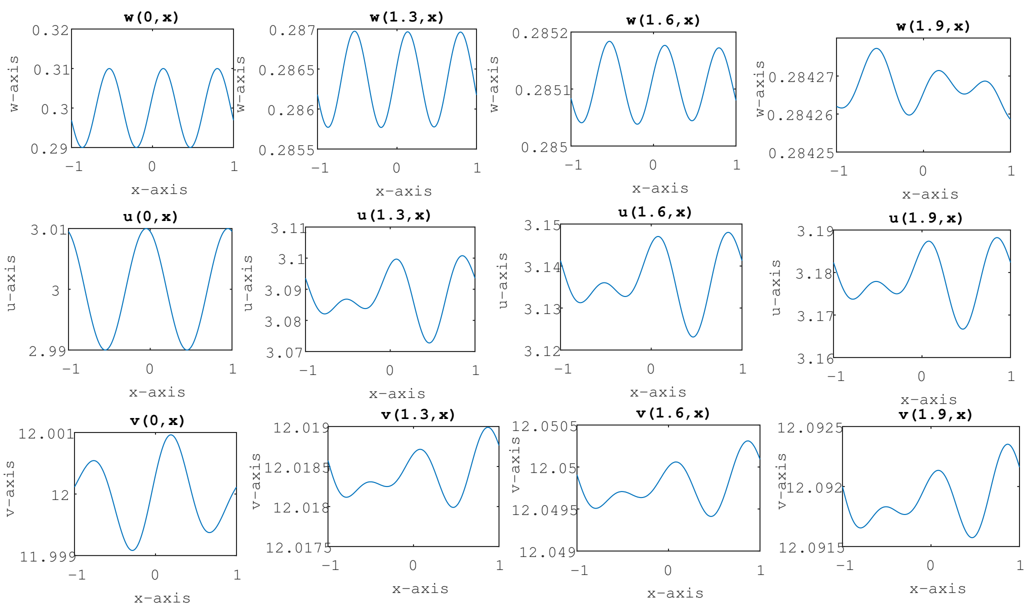

FIG. 3. *The snapshots of the solutions* $(w(t,x), u(t,x), v(t,x))$ *of* (1) *in a one-dimensional spatial domain with* $t = 0, 1.3, 1.6, 1.9$*, which converges to the infection steady state* $(0.25, 3.5, 14)$.

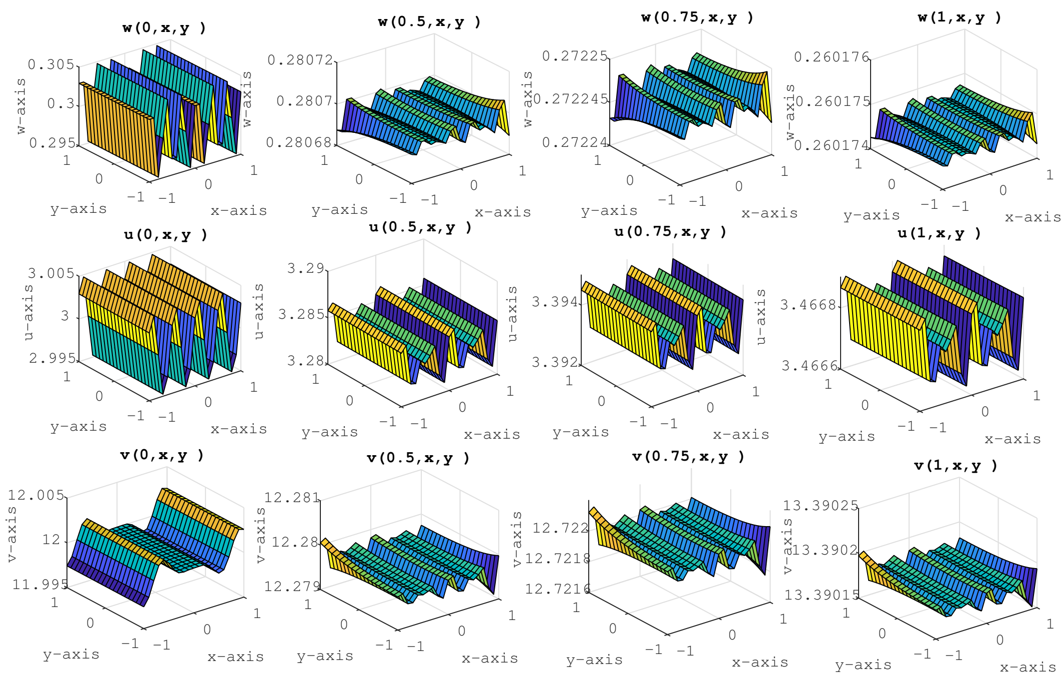

FIG. 4. *The snapshots of the solutions* $(w(t,x,y), u(t,x,y), v(t,x,y))$ *of* (1) *converging to the infection steady state* $(0.25, 3.5, 14)$ *in a two-dimensional spatial domain with* $t = 0, 0.5, 0.75, 1.0$.

