Traveling Wave Solutions in a Two-group Epidemic Model with Latent Period

LIN ZHAO, ZHI-CHENG WANG*

School of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu 730000, People's Republic of China

SHIGUI RUAN†

Department of Mathematics, University of Miami, Coral Gables, FL 33146, USA

December 4, 2016

Abstract

In this paper, we propose a susceptible-infective-recovered (SIR) epidemic model to describe the geographic spread of an infectious disease in two groups/sub-populations living in a spatially continuous habitat. It is assumed that the susceptibility of individuals for infection and the infectivity of individuals are distinct between these two groups/sub-populations. It is also assumed that the infectious disease has a fixed latent period and the latent individuals may diffuse. We investigate the traveling wave solutions and obtain complete information about the existence and nonexistence of nontrivial traveling wave solutions. We prove that when the basic reproduction number $R_0(S_1^0, S_2^0) > 1$ at the disease free equilibrium $(S_1^0, S_2^0, 0, 0)$, there exists a critical number $c^* > 0$ such that for each $c > c^*$, the system admits a nontrivial traveling wave solution with wave speed c, and for $c < c^*$, the system admits no nontrivial traveling wave solution. When $R_0(S_1^0, S_2^0) \leq 1$, we show that there exists no nontrivial traveling wave solution. In addition, for the case $R_0(S_1^0, S_2^0) > 1$ and $c > c^*$, we also find that the final sizes of susceptible individuals, denoted by $(S_{1,0}, S_{2,0})$, satisfy $R_0(S_{1,0}, S_{2,0}) < 1$, which means that there is no outbreak of this infectious disease anymore. Finally, we analyze and simulate the continuous dependence of the minimal speed c^* on the model parameters.

*Research was partially supported by NNSF of China (11371179).

†Corresponding author. E-mail: ruan@math.miami.edu. Research was partially supported by NSF (DMS-1412454).

Keywords: Traveling wave solutions, epidemic model, different susceptibility, different infectivity, latent period

AMS Subject Classification (2000): 35C07, 35R10, 35B40, 35K57, 92D30

1 Introduction

Various factors, such as biological invasion, global warming, environmental degradation, increased international travel, and economic development, continue to provide more opportunities for the emerging and re-emerging of many infectious diseases, and the spatial spread of the infectious diseases has become a subject of continuing interest to both theoreticians and empiricists (Gao and Ruan [23], Hethcote [28], Martens [46], Murray [47], Rass and Radcliffe [48], Ruan [49]). To better understand the geographic spread of infectious diseases, spatial effects have been extensively included into mathematical models and have been quantitatively studied. In general, an epidemic model with spatial effects can give rise to a moving zone of transition from an infective state to a diseases-free state. Hence, traveling wave solutions play a key role in studying the spatial spread of infectious diseases. In the last three decades, there have been many studies on establishing the existence of traveling wave solutions and discussing the asymptotic speed of propagation in epidemic models, see Aronson [2], Ai and Albashaireh [1], Anderson and May [3], Barbour [4], Brown and Carr [7], Diekmann [11, 12], Ducrot [13], Ducrot et al. [15, 16], Fu and Tsai [22], Huang [31, 32], Hosono and Ilyas [30], Kenndy and Aris [37], Li et al. [40], Murray [47], Rass and Radcliffe [48], Ruan [49], Ruan and Wu [50], Smith and Zhao [51], Yang et al. [66], Zhang et al. [67, 68], Zhang and Xu [69], Zhao and Wang [70], and the references therein.

On the other hand, many infectious diseases have latency, namely, the infected individuals do not infect other susceptible individuals until some time later, see Anderson and May [3], Guo et al. [27], Jones et al. [34–36], Li and Zou [38], Lou and Zhao [43], Wang and Zhao [61], Xu and Zhao [64], Zhang and Xu [69], and the references therein. During the latent period the individuals may drift from one spatial point at one time to another spatial point at the other time, and may disperse from a domain to a larger domain. In order to construct more realistic models, the factors of latency of the infectious disease and mobility of the individuals in the latent period should be incorporated into the model. Li and Zou [38] derived a reaction-diffusion system with non-locality and discrete delay by incorporating these two factors into a SIR disease model in a spatially continuous environment. They proved the existence, uniqueness and positivity of solutions to the initial-value problem for the system. In particular, they investigated traveling wave solutions of the system and obtained a critical value which is a lower bound for the wave speed of the traveling wave fronts. However, their discussion is rough and there is no rigorous proof. Ducrot and Magal [17] also studied the existence of traveling wave solutions for a class of epidemic models structured in space and with respect to the age of infection. They obtained a necessary and sufficient condition for the existence of traveling wave solutions for such a class of problems. Wang and Wu [58] further considered a general class of diffusive Kermack-McKendrick SIR models with nonlocal and delayed transmission and showed that the existence and non-existence of traveling wave solutions is completely determined by the basic reproduction number. In particular, they gave the minimal wave speed and discussed how the model parameters (such as the latent period of disease, non-local interaction between the infective and susceptible individuals, and the diffusion rate of infective individuals) affect the minimal wave speed. Besides the above mentioned studies, recently there have been many other papers studying the spatial dynamics of epidemic models with diffusion and latency, see Ducrot and Magal [18], Gao and Ruan [23], Guo et al. [27], Li et al. [41], Li and Zou [39], Lou and Zhao [43], Wang and Wu [59], Wang and Zhao [61], Xu and Zhao [64], Zhang and Xu [69], and the references therein.

As reported by Bonzi et al. [6], Guo et al. [26], Hyman and Li [33], Yang et al. [65] and so on, genetic variability of susceptible individuals may lead to their differentiation in susceptibility to infection. Similarly, individuals may admit differentiation in infectivity. Moreover, many other factors can lead to multi-group epidemic models, such as different social behaviors, different species, different geography and nature, different genders, etc. In fact, many diseases with multiple groups have been described by ODE models or PDE models, for example, HBV, HIV, syphilis, human respiratory syncytial and avian influenza, see Bonzi et al. [6], Cai et al. [8], Demasse and Ducrot [10], Fitzgibbon et al. [20, 21], Hyman and Li [33], Li and Zou [38], Magal and Mccluskey [45], Martcheva and Li [44], Vaidya et al. [55], van den Driessche et al. [56, 57], Yang et al. [65], and references cited therein. Furthermore, such models can better reflect the variance of within group transmission rates and the transmission rates between different groups. Recently, there have also been some studies concerned with traveling wave solutions of diffusive epidemic models with differential susceptibility and differential infectivity. Weng and Zhao [62] proved the existence of the spreading speed and traveling waves for a multi-type SIS epidemic model. Wang et al. [60] established the existence and nonexistence of traveling waves of a reaction-advection-diffusion epidemic model, which describes the spatio-temporal spread of H5N1 avian influenza in an ecosystem involving the virus in the environment and a wide range of bird species. Ducrot et al. [19] studied traveling wave solutions for a multigroup age-structured SIR epidemic models and proved that the existence and nonexistence of traveling wave solutions of the system is also determined by the basic reproduction number. Their results can be applied to the crisscross transmission of feline immunodeficiency virus and some sexual transmission diseases (Fitzgibbon et al. [20, 21]).

The purpose of this paper is to incorporate the latency of the disease, the mobility of individuals in the latent period, and differential susceptibility and differential infectivity of individuals into a SIR disease model in a spatially continuous environment. Without loss of generality, we only consider the spread of an infectious disease in two groups/sub-populations living in a one-dimensional spatial domain $\mathbb{R}$. It is also assumed that the infectious disease has a fixed latent period and the latent individuals diffuse. In Section 2, we derive a reaction-diffusion system with non-locality and time delay. In Section 3, when the basic reproduction number $R_0(S_1^0, S_2^0) > 1$ we prove that there exists a positive number c^* such that for each $c > c^*$, the system admits a nontrivial traveling wave solution with wave speed c. In particular, we prove that the final sizes of susceptible individuals, denoted by $(S_{1,0}, S_{2,0})$, satisfy $R_0(S_{1,0}, S_{2,0}) < 1$, which means that the infectious disease cannot break out again. In Section 4, we prove the nonexistence of nontrivial traveling wave solutions when $R_0(S_1^0, S_2^0) \le 1$ or $R_0(S_1^0, S_2^0) > 1$ and $c < c^*$. In Section 5, we analyze and simulate the continuous dependence of the minimal wave speed c^* on the model parameters. In section 6, we give a brief discussion.

2 Model formulation

Assume that an infectious disease spreads between two groups/sub-populations living in a one-dimensional spatial domain $\mathbb{R}$. In the following we always denote the two groups/sub-populations by subscripts 1 and 2, respectively. Assume that the infectious disease has a fixed latent period and the latent individuals diffuse in the domain. The fixed latent period can be treated as an approximation of the mean latency, denoted by τ. More precisely, newly infected individuals do not infect others immediately but do so after a period τ. We divide each group/sub-population into four sub-groups: the susceptible group, the latent group, the infectious group, and the removed group. The susceptible group consists of individuals who can be infected by the disease; the latent group consists of those who have been infected and do not have an influence on other susceptible individuals; the infective individuals include those who are capable of infecting others; and the removed group includes recovered ones with full immunity, or isolated, or sadly dead. We denote the densities of four groups at time t and location x by $S_i(t,x)$, $L_i(t,x)$, $I_i(t,x)$ and $R_i(t,x)$, respectively, where $i = 1, 2$ and $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$.

Let $E_1(t,a,x)$ and $E_2(t,a,x)$ be the densities of the two groups/sub-populations at the time $t \geqslant 0$ with infection age $a \geqslant 0$ and location $x \in \mathbb{R}$. $D_i(a)$, $\gamma_i(a)$ and $\sigma_i(a)$

represent the diffusion rate, the disease induced mortality rate and the recovery rate at the infection age variable a, respectively. Then the functions $E_1(t,a,x)$ and $E_2(t,a,x)$ satisfy the equations

$$\frac{\partial E_i}{\partial t}(t,a,x)+\frac{\partial E_i}{\partial a}(t,a,x)=D_i(a)\frac{\partial^2 E_i(t,a,x)}{\partial x^2}-(\sigma_i(a)+\gamma_i(a))E_i(t,a,x),\quad i=1,2 \tag{2.1}$$

and the following boundary conditions

$$E_i(t,a,\pm\infty)<\infty \quad \text{and} \quad E_i(t,\infty,x)=0,\quad i=1,2.$$

By the definitions of $L_i(t,x)$ and $I_i(t,x)$, we can conclude that

$$I_i(t,x)=\int_\tau^\infty E_i(t,a,x)da,\quad L_i(t,x)=\int_0^\tau E_i(t,a,x)da,\qquad i=1,2. \tag{2.2}$$

Here ∞ is just a notation which can be replaced by a finite number larger than τ. Differentiating (2.2) with respect to t yields

$$\begin{aligned}\frac{\partial I_i}{\partial t}&=\int_\tau^\infty \frac{\partial E_i}{\partial t}(t,a,x)da\\ &=E_i(t,\tau,x)+\int_\tau^\infty \big[D_i(a)\frac{\partial^2 E_i(t,a,x)}{\partial x^2}-(\sigma_i(a)+\gamma_i(a))E_i(t,a,x)\big]da\end{aligned}$$

and

$$\begin{aligned}\frac{\partial L_i}{\partial t}&=\int_0^\tau \frac{\partial E_i}{\partial t}(t,a,x)da\\ &=E_i(t,0,x)-E_i(t,\tau,x)+\int_0^\tau \big[D_i(a)\frac{\partial^2 E_i(t,a,x)}{\partial x^2}-(\sigma_i(a)+\gamma_i(a))E_i(t,a,x)\big]da.\end{aligned}$$

Here we have used the fact that $E_i(t,\infty,x)=0$ $(i=1,2)$, which has a realistic meaning.

To process further, we assume that $D_i(a)=D_i$, $\sigma_i(a)=\sigma_i$, $\gamma_i(a)=\gamma_i$ for $a\in(\tau,\infty)$. Assume that the two groups/sub-populations can be crisscrossly infected due to the interactions between infectious individuals and susceptible individuals. In addition, we adopt the mass action infection mechanism that the lost of susceptible individuals by infection is at a rate proportional to the number of infectious and susceptible individuals. Let constants β_{ij} $(i,j=1,2)$ be the infection rates, then we have the following conditions

$$E_1(t,0,x)=\beta_{11}S_1I_1+\beta_{12}S_1I_2,\quad E_2(t,0,x)=\beta_{21}S_2I_1+\beta_{22}S_1I_2. \tag{2.3}$$

Thus, the disease dynamics can be described by the following equations

$$\begin{cases}\frac{\partial S_i(t,x)}{\partial t} = & d_i\Delta S_i(t,x)-\beta_{i1}S_i(t,x)I_1(t,x)-\beta_{i2}S_i(t,x)I_2(t,x),\\ \frac{\partial L_1(t,x)}{\partial t} = & \beta_{i1}S_i(t,x)I_1(t,x)+\beta_{i2}S_i(t,x)I_2(t,x)-E_i(t,\tau,x)\\ & +\int_0^\tau \big[D_i(a)\frac{\partial^2 E_i(t,a,x)}{\partial x^2}-(\sigma_i(a)+\gamma_i(a))E_i(t,a,x)\big]da,\\ \frac{\partial I_i(t,x)}{\partial t} = & D_i\Delta I_i(t,x)-(\sigma_i+\gamma_i)I_i(t,x)+E_i(t,\tau,x),\\ \frac{\partial R_i(t,x)}{\partial t} = & D_{R_i}\Delta R_i(t,x)+\int_0^\tau r_{L_i}(a)E_i(t,a,x)da-\gamma_iI_i(t,x)-d_iR_i(t,x)\end{cases} \tag{2.4}$$

whenever $i = 1, 2$. The remaining task is to derive the explicit expressions of $E_i(t, \tau, x)$.

Let

$$\omega_i^s(t, x) = E_i(t, t - s, x), \qquad i = 1, 2 \tag{2.5}$$

for any fixed $s \geqslant 0$ and $s \leqslant t \leqslant s + \tau$. By (2.1), we can obtain that

$$\begin{aligned}\frac{\partial \omega_i^s(t, x)}{\partial t} =& \frac{\partial E_i(t, t - s, x)}{\partial t} + \frac{\partial E_i(t, t - s, x)}{\partial a} \\ =& D_i(t - s)\frac{\partial^2 \omega_i^s(t, x)}{\partial x^2} - [\sigma_i(t - s) + \gamma_i(t - s)]\omega_i^s(t, x)\end{aligned} \tag{2.6}$$

and the corresponding boundary conditions

$$| \, \omega_i^s(t, \pm\infty) \, | < \infty, \quad i = 1, 2. \tag{2.7}$$

Using the standard theory of Fourier transform to (2.6) and (2.7), we can show that

$$\omega_i^s(t, x) = \int_{-\infty}^{+\infty} k_i(s, \omega) \exp\left(-\int_0^{t-s} [\omega^2 D_i(a) + \sigma_i(a) + \gamma_i(a)]da\right) e^{-i\omega x} d\omega, \tag{2.8}$$

where $i = 1, 2$. By (2.3) and (2.5), we have that

$$\beta_{i1} S_i(s, x) I_1(s, x) + \beta_{i2} S_i(s, x) I_2(s, x) = E_i(s, 0, x) = \int_{-\infty}^{+\infty} k_i(s, \omega) e^{-i\omega x} d\omega.$$

Therefore, we obtain

$$k_i(s, \omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} (\beta_{i1} S_i(s, y) I_1(s, y) + \beta_{i2} S_i(s, y) I_2(s, y) e^{i\omega y} dy$$

by the inverse Fourier transform.

For simplicity, we make the following assumptions

$$\begin{aligned} &D_i(a) = D_{L_i}(a) = D_{L_i}, && \sigma_i(a) = \sigma_{L_i}(a) = \sigma_{L_i}, \\ &\gamma_i(a) = \gamma_{L_i}(a) = \gamma_{L_i}, && M_i = \sigma_{L_i} + \gamma_{L_i}, \\ &\alpha_i = \int_0^\tau D_i(a) da = \tau D_{L_i}, && \epsilon_i = e^{-M_i \tau} \end{aligned}$$

for $i = 1, 2$ and $a \in [0, \tau]$. Then we have

$$\begin{aligned} &E_i(t, \tau, x) = \omega_i^{t-\tau}(t, s) \\ =& \int_{-\infty}^{+\infty} k_i(t - \tau, \omega) \exp\left(-\int_0^\tau \left[\omega^2 D_{L_i}(a) + M_i\right] da\right) e^{-i\omega x} d\omega \\ =& \int_{-\infty}^{+\infty} \frac{1}{2\pi} \int_{-\infty}^{+\infty} (\beta_{i1} S_i(t - \tau, y) I_1(t - \tau, y) + \beta_{i2} S_i(t - \tau, y) I_2(t - \tau, y)) e^{i\omega y} dy \\ & \times \exp\left(-\int_0^\tau \left[\omega^2 D_{L_i}(a) + M_i\right] da\right) e^{-i\omega x} d\omega \\ =& \frac{\epsilon_i}{\sqrt{4\pi\alpha_i}} \int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4\alpha_i}} \left[\beta_{i1} S_1(t - \tau, y) I_1(t - \tau, y) + \beta_{i2} S_i(t - \tau, y) I_2(t - \tau, y)\right] dy. \end{aligned} \tag{2.9}$$

Plugging (2.9) into the I_i equation of system (2.4) results in

$$\begin{aligned}\frac{\partial I_i(t,x)}{\partial t} =& D_i \Delta I_i(t,x) - r_i I_i(t,x) \\ &+ \int_{-\infty}^{+\infty} f_i(x-y)[\beta_{i1} S_i(t-\tau,y) I_1(t-\tau,y) + \beta_{i2} S_i(t-\tau,y) I_2(t-\tau,y)]dy\end{aligned}$$

where $i = 1,2$, $f_i(x-y) = \frac{\epsilon_i}{\sqrt{4\pi\alpha_i}} e^{-\frac{(x-y)^2}{4\alpha_i}}$ and $r_i = \sigma_i + \gamma_i$. Similarly, the L_i equation becomes

$$\begin{aligned}\frac{\partial L_i(t,x)}{\partial t} =& D_{L_i} \Delta L_i + [\beta_{i1} S_i(t,x) I_1(t,x) + \beta_{i2} S_i(t,x) I_2(t,x)] - M_i L_i(t,x) \\ &- \int_{-\infty}^{+\infty} f_i(x-y)[\beta_{i1} S_i(t-\tau,y) I_1(t-\tau,y) + \beta_{i2} S_i(t-\tau,y) I_2(t-\tau,y)]dy.\end{aligned}$$

Therefore, the full model (2.4) becomes ($i = 1,2$)

$$\begin{cases} \frac{\partial S_i(t,x)}{\partial t} = & d_i \Delta S_i(t,x) - \beta_{i1} S_i(t,x) I_1(t,x) - \beta_{i2} S_i(t,x) I_2(t,x), \\ \frac{\partial L_i(t,x)}{\partial t} = & D_{L_i} \Delta L_i + [\beta_{i1} S_i(t,x) I_1(t,x) + \beta_{i2} S_i(t,x) I_2(t,x)] - M_i L_i(t,x) \\ & - \int_{-\infty}^{+\infty} f_i(x-y)[\beta_{i1} S_i(t-\tau,y) I_1(t-\tau,y) + \beta_{i2} S_2(t-\tau,y) I_2(t-\tau,y)]dy, \\ \frac{\partial I_i(t,x)}{\partial t} = & D_i \Delta I_i - r_i I_i(t,x) \\ & + \int_{-\infty}^{+\infty} f_i(x-y)[\beta_{i1} S_i(t-\tau,y) I_1(t-\tau,y) + \beta_{i2} S_i(t-\tau,y) I_2(t-\tau,y)]dy, \\ \frac{\partial R_i(t,x)}{\partial t} = & D_{R_i} \Delta R_i(t,x) + \int_0^\tau r_{L_i} E_i(t,a,x)da - \gamma_i I_i(t,x) - d_i R_i(t,x). \end{cases} \tag{2.10}$$

From (2.10), it is obvious that the equations for $S_i(t,x)$ and $I_i(t,x)$ are fully decoupled from $L_i(t,x)$ and $R_i(t,x)$ ($i = 1,2$). Thus, we only need to consider the following subsystem:

$$\begin{cases} \frac{\partial S_1(t,x)}{\partial t} = & d_1 \Delta S_1(t,x) - \beta_{11} S_1(t,x) I_1(t,x) - \beta_{12} S_1(t,x) I_2(t,x), \\ \frac{\partial S_2(t,x)}{\partial t} = & d_2 \Delta S_2(t,x) - \beta_{21} S_2(t,x) I_1(t,x) - \beta_{22} S_2(t,x) I_2(t,x), \\ \frac{\partial I_1(t,x)}{\partial t} = & D_1 \Delta I_1 - r_1 I_1(t,x) \\ & + \int_{-\infty}^{+\infty} f_1(x-y) S_1(t-\tau,y) \left[\beta_{11} I_1(t-\tau,y) + \beta_{12} I_2(t-\tau,y)\right] dy, \\ \frac{\partial I_2(t,x)}{\partial t} = & D_2 \Delta I_2 - r_2 I_2(t,x) \\ & + \int_{-\infty}^{+\infty} f_2(x-y) S_2(t-\tau,y) \left[\beta_{21} I_1(t-\tau,y) + \beta_{22} I_2(t-\tau,y)\right] dy. \end{cases} \tag{2.11}$$

3 Existence of traveling wave solutions

In this section, we establish the existence of traveling wave solutions of model (2.11). The main strategy of the proof comes from Ducrot and Magal [17] and Ducrot et al. [19], see also Wang and Wu [59] and Wang et al. [60]. A traveling wave solution of (2.11) is a special solution with the form $(S_1(\xi), S_2(\xi), I_1(\xi), I_2(\xi)), \xi = x + ct \in \mathbb{R}$. The parameter c is called the wave speed. For any $u \in C(\mathbb{R})$, define the convolution

$$(f_i * u)(x) := \int_{-\infty}^{+\infty} f_i(y)u(x-y)dy, \quad x \in \mathbb{R},\ i = 1, 2.$$

Substituting the ansatz into (2.11), a traveling wave satisfies the following systems of second order differential equations

$$\begin{cases} d_1 S_1''(\xi) - cS_1'(\xi) - \beta_{11}S_1(\xi)I_1(\xi) - \beta_{12}S_1(\xi)I_2(\xi) = 0, \\ d_2 S_2''(\xi) - cS_2'(\xi) - \beta_{21}S_2(\xi)I_1(\xi) - \beta_{22}S_2(\xi)I_2(\xi) = 0, \\ D_1 I_1''(\xi) - cI_1'(\xi) - r_1 I_1(\xi) + (f_1 * (\beta_{11}S_1I_1 + \beta_{12}S_1I_2))(\xi - c\tau) = 0, \\ D_2 I_2''(\xi) - cI_2'(\xi) - r_2 I_2(\xi) + (f_2 * (\beta_{21}S_2I_1 + \beta_{22}S_2I_2))(\xi - c\tau) = 0, \end{cases} \xi \in \mathbb{R}. \quad (3.1)$$

Let $(S_1^0, S_2^0, 0, 0)$ be the initial disease-free equilibrium with $S_i^0 > 0$ $(i = 1, 2)$. We intend to find a traveling wave solution $(S_1(\xi), S_2(\xi), I_1(\xi), I_2(\xi))$ of (3.1) which is nonnegative and satisfies the following boundary conditions

$$S_i(-\infty) = S_i^0, \quad S_i(+\infty) = S_{i,0}, \quad I_i(\pm\infty) = 0, \quad (3.2)$$

where $S_i^0 > S_{i,0}$ and $i = 1, 2$.