7. Discussion. Recent studies suggest that spatial heterogeneity plays an important role in the within-host infection of viruses such as HBV, HCV, and HIV (Graw and Perelson [16], Haase [18], Shulla and Randall [30]). Thus, basic ODE models are not able to capture the spatial aspects of viral infections and spatial models may be more realistic. Under the assumption that target cells and infected cells were stationary while viruses were capable of migrating from one grid site to a neighboring site, Funk et al. [15] used a discrete ordinary differential equation model to study the interactions of target cells, infected cells, and viral load at anatomical sites where each grid site represents different anatomical sites inside the host. Strain et al. [31] introduced a cellular automaton model of viral propagation based on the known biophysical properties of HIV including the competition between viral lability and Brownian motion. Wang and Wang [32] proposed a spatial HBV model of two ODEs coupled with a parabolic PDE for the virus particles and proved the existence of traveling waves.

Nonlocal (convolution) diffusion operators have been used in nonlinear diffusion models to describe the spatial movement of particles or individuals, in which the convolutions represent the rates at which individuals are arriving at one position from other places and are leaving one location to travel to other sites. Such models have been used to study problems in materials science (Bates [3]) and epidemiology (Ruan [28]). In this paper, we proposed a spatial model of viral dynamics with a nonlocal (convolution) diffusion operator describing the spatial spread of virions between cells. The model is a spatial generalization of the ODE model of Nowak and Bangham [22] and a counterpart of the spatially discrete model of Funk et al. [15] in which viron movement is spatially continuous. In section 3, we considered positive stationary solutions of the model and showed that the existence of infection steady states depends upon the sign of the principal eigenvalue of a nonlocal operator. More precisely, when the principal eigenvalue is less than or equal to zero, the only non-negative steady state is the infection-free steady state, which is stable; when the principal eigenvalue is great than zero there is a unique infection steady state, which is stable. In section 4, we studied how the infection steady state depends on the dispersal rate. In section 5, we discussed the asymptotical stability of the infection-free steady state in invariant regions. Therefore, we established threshold dynamics for the nonlocal evolution model of viral infection.

Compared to spatially discrete ODE models (Funk et al. [15]), cellular automaton models (Strain et al. [31]), and diffusive models (Wang and Wang [32]), our model (1) is a first spatial model with a nonlocal (convolution) diffusion operator describing the spatial spread of viruses between cells. The existing studies on other nonlocal evolution models in materials science (Bates [3]) and epidemiology (Ruan [28]) are either concerned with the stability of scalar equations or focused on the existence of traveling waves, while we studied the stability of the steady states for a system of three coupled equations using spectral theory of linear operators. We believe that the modeling approach and analysis technique can be used to investigate other nonlocal diffusion problems.

Acknowledgement. We would like to thank the two anonymous reviewers for their helpful comments and suggestions which helped us to improve the paper significantly.