The corresponding kinetic system of (2.11) is as follows

$$\begin{cases} \frac{dS_1(t)}{dt} = -\beta_{11}S_1(t)I_1(t) - \beta_{12}S_1(t)I_2(t), \\ \frac{dS_2(t)}{dt} = -\beta_{21}S_2(t)I_1(t) - \beta_{22}S_2(t)I_2(t), \\ \frac{dI_1(t)}{dt} = \epsilon_1\beta_{11}S_1(t-\tau)I_1(t-\tau) + \epsilon_1\beta_{12}S_1(t-\tau)I_2(t-\tau) - r_1I_1(t) \\ \frac{dI_2(t)}{dt} = \epsilon_2\beta_{21}S_2(t-\tau)I_1(t-\tau) + \epsilon_2\beta_{22}S_2(t-\tau)I_2(t-\tau) - r_2I_2(t). \end{cases} \quad (3.3)$$

Let

$$F = \begin{pmatrix} \beta_{11}S_1^0\epsilon_1 & \beta_{12}S_1^0\epsilon_1 \\ \beta_{21}S_2^0\epsilon_2 & \beta_{22}S_2^0\epsilon_2 \end{pmatrix}, \quad V = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}.$$

From [57], we know that the basic reproduction number of system (3.3) at the disease-free equilibrium $(S_1^0, S_2^0, 0, 0)$, denoted by $R_0(S_1^0, S_2^0)$, can be calculated by

$$R_0(S_1^0, S_2^0) = \rho(V^{-1}F),$$

where $\rho(V^{-1}F)$ denotes the principle eigenvalue of the matrix $V^{-1}F$. It is not difficult to show that $R_0(S_1^0, S_2^0) = \rho(V^{-1}F)$ too.

Linearizing the I_i equations of (3.1) at $(S_1^0, S_2^0, 0, 0)$, we obtain the linearized system

$$\begin{cases} D_1 u_1''(\xi) - c u_1'(\xi) - r_1 u_1(\xi) + S_1^0 \left((f_1 * (\beta_{11} u_1 + \beta_{12} u_2)) (\xi - c\tau) = 0, \right. \\ D_2 u_2''(\xi) - c u_2'(\xi) - r_2 u_2(\xi) + S_2^0 \left((f_2 * (\beta_{21} u_1 + \beta_{22} u_2)) (\xi - c\tau) = 0, \right. \end{cases} \xi \in \mathbb{R}.$$

Letting $\binom{u_1(\xi)}{u_2(\xi)} = e^{\lambda \xi} \binom{\eta_1}{\eta_2}$ yields the characteristic equations

$$\begin{cases} D_1 \eta_1 \lambda^2 - c \eta_1 \lambda + S_1^0 (\beta_{11} \eta_1 + \beta_{12} \eta_2) J_1(\lambda, c) - r_1 \eta_1 = 0, \\ D_2 \eta_2 \lambda^2 - c \eta_2 \lambda + S_2^0 (\beta_{21} \eta_1 + \beta_{22} \eta_2) J_2(\lambda, c) - r_2 \eta_2 = 0, \end{cases} \tag{3.4}$$

where $J_i(\lambda, c) = \int_{-\infty}^{+\infty} f_i(y) e^{-\lambda y - c\lambda\tau} dy, \quad i = 1, 2$. Denote

$$\tilde{A} = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}, \quad \tilde{D} = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$$

and

$$\tilde{F} = \begin{pmatrix} \beta_{11} S_1^0 J_1(\lambda, c) & \beta_{12} S_1^0 J_1(\lambda, c) \\ \beta_{21} S_2^0 J_2(\lambda, c) & \beta_{22} S_2^0 J_2(\lambda, c) \end{pmatrix},$$

Let $\Theta(\lambda, c) := \lambda^2 \tilde{A} - \lambda \tilde{B} - \tilde{D} + \tilde{F}$. Then system (3.4) reduces to

$$\Theta(\lambda, c) \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = 0. \tag{3.5}$$

Define $A = \tilde{D}^{-1} \tilde{A}$, $B = \tilde{D}^{-1} \tilde{B}$ and $F = \tilde{D}^{-1} \tilde{F}$. Then (3.5) becomes

$$(-A\lambda^2 + B\lambda + I)^{-1} F \eta = \eta, \tag{3.6}$$

where $\eta = \binom{\eta_1}{\eta_2}$, $m_i(\lambda, c) = -D_i \lambda^2 + c\lambda + r_i$ $(i = 1, 2)$ and

$$(-A\lambda^2 + B\lambda + I)^{-1} F = \begin{pmatrix} \frac{\beta_{11} S_1^0 J_1(\lambda,c)}{m_1(\lambda,c)} & \frac{\beta_{12} S_1^0 J_1(\lambda,c)}{m_1(\lambda,c)} \\ \frac{\beta_{21} S_2^0 J_2(\lambda,c)}{m_2(\lambda,c)} & \frac{\beta_{22} S_2^0 J_2(\lambda,c)}{m_2(\lambda,c)} \end{pmatrix}.$$

Let $M(\lambda, c) = (-A\lambda^2 + B\lambda + I)^{-1} F$, then (3.6) turns to

$$M(\lambda, c) \eta = \eta.$$

Let $\rho(\lambda, c)$ be the principal eigenvalue of $M(\lambda, c)$ and $\lambda(c) = \min_{i=1,2} \frac{c + \sqrt{c^2 + 4D_i r_i}}{2D_i}$. For $c \geq 0$ and $\lambda \in [0, \lambda(c))$, a straightforward computation shows that

$$\begin{aligned} \rho(\lambda, c) = & \frac{1}{2} \Bigg\{ \left(\frac{\beta_{11} S_1^0 J_1(\lambda, c)}{m_1(\lambda, c)} + \frac{\beta_{22} S_2^0 J_2(\lambda, c)}{m_2(\lambda, c)} \right) \\ & + \Bigg[\left(\frac{\beta_{11} S_1^0 J_1(\lambda, c)}{m_1(\lambda, c)} - \frac{\beta_{22} S_2^0 J_2(\lambda, c)}{m_2(\lambda, c)} \right)^2 \\ & + \frac{4 \beta_{12} \beta_{21} S_1^0 S_2^0 J_1(\lambda, c) J_2(\lambda, c)}{m_1(\lambda, c) m_2(\lambda, c)} \Bigg]^{\frac{1}{2}} \Bigg\}. \end{aligned} \tag{3.7}$$

We have the following results.

Proposition 3.1 *We have the following statements*

(i) $\lambda(c)$ *is strictly increasing in* $c \in [0, \infty)$ *and* $\lim_{c\to\infty} \lambda(c) = +\infty$.

(ii) $\rho(0, c) = R_0(S_1^0, S_1^0)$ *for any* $c \in [0, \infty)$, $\rho(\lambda, 0)$ *is strictly increasing in* $\lambda \in [0, \lambda(0))$; *and* $\lim_{\lambda\to\lambda(c)-0} \rho(\lambda, c) = +\infty$ *for any* $c \geq 0$.

(iii) $\frac{\partial}{\partial c}\rho(\lambda, c) < 0$ *for* $\lambda \in (0, \lambda(c))$.

Proof. We only prove (ii) and (iii).

Fix $\lambda \in (0, \lambda(c))$. Differentiating ρ with respect to c yields

$$\begin{aligned}\frac{\partial \rho}{\partial c} =& \frac{1}{2}\left\{\left[\beta_{11}S_1^0 \frac{\partial}{\partial c}\left(\frac{J_1(\lambda,c)}{m_1(\lambda,c)}\right) + \beta_{22}S_2^0 \frac{\partial}{\partial c}\left(\frac{J_2(\lambda,c)}{m_2(\lambda,c)}\right)\right]\right. \\ &+ p(\lambda,c)\left[\left(\beta_{11}S_1^0\frac{J_1(\lambda,c)}{m_1(\lambda,c)} - \beta_{22}S_2^0\frac{J_2(\lambda,c)}{m_2(\lambda,c)}\right)^2\right. \\ &\left.\left.+4\beta_{12}S_1^0\beta_{21}S_2^0\frac{J_1(\lambda,c)}{m_1(\lambda,c)}\frac{J_2(\lambda,c)}{m_2(\lambda,c)}\right]^{-\frac{1}{2}}\right\}, \end{aligned} \tag{3.8}$$

where

$$\begin{aligned}p(\lambda,c) =& \left[\beta_{11}S_1^0 \frac{\partial}{\partial c}\left(\frac{J_1(\lambda,c)}{m_1(\lambda,c)}\right) - \beta_{22}S_2^0 \frac{\partial}{\partial c}\left(\frac{J_2(\lambda,c)}{m_2(\lambda,c)}\right)\right] \\ &\times\left[\beta_{11}S_1^0\frac{J_1(\lambda,c)}{m_1(\lambda,c)} - \beta_{22}S_2^0\frac{J_2(\lambda,c)}{m_2(\lambda,c)}\right] \\ &+2\beta_{12}\beta_{21}S_1^0S_2^0\frac{\partial}{\partial c}\left(\frac{J_1(\lambda,c)}{m_1(\lambda,c)}\right)\frac{J_{21}(\lambda,c)}{m_2(\lambda,c)} + 2\beta_{12}\beta_{21}S_1^0S_2^0\frac{J_1(\lambda,c)}{m_1(\lambda,c)}\frac{\partial}{\partial c}\left(\frac{J_2(\lambda,c)}{m_2(\lambda,c)}\right).\end{aligned}$$

Furthermore, (3.8) reduces to

$$\begin{aligned}\frac{\partial \rho}{\partial c} =& \left\{\beta_{11}S_1^0 \frac{\partial}{\partial c}\left(\frac{J_1(\lambda,c)}{m_1(\lambda,c)}\right)[p_1(\lambda,c)+n(\lambda,c)]\right. \\ &+\beta_{22}S_2^0 \frac{\partial}{\partial c}\left(\frac{J_2(\lambda,c)}{m_2(\lambda,c)}\right)[p_1(\lambda,c)-n(\lambda,c)] \\ &+2\beta_{12}\beta_{21}S_1^0S_2^0\frac{\partial}{\partial c}\left(\frac{J_1(\lambda,c)}{m_1(\lambda,c)}\right)\frac{J_2(\lambda,c)}{m_2(\lambda,c)} \\ &\left.+2\beta_{12}\beta_{21}S_1^0S_2^0\frac{J_1(\lambda,c)}{m_1(\lambda,c)}\frac{\partial}{\partial c}\left(\frac{J_2(\lambda,c)}{m_2(\lambda,c)}\right)\right\}(2p_1(\lambda,c))^{-1},\end{aligned}$$

where

$$n(\lambda,c) = \beta_{11}S_1^0\frac{J_1(\lambda,c)}{m_1(\lambda,c)} - \beta_{22}S_2^0\frac{J_2(\lambda,c)}{m_2(\lambda,c)}$$

and

$$p_1(\lambda,c) = \left[\left(\beta_{11}S_1^0\frac{J_1(\lambda,c)}{m_1(\lambda,c)} - \beta_{22}S_2^0\frac{J_2(\lambda,c)}{m_2(\lambda,c)}\right)^2 + 4\beta_{12}\beta_{21}S_1^0S_2^0\frac{J_1(\lambda,c)}{m_1(\lambda,c)}\frac{J_2(\lambda,c)}{m_2(\lambda,c)}\right]^{\frac{1}{2}}.$$

Since $m_i(\lambda,c)>0$ and $J_i(\lambda,c)>0$ for $\lambda\in(0,\lambda(c))$, we have that $p_1(\lambda,c)+n(\lambda,c)>0$, $p_1(\lambda,c)-n(\lambda,c)>0$ and

$$\frac{\partial}{\partial c}\left(\frac{J_i(\lambda,c)}{m_i(\lambda,c)}\right)=-\lambda J_i(\lambda,c)\left(\tau m_i(\lambda,c)+1\right)\left(m_i(\lambda,c)\right)^{-2}<0,\quad i=1,2.$$

Consequently, we conclude that $\frac{\partial\rho}{\partial c}<0$, which implies that (iii) holds.

It is obvious that $\rho(0,c)=R_0(S_1^0,S_2^0)$ for any $c\geq 0$. Differentiating $\rho(\lambda,0)$ with respect to $\lambda\in(0,\lambda(0))$ gives

$$\begin{aligned}
&\frac{\partial}{\partial\lambda}\rho(\lambda,0)\\
=&\left\{\beta_{11}S_1^0\frac{\partial}{\partial\lambda}\left(\frac{J_1(\lambda,0)}{m_1(\lambda,0)}\right)(p_1(\lambda,0)+n(\lambda,0))+\beta_{22}S_2^0\frac{\partial}{\partial\lambda}\left(\frac{J_2(\lambda,0)}{m_2(\lambda,0)}\right)(p_1(\lambda,0)-n(\lambda,0))\right.\\
&\left.+2\beta_{12}\beta_{21}S_1^0S_2^0\frac{\partial}{\partial\lambda}\left(\frac{J_1(\lambda,0)}{m_1(\lambda,0)}\right)\frac{J_2(\lambda,0)}{m_2(\lambda,0)}+2\beta_{12}\beta_{21}S_1^0S_2^0\frac{J_1(\lambda,0)}{m_1(\lambda,0)}\frac{\partial}{\partial\lambda}\left(\frac{J_2(\lambda,0)}{m_2(\lambda,0)}\right)\right\}\\
&\times\left\{2\left[\left(\beta_{11}S_1^0\frac{J_1(\lambda,0)}{m_1(\lambda,0)}-\beta_{22}S_2^0\frac{J_2(\lambda,0)}{m_2(\lambda,0)}\right)^2+4\beta_{12}\beta_{21}S_1^0S_2^0\frac{J_1(\lambda,0)}{m_1(\lambda,0)}\frac{J_2(\lambda,0)}{m_2(\lambda,0)}\right]^{\frac{1}{2}}\right\}^{-1}.
\end{aligned}$$

Since $m_i(\lambda,0)>0$, $p_1(\lambda,0)+n(\lambda,0)>0$, $p_1(\lambda,0)-n(\lambda,0)>0$ and

$$\frac{\partial}{\partial\lambda}\left(\frac{J_i(\lambda,0)}{m_i(\lambda,0)}\right)=\frac{\partial}{\partial\lambda}\left(J_i(\lambda,0)\right)\left(m_i(\lambda,0)\right)^{-1}+2D_i\lambda J_i(\lambda,0)\left(m_i(\lambda,0)\right)^{-2}>0,$$

we have $\frac{\partial\rho}{\partial\lambda}(\lambda,0)>0$ for $\lambda\in(0,\lambda(0))$. Due to the fact

$$\lim_{\lambda\to\lambda(c)-0}\max\left\{\frac{1}{m_1(\lambda,c)},\frac{1}{m_2(\lambda,c)}\right\}=+\infty\quad\text{for}\quad c\geq 0,$$

it is easy to see that $\lim_{\lambda\to\lambda(c)-0}\rho(\lambda,c)=+\infty$. This completes the proof of (ii). $\square$

Following from Proposition 3.1, we define

$$\Lambda(c)=\min_{\lambda\in[0,\lambda(c))}\rho(\lambda,c)\quad\text{for}\quad c\geq 0.$$

Then we have $\Lambda(0)=R_0(S_1^0,S_2^0)$, $\lim_{c\to\infty}\Lambda(c)=0$ and $\Lambda(c)$ is continuous and strictly decreasing in $c\in[0,\infty)$. Assume $R_0(S_1^0,S_2^0)>1$. It follows that there exists a $c^*>0$ such that $\Lambda(c^*)=1$, $\Lambda(c)>1$ for $c\in[0,c^*)$ and $\Lambda(c)<1$ for $c\in(c^*,\infty)$. Let

$$\lambda_*=\inf\left\{\lambda\in[0,\lambda(c^*)):\rho(\lambda,c^*)=1\right\}.$$

It follows that $\rho(\lambda_*,c^*)=1$ and $\rho(\lambda_*,c)<1$ for any $c>c^*$. Define

$$\lambda_1(c)=\sup\left\{\lambda\in(0,\lambda_*):\rho(\lambda,c)=1,\ \rho(\lambda',c)\geq 1\text{ for any }\lambda'\in(0,\lambda)\right\}.$$

Since $\rho(\lambda_*,c)<1$ for any $c>c^*$, we have the following lemma.

Proposition 3.2 *Assume $R_0(S_1^0, S_2^0) > 1$. Then there exist $c^* > 0$ and $\lambda_* \in (0, \lambda(c^*))$ such that*

(i) $\rho(\lambda, c) > 1$ *for any* $0 \le c < c^*$ *and* $\lambda \in (0, \lambda(c))$*;*

(ii) $\rho(\lambda_*, c^*) = 1$*,* $\rho(\lambda, c^*) > 1$ *for* $\lambda \in (0, \lambda_*)$ *and* $\rho(\lambda, c^*) \ge 1$ *for* $\lambda \in (0, \lambda(c^*))$*;*

(iii) *for any $c > c^*$, there exists $\lambda_1(c) \in (0, \lambda_*)$ such that $\rho(\lambda_1(c), c) = 1$, $\rho(\lambda, c) \ge 1$ for $\lambda \in (0, \lambda_1(c))$ and $\rho(\lambda_1(c) + \varepsilon_n(c), c) < 1$ for some decreasing sequence $\{\varepsilon_n(c)\}$ satisfying $\lim_{n\to\infty} \varepsilon_n = 0$ and $\varepsilon_n + \lambda_1(c) < \lambda_*$ for any $n \in \mathbb{N}$. Especially, $\lambda_1(c)$ is strictly decreasing in $c \in (c^*, \infty)$.*

Since the matrix $M(\lambda, c)$ is nonnegative and irreducible for $\lambda \in [0, \lambda(c))$, using the Perron-Frobenius theorem yields the following proposition.

Proposition 3.3 *Assume $R_0(S_1^0, S_2^0) > 1$. For $c > c^*$, there exist positive unit vectors $\eta(c) = (\eta_1(c), \eta_2(c))^T$ and $\zeta^n(c) = (\zeta_1^n(c), \zeta_2^n(c))^T$ $(n \in \mathbb{N})$ such that*

$$M(\lambda_1(c), c)\eta(c) = \eta(c), \quad M(\lambda_1(c) + \varepsilon_n(c), c)\zeta^n(c) = \rho(\lambda_1(c) + \varepsilon_n(c), c)\zeta^n(c), \quad n \in \mathbb{N}.$$

In the rest of this section, we always assume that $R_0(S_1^0, S_2^0) > 1$. Fix $c > c^*$. Let $\lambda_1(c)$, $\eta(c) = (\eta_1(c), \eta_2(c))^T$, $\varepsilon_n(c)$, and $\zeta^n(c) = (\zeta_1^n(c), \zeta_2^n(c))^T$ $(n \in \mathbb{N})$ be defined in Propositions 3.2 and 3.3. For simplicity, we denote $\lambda_1(c)$, $\eta(c) = (\eta_1(c), \eta_2(c))^T$, $\varepsilon_n(c)$, and $\zeta^n(c) = (\zeta_1^n(c), \zeta_2^n(c))^T$ $(n \in \mathbb{N})$ by λ_1, $\eta = (\eta_1, \eta_2)^T$, ε_n and $\zeta^n = (\zeta_1^n, \zeta_2^n)^T$ $(n \in \mathbb{N})$. Since $\rho(\lambda_1 + \varepsilon_n, c) < 1$, it follows from Proposition 3.3 that

$$\begin{cases} -m_1(\lambda_1, c)\eta_1 + S_1^0 \left(\beta_{11}\eta_1 + \beta_{12}\eta_2\right) J_1(\lambda_1, c) = 0, \\ -m_2(\lambda_1, c)\eta_2 + S_2^0 \left(\beta_{21}\eta_1 + \beta_{22}\eta_2\right) J_2(\lambda_1, c) = 0 \end{cases} \tag{3.9}$$

and

$$\begin{cases} -m_1(\lambda_1 + \varepsilon_n, c)\zeta_1^n + S_1^0 \left(\beta_{11}\zeta_1^n + \beta_{12}\zeta_2^n\right) J_1\left(\lambda_1 + \varepsilon_n, c\right) < 0, \\ -m_2(\lambda_1 + \varepsilon_n, c)\zeta_2^n + S_2^0 \left(\beta_{21}\zeta_1^n + \beta_{22}\zeta_2^n\right) J_2\left(\lambda_1 + \varepsilon_n, c\right) < 0 \end{cases} \tag{3.10}$$

for any $n \in \mathbb{N}$.

Lemma 3.4 *The vector function $P(\xi) = (p_1(\xi), p_1(\xi))^T$ with $p_i(\xi) = \eta_i e^{\lambda_1 \xi}$ satisfies*

$$\begin{cases} D_1 p_1''(\xi) - c p_1'(\xi) + \beta_{11} S_1^0 \int_{-\infty}^{+\infty} f_{\alpha 1}(y) p_1(\xi - y - c\tau) dy \\ +\beta_{12} S_1^0 \int_{-\infty}^{+\infty} f_{\alpha 1}(y) p_2(\xi - y - c\tau) dy - r_1 p_1(\xi) = 0, \\ D_2 p_2''(\xi) - c p_2'(\xi) + \beta_{21} S_2^0 \int_{-\infty}^{+\infty} f_{\alpha 2}(y) p_1(\xi - y - c\tau) dy \\ +\beta_{22} S_2^0 \int_{-\infty}^{+\infty} f_{\alpha 2}(y) p_2(\xi - y - c\tau) dy - r_2 p_2(\xi) = 0. \end{cases}$$

Lemma 3.5 *There exist $0<\alpha<\frac{\lambda_1}{2}$ small enough and $\sigma>\max\left\{S_1^0,S_2^0,1\right\}$ large enough such that the vector value map $Q(\xi)=(q_1(\xi),q_2(\xi))^T$ defined by $q_i(\xi)=\max\{S_i^0(1-\sigma e^{\xi\alpha}),0\}$ satisfies*

$$d_i q_i''(\xi)-cq_i'(\xi)-\beta_{i1}q_i(\xi)p_1(\xi)-\beta_{i2}q_i(\xi)p_2(\xi)\geq 0,\quad i=1,2 \tag{3.11}$$

for any $\xi<\frac{1}{\alpha}\ln\frac{1}{\sigma}$.