REFERENCES

[1] F. Andreu, J. M. Mazón, J. D. Rossi and J. Toledo, A nolocal p-Laplacian evolution equation with Neumann boundary conditions, *J. Math. Pures Appl.* **90** (2008), 201-227.
[2] J. Appell, E. D. Pascale and A. Vignoli, *Nonlinear Spectral Theory*, Walter de Gruyter, Berlin, 2004.
[3] P. W. Bates, On some nonlocal evolution equations arising in materials science, in *"Nonlinear Dynamics and Evolution Equations"*, H. Brunner, X.-Q. Zhao and X. Zou (eds.), *Fields Inst. Commun.* **48** (2006), 13-52.
[4] P. W. Bates, P. Fife, X. Ren, and X. Wang, Traveling waves in a convolution model for phase transitions, *Arch. Ration. Mech. Anal.* **138** (1997), 105-136.
[5] P. W. Bates and G. Zhao, Existence, uniqueness and stability of the stationary solution to a nonlocal evoluation equation arising in population dispersal, *J. Math. Anal. Appl.* **332** (2007), 428-440.
[6] P. W. Bates and G. Zhao, Spectral convergence and Turing Patterns for nonlocal diffusion systems, preprint.
[7] R. S Cantrell, C. Cosner, Y. Lou and D. Ryan, Evolutionary stability of ideal free dispersal in spatial population models with nonlocal dispersal, *Can. Appl. Math. Quart.* **20** (2012), 15-38.
[8] R. S Cantrell, C. Cosner and V. Huston, Ecological models, permanence and spatial heterogeneity, *Rocky Mountain J. Math.* **26** (1996), 1-35.
[9] C. Cortazar, M. Elgueta, J. D. Rossi, N. Wolanski, How to approximate the heat equation with Neumann boundary condition by nonlocal diffusion problems, *Arch. Ration. Mech. Anal.***187** (2008), 137-156.
[10] J. Coville, On the principal eigenvalue of some inhomogeneous nonlocal operator in general domains, *J. Differential Equations* **249** (2010), 2921-2953.
[11] N.M. Dixit, J.E. Layden-Almer, T.J. Layden and A.S. Perelson, Modelling how ribavirin improves interferon response rates in hepatitis C virus infection, *Nature* **432** (2004), 922-924.
[12] Q. Du, M. Gunzburger, R. B. Lehoucq and K. Zhou, Analysis and approximation of nonlocal diffusion problems with volume constraints, *SIAM Rev.* **54** (2012), 667-696.
[13] Y. Du, Fixed points of increasing operators in ordered Banach spaces and applications, *Appl. Anal.* **38** (1990), 1-20.
[14] K.-J. Engel and R. Nagel, *One-parameter Semigroups for Linear Evolution Equations,* Springer, New York, 2000.
[15] G. A. Funk, V. A. Jansen, S. Bonhoeffer and T. Killingback, Spatial models of virus-immune dynamics, *J. Theor. Biol.* **233** (2005), 221-236.
[16] F. Graw and A. S. Perelson, Spatial aspects of HIV infection, in *"Mathematical Methods and Models in Biomedicine"*, U. Ledzewicz, H. Schẗtler, A. Friedman and E. Kashdan (eds.), Lecture Notes on Mathematical Modelling in the Life Sciences, Springer, New York, 2013, pp. 3-31.
[17] J. E. F. Green, S. L. Waters, J. P. Whiteley, L. Edelstein-Keshete, K. M. Shakeshefff, and H. M. Byrneb, Non-local models for the formation of hepatocyte-stellate cell aggregates, *J. Theor. Biol.* **267** (2010), 106-120.
[18] A. T. Haase, Targeting early infection to prevent HIV-1 mucosal transmission, *Nature* **464** (2010), 217-223.
[19] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Springer Verlag, New York, 1981.
[20] V. Hutson, S. Martínez, K. Mischaikow and G.T. Vickers, The evolution of dispersal, *J. Math. Biol.* **47** (2003), 483-517.
[21] C. Kao, Y. Lou, and W. Shen, Random dispersal vs. non-local dispersal, *Discrete Contin. Dyn. Syst.* **26** (2010), 551-596.
[22] M. A. Nowak and C. R. Bangham, Population dynamics of immune responses to persistent viruses, *Science* **272** (1996), 74-79.
[23] M. A. Nowak, S. Bonhoffer, A. M. Hill, R. Boehme, H. C. Thomas and H. McDade, Viral dynamics in hepatitis B virus infection, *Proc. Natl. Acad. Sci. USA* **93** (1996), 4398-4402.
[24] M. A. Nowak and R. M. May, *Virus Dynamics: Mathematical Principles of Immunology and Virology,* Oxford University Press, Oxford, 2000.
[25] A. S. Perelson, Modelling viral and immune system dynamics, *Nat. Rev. Immunol.* **2** (2002), 28-36.
[26] N. Rawal and W. Shen, Criteria for the existence and lower bounds of principal eigenvalues of time periodic nonlocal dispersal operators and applications, *J. Dynam. Differential Equations* **24** (2012), 927-954.
[27] M. Reed and B. Simon, *Method of Modern Mathematical Physics, IV: Analysis of Operators*, Academic Press, New York, 1978.

[28] S. Ruan, Spatial-temporal dynamics in nonlocal epidemiological models, in *"Mathematics for Life Science and Medicine"*, Y. Takeuchi, K. Sato and Y. Iwasa (eds.), Springer-Verlag, Berlin, 2007, pp. 99-122 .

[29] K. Schmüdgen, *Unbounded Self-adjoint Operators on Hilbert Space*, Graduate Texts in Mathematics, Springer, New York, 2012.

[30] A. Shulla and G. Randall, Spatiotemporal analysis of hepatitis C virus infection, *PLoS Pathog.* **11** (2015)(3): e1004758. doi:10.1371/journal.ppat.1004758.

[31] M. C. Strain, D. D. Richman, J. K. Wong and H. Levine, Spatiotemporal dynamics of HIV propagation, *J. Theoret. Biol.* **218** (2002), 85-96.

[32] K. Wang and W. Wang, Propagation of HBV with spatial dependence, *Math. Biosci.* **210** (2007), 78-95.

[33] Y. Yang, L. Zou and S. Ruan, Global dynamics of a delayed within-host viral infection model with both virus-to-cell and cell-to-cell transmissions, *Math. Biosci.* **270** (2015), 183-191.