The proofs of the last two lemmas are similar to those of Wang and Wu [59, Lemmas 2.1 and 2.2] and Wang et al. [60, Lemmas 3.2 and 3.3], we omit the details.

Lemma 3.6 *Fix $0<\epsilon<\frac{\alpha}{2}$ with $\epsilon=\varepsilon_{n_0}$ for some $n_0\in\mathbb{N}$. Denote the eigenvector $\zeta^{n_0}=(\zeta_1^{n_0},\zeta_2^{n_0})^T$ by $\zeta=(\zeta_1,\zeta_2)^T$. Then the function $H(\xi)=(h_1(\xi),h_2(\xi))^T$ with $h_i(\xi)=\max\{\eta_i e^{\lambda_1\xi}-M\zeta_i e^{(\lambda_1+\epsilon)\xi}),0\}$ satisfies*

$$ch_1'\leq D_1h_1''-r_1h_1+\beta_{11}(f_1*(q_1h_1)(\xi-c\tau)+\beta_{12}(f_1*(q_1h_2))(\xi-c\tau),\quad \xi<\frac{1}{\epsilon}\ln\frac{\eta_1}{M\zeta_1} \tag{3.12}$$

and

$$ch_2'\leq D_2h_2''-r_2h_2+\beta_{21}(f_2(q_2h_1))(\xi-c\tau)+\beta_{22}(f_2*(q_2h_2))(\xi-c\tau),\quad \xi<\frac{1}{\epsilon}\ln\frac{\eta_2}{M\zeta_2}, \tag{3.13}$$

where $M>0$ is large enough so that $\min\left\{\frac{1}{\epsilon}\ln\frac{M\zeta_1}{\eta_1},\frac{1}{\epsilon}\ln\frac{M\zeta_2}{\eta_2}\right\}>\frac{1}{\alpha}\ln\sigma$.

Proof. Firstly, we show only the inequality (3.12). When $\xi<\frac{1}{\epsilon}\ln\frac{\eta_1}{M\zeta_1}$, $h_1(\xi)=\eta_1e^{\lambda_1\xi}-M\zeta_1e^{(\lambda_1+\epsilon)\xi}$. To prove (3.12) for $\xi<\frac{1}{\epsilon}\ln\frac{\eta_1}{M\zeta_1}$, it is sufficient to show the following inequality

$$-\zeta_1Mm_1\left(\lambda_1+\epsilon,c\right)e^{(\lambda_1+\epsilon)\xi}+\eta_1m_1\left(\lambda_1,c\right)e^{\lambda_1\xi}-(f_1*(\beta_{11}q_1h_1+\beta_{12}q_1h_2))(\xi-c\tau)\leq 0. \tag{3.14}$$

Using the first equality of (3.9), the inequality becomes (3.14)

$$\begin{aligned}-\zeta_1Mm_1(\lambda_1+\epsilon,c)e^{(\lambda_1+\epsilon)\xi}&+S_1^0\left(\beta_{11}\eta_1+\beta_{12}\eta_2\right)e^{\lambda_1\xi}J_1(\lambda_1,c)\\&-(f_1*(\beta_{11}q_1h_1+\beta_{12}q_1h_2))(\xi-c\tau)\leq 0.\end{aligned}$$

Since $S_1^0-q_1(\xi)\leq S_1^0\sigma e^{\alpha\xi}$ and $\eta_je^{\lambda_1\xi}-h_j(\xi)\leq\zeta_jMe^{(\lambda_1+\epsilon)\xi}$ for all $\xi\in\mathbb{R}$, we obtain that

$$\begin{aligned}&\beta_{1j}S_1^0\eta_je^{\lambda_1\xi}J_1(\lambda_1,c)-\beta_{1j}(f_1*(q_1h_j))(\xi-c\tau)\\=\;&\beta_{1j}S_1^0\eta_j\int_{-\infty}^{+\infty}f_1(y)e^{\lambda_1(\xi-y-c\tau)}dy-\beta_{1j}S_1^0\int_{-\infty}^{+\infty}f_1(y)h_j(\xi-y-c\tau)dy\\&+\beta_{1j}S_1^0\int_{-\infty}^{+\infty}f_1(y)h_j(\xi-y-c\tau)dy\\&-\beta_{1j}\int_{-\infty}^{+\infty}f_1(y)q_1(\xi-y-c\tau)h_j(\xi-y-c\tau)dy\\\leq\;&\beta_{1j}S_1^0\zeta_jMJ_1(\lambda_1+\epsilon,c)e^{(\lambda_1+\epsilon)\xi}+\beta_{1j}\sigma S_1^0\eta_jJ_1(\lambda_1+\alpha,c)e^{(\lambda_1+\alpha)\xi},\end{aligned} \tag{3.15}$$

where $j=1,2$. Due to the inequality (3.15), we need only to show the following inequality

$$\begin{aligned} M\left[-m_1(\lambda_1+\epsilon,c)\zeta_1+S_1^0\left(\beta_{11}\zeta_1+\beta_{12}\zeta_2\right)J_1(\lambda_1+\epsilon,c)\right] \\ +\sigma S_1^0\left(\beta_{11}\eta_1+\beta_{12}\eta_2\right)J_1(\lambda_1+\alpha,c)e^{(\alpha-\epsilon)\xi}\leq 0 \end{aligned} \tag{3.16}$$

for $\xi<\frac{1}{\epsilon}\ln\frac{\eta_1}{M\zeta_1}$. It follows from the first inequality of (3.10) that

$$-m_1(\lambda_1+\epsilon,c)\zeta_1+S_1^0\left(\beta_{11}\zeta_1+\beta_{12}\zeta_2\right)J_1(\lambda_1+\epsilon,c)<0.$$

For $\xi<\frac{1}{\epsilon}\ln\frac{\eta_1}{M\zeta_1}$, we have that

$$e^{(\alpha-\epsilon)\xi}\leq\left(\frac{M\zeta_1}{\eta_1}\right)^{-\frac{\alpha-\epsilon}{\epsilon}}\leq\left(\frac{M\zeta_1}{\eta_1}\right)^{-2}\to 0\quad\text{as}\quad M\to\infty.$$

Thus the inequality (3.16) holds for $M>\frac{\eta_1}{\zeta_1}$ large enough. This completes the proof. □

Now, we consider the system (2.11) on a large bounded domain $[-X,X]$ with $X>\max\left\{\frac{1}{\epsilon}\ln\frac{M\zeta_1}{\eta_1},\frac{1}{\epsilon}\ln\frac{M\zeta_2}{\eta_2}\right\}$. Let

$$\Gamma_X=\left\{(\chi_1(\cdot),\chi_2(\cdot),\varphi_1(\cdot),\varphi_2(\cdot))\in C([-X,X],\mathbb{R}^4)\,\middle|\,\begin{array}{l}\chi_i(\pm X)=q_i(\pm X);\\ \varphi_i(\pm X)=h_i(\pm X);\\ q_i(\xi)\leq\chi_i(\xi)\leq S_i^0,\ i=1,2;\\ h_i(\xi)\leq\varphi_i(\xi)\leq p_i(\xi),\ i=1,2.\end{array}\right\}.$$

It is easy to see that Γ_X is closed and convex. Define

$$\hat{\chi}_i(\xi)=\begin{cases}\chi_i(\xi), & |\xi|\leqslant X,\\ q_i(\xi), & |\xi|>X\end{cases}\quad\text{and}\quad\hat{\varphi}_i(\xi)=\begin{cases}\varphi_i(\xi), & |\xi|\leqslant X,\\ h_i(\xi), & |\xi|>X,\end{cases}\qquad i=1,2$$

for any $(\chi_1(\xi),\chi_2(\xi),\varphi_1(\xi),\varphi_2(\xi))\in\Gamma_X$. Then we consider the following boundary value problem

$$\begin{cases}-d_1S''_{1,X}+cS'_{1,X}+\beta_{11}\varphi_1S_{1,X}+\beta_{12}\varphi_2S_{1,X}=0,\\ -d_2S''_{2,X}+cS'_{2,X}+\beta_{21}\varphi_1S_{2,X}+\beta_{12}\varphi_2S_{2,X}=0,\\ -D_1I''_{1,X}+cI'_{1,X}+r_1I_{1,X}=\beta_{11}(f_1*(\hat{\chi}_1\hat{\varphi}_1))(\xi-c\tau)+\beta_{12}(f_1*(\hat{\chi}_1\hat{\varphi}_2))(\xi-c\tau),\\ -D_2I''_{2,X}+cI'_{2,X}+r_2I_{2,X}=\beta_{21}(f_2*(\hat{\chi}_2\hat{\varphi}_1))(\xi-c\tau)+\beta_{22}(f_2*(\hat{\chi}_2\hat{\varphi}_2))(\xi-c\tau)\end{cases} \tag{3.17}$$

with

$$S_{i,X}(\pm X)=q_i(\pm X),\quad I_{i,X}(\pm X)=h_i(\pm X),\quad i=1,2. \tag{3.18}$$

Note that the problem (3.17)-(3.18) admits a unique solution

$$(S_{1,X},S_{2,X},I_{1,X},I_{2,X})$$

satisfying $S_{1,X}, S_{2,X}, I_{1,X}, I_{2,X} \in W^{2,p}((-X,X),\mathbb{R}) \cap C([-X,X],\mathbb{R})$ for any $p>1$ (see Gilbarg and Trudinger [24, Corollary 9.18]). By the embedding theorem (see Gilbarg and Trudinger [24, Theorem 7.26]), we have $S_{i,X}(\cdot),\ I_{i,X}(\cdot) \in W^{2,p}(-X,X) \hookrightarrow C^{1+\alpha}[-X,X]$ for some $\alpha \in (0,1)$ and $i=1,2$. Therefore, we can define an operator $\mathcal{T}=(\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3,\mathcal{T}_4):\Gamma_X \to C[-X,X]$ by

$$S_{i,X}=\mathcal{T}_i(\chi_1,\chi_2,\varphi_1,\varphi_2), \quad I_{i,X}=\mathcal{T}_{i+2}(\chi_1,\chi_2,\varphi_1,\varphi_2), \quad \forall\,(\chi_1(\cdot),\chi_2(\cdot),\varphi_1(\cdot),\varphi_2(\cdot)) \in \Gamma_X,$$

and have the following result.

Theorem 3.7 *The operator T maps Γ_X into Γ_X.*

Proof. It is obvious that 0 is a subsolution of the first equation and the second equation of (3.17) on $[-X,X]$, respectively. In addition, S_1^0 and S_2^0 are supersolutions of the first equation and the second equation of (3.17), respectively. Using the maximum principle (see Gilbarg and Trudinger [24, Theorem 9.6]), we have that $0 \le S_{i,X}(\xi) \le S_i^0$ for any $\xi \in [-X,X]$, where $i=1,2$. It follows from (3.11) that $q_1(\xi)=S_1^0(1-\sigma e^{\alpha\xi})$ satisfies

$$\begin{aligned} 0 &\ge -d_1 q_1''(\xi)+cq_1'(\xi)+\beta_{11}q_1(\xi)p_1(\xi)+\beta_{12}q_1(\xi)p_2(\xi) \\ &\ge -d_1 q_1''(\xi)+cq_1'(\xi)+\beta_{11}\varphi_1(\xi)q_1(\xi)+\beta_{12}\varphi_2(\xi)q_1(\xi) \end{aligned}$$

for $\xi \in [-X,X']$ with $X'=\frac{1}{\alpha}\ln\frac{1}{\sigma}$, which implies that $q_1(\xi)$ is a subsolution of the first equation of (3.17) on $[-X,X']$. Since $S_{1,X}(-X)=q_1(-X)$ and $S_{1,X}(X') \ge q_1(X')=0$, we obtain that $q_1(\xi) \leqslant S_{1,X}(\xi)$ for $\xi \in [-X,X']$ by using the maximum principle. Thus, $q_1(\xi) \leqslant S_{1,X}(\xi) \le S_1^0$ for $\xi \in [-X,X]$. By a similar argument, we have $q_2(\xi) \leqslant S_{2,X}(\xi) \leqslant S_2^0$ for any $\xi \in [-X,X]$.

We consider $I_{1,X}(\xi)$ and $I_{2,X}(\xi)$. Firstly, we obtain that $I_{1,X}(\xi) \geqslant 0$ and $I_{2,X}(\xi) \ge 0$ for any $\xi \in [-X,X]$. Since $q_i(\xi) \le \hat{\chi}_i(\xi) \leqslant S_i^0$ and $h_i(\xi) \le \hat{\varphi}_i(\xi) \leqslant p_i(\xi)$ for any $\xi \in \mathbb{R}$, one has

$$(f_i * (\beta_{i1}\hat{\chi}_i\hat{\varphi}_1+\beta_{i2}\hat{\chi}_i\hat{\varphi}_2))(\xi-c\tau) \le \beta_{i1}S_i^0(f_i*p_1)(\xi-c\tau)+\beta_{i2}S_i^0(f_i*p_2)(\xi-c\tau)$$

for $\xi \in \mathbb{R}$ and $i=1,2$, which combing (4.6) implies that $p_1(\xi)$ and $p_2(\xi)$ are supersolutions of the last two equations of (3.17), respectively. Consequently, using the maximum principle yields

$$I_{1,X}(\xi) \leqslant p_1(\xi), \qquad I_{2,X}(\xi) \leqslant p_2(\xi), \qquad \forall \xi \in [-X,X].$$

On the other hand, since

$$(f_i * (\beta_{i1}\hat{\chi}_i\hat{\varphi}_1+\beta_{i2}\hat{\chi}_i\hat{\varphi}_2))(\xi-c\tau) \ge (f_i * (\beta_{i1}q_ih_1+\beta_{i2}q_ih_2))(\xi-c\tau)$$

for $\xi \in \mathbb{R}$ and $i = 1, 2$, using the maximum principle and combining (3.12) and (3.13) we obtain that

$$I_{1,X}(\xi) \geqslant h_1(\xi), \quad I_{2,X}(\xi) \geqslant h_2(\xi), \quad \forall \xi \in [-X, X].$$

Thus, we have that

$$p_1(\xi) \geqslant I_{1,X}(\xi) \geqslant h_1(\xi), \quad p_2(\xi) \geqslant I_{2,X}(\xi) \geqslant h_2(\xi), \quad \forall \xi \in [-X, X].$$

This completes this proof. □

Theorem 3.8 *The operator $\mathcal{T} : \Gamma_X \to \Gamma_X$ is completely continuous.*

Proof. Firstly, we obtain that $\mathcal{T}$ is compact by using the globally elliptic estimate and the embedding theorem.

It is obvious that $\mathcal{T}_3$ and $\mathcal{T}_4$ are continuous. Consider $\mathcal{T}_1$ and $\mathcal{T}_2$. Assume that $(\chi_1^1(\cdot), \chi_2^1(\cdot), \varphi_1^1(\cdot), \varphi_2^1(\cdot)) \in \Gamma_X$ and $(\chi_1^2(\cdot), \chi_2^2(\cdot), \varphi_1^2(\cdot), \varphi_2^2(\cdot)) \in \Gamma_X$. Let

$$S_{i,X}^j = \mathcal{T}_i[\chi_1^j, \chi_2^j, \varphi_1^j, \varphi_2^j], \quad i, j = 1, 2.$$

Then we have that

$$\begin{aligned} &- d_1[S_{1,X}^1 - S_{1,X}^2]''(\xi) + c[S_{1,X}^1 - S_{1,X}^2]'(\xi) \\ &\quad + \beta_{11}\varphi_1^1(\xi)[S_{1,X}^1 - S_{1,X}^2](\xi) + \beta_{12}\varphi_2^1(\xi)[S_{1,X}^1 - S_{1,X}^2](\xi) \\ &\quad = \beta_{11}[\varphi_1^2 - \varphi_1^1]S_{1,X}^2(\xi) + \beta_{12}[\varphi_2^2 - \varphi_2^1]S_{1,X}^2(\xi) \end{aligned}$$

for any $\xi \in (-X, X)$. Since $\varphi_1^1, \varphi_2^1, S_{1,X}^2 \in \Gamma_X$ are uniformly bounded in $C([-X, X])$, we have that the operator $\mathcal{T}_1$ is continuous on Γ_X by using the globally elliptic estimate (see Gilbarg and Trudinger [24, Lemma 9.17]) and the embedding theorem again. Similarly, we can show that $\mathcal{T}_2$ is also continuous on Γ_X. This completes this proof. □

Now applying the Schauder's fixed point theorem to the operator $\mathcal{T}$ yields that there exists a vector function $(S_{1,X}, S_{2,X}, I_{1,X}, I_{2,X}) \in \Gamma_X$ such that

$$(S_{1,X}, S_{2,X}, I_{1,X}, I_{2,X}) = \mathcal{T}(S_{1,X}, S_{2,X}, I_{1,X}, I_{2,X})$$

on $[-X, X]$. Next, we give some estimates for $S_{i,X}$ and $I_{i,X}$, $i = 1, 2$.

Theorem 3.9 *There exists a constant $C_0 > 0$ such that*

$$\|S_{i,X}\|_{C^3(-X,X)} < C_0, \quad \|I_{i,X}\|_{C^3(-X,X)} < C_0, \quad i = 1, 2$$

for any $X > X_0 := \max\left\{\frac{1}{\epsilon}\ln\frac{M\zeta_1}{\eta_1}, \frac{1}{\epsilon}\ln\frac{M\zeta_2}{\eta_2}\right\}$. In particular, one has $(S_{1,X})' \leq 0$ and $(S_{2,X})' \leq 0$ for any $\xi \in (-X, X)$.

Proof. Firstly, we have that

$$\begin{cases} -d_1 S''_{1,X}(\xi) + cS'_{1,X}(\xi) + \beta_{11} I_{1,X}(\xi) S_{1,X}(\xi) + \beta_{12} I_{2,X}(\xi) S_{1,X}(\xi) = 0, \\ -d_2 S''_{2,X}(\xi) + cS'_{2,X}(\xi) + \beta_{21} I_{1,X}(\xi) S_{2,X}(\xi) + \beta_{22} I_{2,X}(\xi) S_{2,X}(\xi) = 0, \\ -D_1 I''_{1,X}(\xi) + cI'_{1,X}(\xi) + r_1 I_{1,X}(\xi) = \left(f_1 * \left(\beta_{11}\hat{S}_{1,X}\hat{I}_{1,X} + \beta_{12}\hat{S}_{1,X}\hat{I}_{2,X}\right)\right)(\xi - c\tau), \\ -D_2 I''_{2,X}(\xi) + cI'_{2,X}(\xi) + r_2 I_{2,X}(\xi) = \left(f_2 * \left(\beta_{21}\hat{S}_{2,X}\hat{I}_{1,X} + \beta_{22}\hat{S}_{2,X}\hat{I}_{2,X}\right)\right)(\xi - c\tau) \end{cases} \tag{3.19}$$

for a.e. $\xi \in (-X, X)$, where

$$\hat{S}_{i,X}(\xi) = \begin{cases} S_{i,X}(\xi), & |\xi| < X, \\ q_i(\xi), & |\xi| \geqslant X, \end{cases} \qquad \hat{I}_{i,X}(\xi) = \begin{cases} I_{i,X}(\xi), & |\xi| < X, \\ h_i(\xi), & |\xi| \geqslant X, \end{cases} \qquad i = 1, 2.$$

Since $S_{i,X}, I_{i,X} \in W^{2,p}\left((-X,X),\mathbb{R}\right) \cap C^0\left([-X,X],\mathbb{R}\right)$ for any $p > 1$, the embedding theorem implies that $S_{i,X}, I_{i,X} \in C^{1+\alpha}[-X,X]$ for some $\alpha \in (0,1)$, $i = 1,2$. Consequently, we have $S_{i,X}, I_{i,X}(\xi) \in C^{2+\alpha}[-X,X]$ (see [24, Theorem 6.14]), $i = 1,2$. Due to the differentiability of $f_i(\cdot)$, we further have $S_{1,X}, S_{2,X}, I_{1,X}(\xi), I_{2,X}(\xi) \in C^3[-X,X]$.

Following (3.19), we know that

$$-d_1 S''_{1,X}(\xi) + cS'_{1,X}(\xi) + \beta_{11} I_{1,X}(\xi) S_{1,X}(\xi) + \beta_{12} I_{2,X}(\xi) S_{1,X}(\xi) = 0, \tag{3.20}$$

which reduces to

$$\left(e^{-\frac{c}{d_1}\xi} S'_{1,X}(\xi)\right)' = \frac{1}{d_1} e^{-\frac{c}{d_1}\xi} \left(\beta_{11} I_{1,X}(\xi) + \beta_{12} I_{2,X}(\xi)\right) S_{1,X}(\xi) \quad \forall \xi \in [-X, X]. \tag{3.21}$$

Integrating the equality (3.21) from $\xi \in [-X, X]$ to X leads to

$$S'_{1,X}(\xi) = e^{-\frac{c}{d_1}(X-\xi)} S'_{1,X}(X) - \frac{1}{d_1}\int_\xi^X e^{\frac{c}{d_1}(\xi - z)} \left(\beta_{11} I_{1,X}(z) + \beta_{12} I_{2,X}(z)\right) S_{1,X}(z) dz.$$

Since $S_{1,X}(\xi) \geq 0 = S_{1,X}(X)$ for $\xi \in [-X, X]$ and $S'_{1,X}(X) \leq 0$, then $S'_{1,X}(\xi) \leq 0$ for $\xi \in [-X, X]$. In particular, $S'_{1,X}(\xi) \not\equiv 0$. Similarly, we get $S'_{2,X}(\xi) \leq 0$ and $S'_{1,X}(\xi) \not\equiv 0$ for $\xi \in [-X, X]$.

Since $S_{1,X}(-X) \leqslant S_1^0$ and $S'_{1,X}(-X) \geqslant q'_1(-X)$, integrating (3.20) on $[-X, X]$ yields that

$$\begin{aligned} \beta_{11} \int_{-X}^X & S_{1,X}(\xi) I_{1,X}(\xi) d\xi + \beta_{12} \int_{-X}^X S_{1,X}(\xi) I_{2,X}(\xi) d\xi \\ &= d_1\big[S'_{1,X}(X) - S'_{1,X}(-X)\big] - c\big[S_{1,X}(X) - S_{1,X}(-X)\big] \\ &\leq cS_1^0 - d_1 q'_1(-X). \end{aligned}$$

By the definitions of $\hat{S}_{i,X}$ and $\hat{I}_{i,X}$, we can show that

$$\begin{aligned}\beta_{11}\int_{-\infty}^{\infty}\hat{S}_{1,X}(\xi)\hat{I}_{1,X}(\xi)d\xi &= \beta_{11}\int_{-\infty}^{-X}q_1(\xi)h_1(\xi)d\xi+\beta_{11}\int_{-X}^{X}S_{1,X}(\xi)I_{1,X}(\xi)d\xi\\ &\le \beta_{11}\int_{-\infty}^{-\frac{1}{\epsilon}\ln\frac{M\zeta_1}{\eta_1}}q_1(\xi)h_1(\xi)d\xi+cS_1^0-d_1q_1'(-X)\end{aligned}$$

and

$$\beta_{12}\int_{-\infty}^{\infty}\hat{S}_{1,X}(\xi)\hat{I}_{2,X}(\xi)d\xi\le\int_{-\infty}^{-\frac{1}{\epsilon}\ln\frac{M\zeta_2}{\eta_2}}q_1(\xi)h_2(\xi)d\xi+cS_1^0-d_1q_1'(-X).$$

Since $I_{1,X}'(X)\le 0$, $I_{1,X}'(-X)\ge h_1'(-X)>0$, $I_{1,X}(-X)=h_1(-X)$, $I_{1,X}(X)=h_1(X)=0$ and $\int_{-\infty}^{+\infty}f_1(y)dy=\epsilon_1$, integrating two sides of the third equation of (3.19) over $[-X,X]$ we have

$$\begin{aligned}r_1\int_{-X}^{X}I_{1,X}(\xi)d\xi &= D_1\big[I_{1,X}'(X)-I_{1,X}'(-X)\big]-c\big[I_{1,X}(X)-I_{1,X}(-X)\big]\\ &\quad+\beta_{11}\int_{-X}^{X}\left(f_1*\left(\hat{S}_{1,X}\hat{I}_{1,X}\right)\right)(\xi-c\tau)d\xi\\ &\quad+\beta_{12}\int_{-X}^{X}\left(f_1*\left(\hat{S}_{1,X}\hat{I}_{2,X}\right)\right)(\xi-c\tau)d\xi\\ &\le ch_1(-X)+\beta_{11}\epsilon_1\int_{-\infty}^{\frac{1}{\epsilon}\ln\frac{\eta_1}{M\zeta_1}}q_1(\xi)h_1(\xi)d\xi\\ &\quad+\beta_{12}\epsilon_1\int_{-\infty}^{\frac{1}{\epsilon}\ln\frac{\eta_2}{M\zeta_2}}q_1(\xi)h_2(\xi)d\xi+2cS_1^0-2d_1q_1'(-X).\end{aligned}$$

Therefore, there exists a constant $C_1>0$ independent of $X>X_0$ such that

$$\int_{-X}^{X}S_{1,X}(\xi)I_{i,X}(\xi)d\xi\le C_1,\quad \int_{-\infty}^{+\infty}\hat{S}_{1,X}(\xi)\hat{I}_{i,X}(\xi)d\xi\le C_1,\quad \int_{-X}^{X}I_{1,X}(\xi)d\xi\le C_1,\quad i=1,2.$$

Similarly, we have

$$\int_{-X}^{X}S_{2,X}(\xi)I_{i,X}(\xi)d\xi\le C_1,\quad \int_{-\infty}^{+\infty}\hat{S}_{2,X}(\xi)\hat{I}_{i,X}(\xi)d\xi\le C_1,\quad \int_{-X}^{X}I_{2,X}(\xi)d\xi\le C_1,\quad i=1,2.$$

Since $I_{1,X}'(-X)>0$, there exists $\xi_0\in(-X,X)$ such that

$$I_{1,X}(\xi_0)=\max_{\xi\in[-X,X]}I_{1,X}(\xi).$$

Integrating both sides of the third equation of system (3.19) from $-X$ to ξ_0, we have

$$cI_{1,X}(\xi_0) = -D_1I_{1,X}'(-X)+cI_1^-(-X)+\beta_{11}\int_{-X}^{\xi_0}\left(f_1*\left(\hat{S}_1\hat{I}_1\right)\right)(\xi-c\tau)d\xi$$

$$
\begin{aligned}
&+\beta_{12}\int_{-X}^{\xi_0}\left(f_1*\left(\hat{S}_1\hat{I}_2\right)\right)(\xi-c\tau)d\xi-r_1\int_{-X}^{\xi_0}I_{1,X}(\xi)d\xi\\
\leq\ & C_0
\end{aligned}
$$

for some $C_0>0$ independent of $X>X_0$. By a similar argument we obtain

$$c\max_{\xi\in[-X,X]}I_{2,X}(\xi)\leq C_0$$

for some $C_0>0$ independent of $X>X_0$.

Integrating the first two equations of (3.19) from $-X$ to $\xi\in[-X,X]$, we get

$$
\begin{aligned}
-d_iS'_{i,X}(\xi) &= -d_iS'_{i,X}(-X)-c[S_{i,X}(\xi)-S_i(-X)]\\
&\quad -\beta_{i1}\int_{-X}^{\xi}S_{i,X}(x)I_{1,X}(x)dx-\beta_{i2}\int_{-X}^{\xi}S_{i,X}(x)I_{2,X}(x)dx\\
&\leq -d_iq'_i(-X)+cS_i^0
\end{aligned}
$$

for any $\xi\in[-X,X]$, which implies that

$$\max_{i=1,2}\max_{\xi\in[-X,X]}\left|S'_{i,X}(\xi)\right|\leq C_0$$

for some $C_0>0$ independent of $X>X_0$. Following the inequality

$$|S''_{i,X}(\xi)|\leq\frac{1}{d_i}\left[|S'_{i,X}(\xi)|+\beta_{i1}S_i^0|I_{1,X}(\xi)|+\beta_{i2}S_i^0|I_{2,X}(\xi)|\right],$$

we know that there exists $C_0>0$ such that $||S_{i,X}||_{C^2([-X,X])}<C_0$ for any $X>X_0$ and $i=1,2$.

Integrating the last two equations of (3.19) from $-X$ to ξ, we have that

$$
\begin{aligned}
D_iI'_{i,X}(\xi) &= D_iI'_{i,X}(-X)+c(I_{i,X}(\xi)-I_{i,X}(-X))+r_i\int_{-X}^{\xi}I_{i,X}(x)dx\\
&\quad -\beta_{i1}\int_{-X}^{\xi}\left(f_i*\left(\hat{S}_{i,X}\hat{I}_{1,X}\right)\right)(x-c\tau)dx\\
&\quad -\beta_{i2}\int_{-X}^{\xi}\left(f_i*\left(\hat{S}_{i,X}\hat{I}_{2,X}\right)\right)(x-c\tau)dx
\end{aligned}
$$

which implies that there exists $C_0>0$ independent of $X>X_0$ such that $I'_{i,X}(\xi)\geq -C_0$ for any $\xi\in[-X,X]$ and $i=1,2$. Similarly, integrating the last two equations of (3.19) from ξ to X, we have that there exists $C_0>0$ independent of $X>X_0$ such that $I'_{i,X}(\xi)\leq C_0$ for any $\xi\in[-X,X]$ and $i=1,2$. Thus, we have $|I'_{i,X}(\xi)|\leq C_0$ for any $\xi\in[-X,X]$. Combining the previous estimates we obtain

$$||I_{1,X}||_{C^2[-X,X]}<C_0,\quad ||I_{2,X}(\xi)||_{C^2[-X,X]}<C_0$$

for some $C_0 > 0$ independent of $X > X_0$.

Differentiating both sides of equations of (3.19), noting that $\int_{-\infty}^{\infty} |y| f_{\alpha i}(y)dy < \infty$, and combining the previous estimates we further get

$$||S_{i,X}(\xi)||_{C^3(-X,X)} < C_0, \quad ||I_{i,X}(\xi)||_{C^3(-X,X)} < C_0, \quad i = 1, 2,$$

where $C_0 > 0$ is independent of $X > X_0$. This completes this proof. □

Now, we show the existence of the nontrivial traveling wave solution of system (2.11) with wave speed $c > c^*$. Let $\{X_n\}$ be an increasing sequence such that $X_n > X_0$ and $\lim_{n\to\infty} X_n = +\infty$. Then the solutions $(S_{1,X_n}, S_{2,X_n}, I_{1,X_n}, I_{2,X_n}) \in \Gamma_{X_n}$ satisfy Theorem 3.9. We can extract a subsequence from the above sequence, still denoted by $(S_{1,X_n}, S_{2,X_n}, I_{1,X_n}, I_{2,X_n})$, such that there exists a vector function $(S_{1,*}, S_{2,*}, I_{1,*}, I_{2,*}) \in C^2(\mathbb{R})$ satisfying

$$S_{1,X_n} \to S_{1,*}, \quad S_{2,X_n} \to S_{2,*}, \quad I_{1,X_n} \to I_{1,*}, \quad I_{2,X_n} \to I_{2,*} \quad in \ \ C^2_{loc}(\mathbb{R}).$$

The Lebesgue's dominated convergence theorem implies that

$$\beta_{ij} \int_{-\infty}^{\infty} f_i(\xi - y - c\tau)\hat{S}_{i,X_n}(y)\hat{I}_{j,X_n}(y)dy \to \beta_{ij} \int_{-\infty}^{\infty} f_i(\xi - y - c\tau)S_{i,*}(y)I_{j,*}(y)dy$$

as $n \to \infty$ for any $\xi \in \mathbb{R}$, where $i, j = 1, 2$. Especially, the function $(S_{1,*}, S_{2,*}, I_{1,*}, I_{2,*})$ satisfies the system (2.11) and

$$q_i(\xi) \le S_{i,*}(\xi) \le S_i^0, \quad h_i(\xi) \le I_{i,*}(\xi) \le \min\{C_0, p_i(\xi)\}, \quad \forall \xi \in \mathbb{R},$$

where C_0 is a positive constant, $i = 1, 2$. Due to $S'_{1,X_n} \le 0$ and $S'_{2,X_n} \le 0$, we have that $S'_{1,*}(\xi) \le 0$ and $S'_{2,*}(\xi) \le 0$ for any $\xi \in \mathbb{R}$.

Let $S_{i,*}(+\infty) = S_{i,0}$, $i = 1, 2$. Then we have the following theorem.

Theorem 3.10 *Let $(S_1^0, S_2^0, 0, 0)$ be the disease-free equilibrium of (2.11). Assume that $R_0(S_1^0, S_2^0) > 1$. Then there exists $c^* > 0$ such that for all $c > c^*$, system (2.11) admits a nontrivial traveling wave solution*

$$(S_{1,*}(x + ct), S_{2,*}(x + ct), I_{1,*}(x + ct), I_{2,*}(x + ct))$$

satisfying

$$\begin{aligned}
&(S_{1,*})'(\xi) < 0, \quad (S_{1,*})'(\xi) < 0, \quad \forall \xi \in \mathbb{R}, \\
&S_{i,*}(-\infty) = S_i^0, \quad S_{i,*}(+\infty) = S_{i,0} < S_i^0, \quad i = 1, 2, \\
&I_{i,*}(\pm\infty) = 0, \quad \int_{-\infty}^{\infty} I_{i,*}(\xi)d\xi = \frac{c\epsilon_i}{r_i}\left(S_i^0 - S_{i,0}\right), \quad i = 1, 2, \\
&I_{1,*}(\xi) \le \epsilon_1\left(S_1^0 - S_{1,0}\right), \quad I_{2,*}(\xi) \le \epsilon_2\left(S_2^0 - S_{2,0}\right), \quad \forall \xi \in \mathbb{R}, \\
&R_0(S_{1,0}, S_{2,0}) < 1.
\end{aligned} \tag{3.22}$$

Proof. We only prove the inequality $R_0(S_{1,0}, S_{2,0}) < 1$, the others can be proved by the arguments similar to [58, pp. 253-254] and [59, pp. 690].

We prove the inequality $R_0(S_{1,0}, S_{2,0}) < 1$ by contradiction. Assume on the contrary that $R_0(S_{1,0}, S_{2,0}) \geq 1$. Then there exists a vector $\tilde{P} = (\tilde{p}_1, \tilde{p}_2)^T \in \mathbb{R}^2$ with $\tilde{p}_1 > 0$ and $\tilde{p}_2 > 0$ such that

$$\begin{pmatrix} \frac{\epsilon_1\beta_{11}S_{1,0}}{r_1} & \frac{\epsilon_1\beta_{12}S_{1,0}}{r_1} \\ \frac{\epsilon_2\beta_{21}S_{2,0}}{r_2} & \frac{\epsilon_2\beta_{22}S_{2,0}}{r_2} \end{pmatrix} \begin{pmatrix} \tilde{p}_1 \\ \tilde{p}_2 \end{pmatrix} = R_0(S_{1,0}, S_{2,0}) \begin{pmatrix} \tilde{p}_1 \\ \tilde{p}_2 \end{pmatrix}.$$

By virtue of $R_0(S_{1,0}, S_{2,0}) \geq 1$, it follows that

$$\left[\frac{\epsilon_1\beta_{11}S_{1,0}}{r_1} - 1\right]\tilde{p}_1 + \frac{\epsilon_1\beta_{12}S_{1,0}}{r_1}\tilde{p}_2 \geqslant 0 \quad \text{and} \quad \frac{\epsilon_2\beta_{21}S_{2,0}}{r_2}\tilde{p}_1 + \left[\frac{\epsilon_2\beta_{22}S_{2,0}}{r_2} - 1\right]\tilde{p}_2 \geqslant 0,$$

which reduces to

$$\frac{\epsilon_1\beta_{11}S_{1,0}}{r_1} - 1 \geqslant -\frac{\epsilon_1\beta_{12}S_{1,0}}{r_1}\frac{\tilde{p}_2}{\tilde{p}_1} \quad \text{and} \quad \frac{\epsilon_2\beta_{22}S_{2,0}}{r_2} - 1 \geqslant -\frac{\epsilon_2\beta_{21}S_{2,0}}{r_2}\frac{\tilde{p}_1}{\tilde{p}_2}. \tag{3.23}$$

Note that $I_{i,*}(\pm\infty) = 0$, $I'_{i,*}(\pm\infty) = 0$ and $(S_{1,*}, S_{2,*}, I_{1,*}, I_{2,*})$ satisfies system (3.1). Integrating both sides of the third equation and the forth equation of (3.1) satisfied by $(S_{1,*}, S_{2,*}, I_{1,*}, I_{2,*})$ from $-\infty$ to ∞ we get

$$\begin{cases} \int_{-\infty}^{+\infty} \left[\epsilon_1\beta_{11}S_{1,*}(\xi) - r_1\right] I_{1,*}(\xi)d\xi + \int_{-\infty}^{+\infty} \epsilon_1\beta_{12}S_{1,*}(\xi)I_{2,*}(\xi)d\xi = 0, \\ \int_{-\infty}^{+\infty} \epsilon_2\beta_{21}S_{2,*}(\xi)I_{1,*}(\xi)d\xi + \int_{-\infty}^{+\infty} \left[\epsilon_2\beta_{22}S_{2,*}(\xi) - r_2\right] I_{2,*}(\xi)d\xi = 0. \end{cases} \tag{3.24}$$

Since $S'_{i,*}(\xi) < 0$ and $I_{i,*}(\xi) > 0$ for any $\xi \in \mathbb{R}$, it follows from (3.24) that

$$\begin{cases} \left[\frac{\epsilon_1\beta_{11}S_{1,0}}{r_1} - 1\right] \int_{-\infty}^{+\infty} I_{1,*}(\xi)d\xi + \frac{\epsilon_1\beta_{12}S_{1,0}}{r_1} \int_{-\infty}^{+\infty} I_{2,*}(\xi)d\xi < 0, \\ \frac{\epsilon_2\beta_{21}S_{2,0}}{r_2} \int_{-\infty}^{+\infty} I_{1,*}(\xi)d\xi + \left[\frac{\epsilon_2\beta_{22}S_{2,0}}{r_2} - 1\right] \int_{-\infty}^{+\infty} I_{2,*}(\xi)d\xi < 0. \end{cases} \tag{3.25}$$

Plugging (3.23) into (3.25), we have

$$\begin{cases} -\tilde{p}_2 \int_{-\infty}^{+\infty} I_{1,*}(\xi)d\xi + \tilde{p}_1 \int_{-\infty}^{+\infty} I_{2,*}(\xi)d\xi < 0, \\ \tilde{p}_2 \int_{-\infty}^{+\infty} I_{1,*}(\xi)d\xi - \tilde{p}_1 \int_{-\infty}^{+\infty} I_{2,*}(\xi)d\xi < 0, \end{cases}$$

which is a contradiction. This completes the proof. □

4 Nonexistence of traveling wave solutions

In this section, we show the nonexistence of traveling wave solutions of system (2.11) for two cases: (i) $R_0(S_1^0, S_2^0) \leqslant 1$; (ii) $R_0(S_1^0, S_2^0) > 1$ and $c \in (0, c^*)$.

4.1 $R_0(S_1^0, S_2^0) \le 1$

Theorem 4.1 *Assume that $R_0(S_1^0, S_2^0) \le 1$. There is no nonnegative traveling wave $(S_1(x+ct), S_2(x+ct), I_1(x+ct), I_2(x+ct))$ of (2.11) satisfying (3.22).*

Proof. We prove the theorem by contradiction. On the contrary, assume that there exists a nonnegative traveling wave solution $(S_1(x+ct), S_2(x+ct), I_1(x+ct), I_2(x+ct))$ satisfying (3.22). In the following we consider two cases $R_0(S_1^0, S_2^0) < 1$ and $R_0(S_1^0, S_2^0) = 1$, respectively.

Assume that $R_0(S_1^0, S_2^0) < 1$. Note that $I_i(\xi)$ satisfies

$$D_i I_i''(\xi) - cI_i'(\xi) - r_i I_i(\xi) + (f_i * (\beta_{i1} S_i I_1 + \beta_{i2} S_i I_2))(\xi - c\tau) = 0 \tag{4.1}$$

for any $\xi \in \mathbb{R}$, where $i = 1, 2$. Let $\Lambda_{i1} = \frac{c - \sqrt{c^2 + 4D_i r_i}}{2D_i}$, $\Lambda_{i2} = \frac{c + \sqrt{c^2 + 4D_i r_i}}{2D_i}$, $\rho_i = D_i(\Lambda_{i2} - \Lambda_{i1})$, $i = 1, 2$. Then one has

$$\begin{aligned} I_i(\xi) =& \frac{1}{\rho_i} \left\{ \int_{-\infty}^{\xi} e^{\Lambda_{i1}(\xi - x)} (f_i * (\beta_{i1} S_i I_1 + \beta_{i2} S_i I_2))(x - c\tau) dx \right. \\ & \left. + \int_{\xi}^{+\infty} e^{\Lambda_{i2}(\xi - x)} (f_i * (\beta_{i1} S_i I_1 + \beta_{i2} S_i I_2))(x - c\tau) dx \right\} \end{aligned} \tag{4.2}$$

for any $\xi \in \mathbb{R}$ and $i = 1, 2$. Integrating both sides of the equality (4.2) yields

$$\begin{aligned} \int_{-\infty}^{+\infty} I_i(\xi) d\xi =& \frac{1}{\rho_i} \left(\int_0^{+\infty} e^{\Lambda_{i1} x} dx + \int_{-\infty}^{0} e^{\Lambda_{i2} x} dx \right) \\ & \times \int_{-\infty}^{+\infty} (f_i * (\beta_{i1} S_i I_1 + \beta_{i2} S_i I_2))(\xi - c\tau) d\xi \\ =& \frac{1}{\rho_i} \left(\frac{1}{\Lambda_{i2}} - \frac{1}{\Lambda_{i1}} \right) \int_{-\infty}^{+\infty} (f_i * (\beta_{i1} S_i I_1 + \beta_{i2} S_i I_2))(\xi) d\xi \\ \le& \frac{\beta_{i1} \epsilon_i}{r_i} S_i^0 \int_{-\infty}^{+\infty} I_1(\xi) d\xi + \frac{\beta_{i2} \epsilon_i}{r_i} S_i^0 \int_{-\infty}^{+\infty} I_2(\xi) d\xi, \end{aligned} \tag{4.3}$$

which implies that

$$\begin{pmatrix} \int_{-\infty}^{+\infty} I_1(\xi) d\xi \\ \int_{-\infty}^{+\infty} I_2(\xi) d\xi \end{pmatrix} \leqslant V^{-1} F \begin{pmatrix} \int_{-\infty}^{+\infty} I_1(\xi) d\xi \\ \int_{-\infty}^{+\infty} I_2(\xi) d\xi \end{pmatrix}.$$

Note that the matrix $V^{-1}F$ is nonnegative and irreducible, and $\rho(V^1 F) = R_0(S_1^0, S_2^0)$. The Perron-Frobenius theorem implies that there exists a vector $P = (p_1, p_2)^T \in \mathbb{R}^2$ with $p_1 > 0$ and $p_2 > 0$ such that $V^{-1} F P = R_0(S_1^0, S_2^0) P$. It is obvious that there exists a large constant $\varrho > 0$ satisfying

$$\begin{pmatrix} \int_{-\infty}^{+\infty} I_1(\xi) d\xi \\ \int_{-\infty}^{+\infty} I_2(\xi) d\xi \end{pmatrix} \le \varrho P.$$

Consequently, we have

$$\begin{pmatrix} \int_{-\infty}^{+\infty} I_1(\xi)d\xi \\ \int_{-\infty}^{+\infty} I_2(\xi)d\xi \end{pmatrix} \leqslant (V^{-1}F)^n \begin{pmatrix} \int_{-\infty}^{+\infty} I_1(\xi)d\xi \\ \int_{-\infty}^{+\infty} I_2(\xi)d\xi \end{pmatrix} \leqslant \varrho (V^{-1}F)^n P = \varrho R_0^n(S_1^0, S_2^0)P \to 0$$

as $n \to \infty$, which contradicts the fact that $\int_{-\infty}^{+\infty} I_1(\xi)d\xi > 0$ and $\int_{-\infty}^{+\infty} I_2(\xi)d\xi > 0$. This completes the proof of the case $R_0(S_1^0, S_2^0) < 1$.

Now consider the case $R_0(S_1^0, S_2^0) = 1$. In this case there exists $P = (p_1, p_2)^T \in \mathbb{R}^2$ with $p_1 > 0$ and $p_2 > 0$ such that $V^{-1}FP = P$. It follows that

$$r_1 = \beta_{11}S_1^0\epsilon_1 + \frac{\beta_{12}S_1^0\epsilon_1 p_2 r_1}{p_1 r_2}, \quad r_2 = \beta_{22}S_2^0\epsilon_2 + \frac{\beta_{21}S_2^0\epsilon_2 p_1 r_2}{p_2 r_1}.$$

By virtue of $I_i(\pm\infty) = 0$ and $I_i'(\pm\infty) = 0$, integrating both sides of (4.1) from $-\infty$ to ∞ we obtain

$$\begin{aligned} \beta_{11}\epsilon_1 \int_{-\infty}^{\infty} \left(S_1^0 - S_1(x)\right) I_1(x)dx + \beta_{12}\epsilon_1 \int_{-\infty}^{\infty} \left(S_1^0 - S_1(x)\right) I_2(x)dx & \\ + \frac{\beta_{12}S_1^0\epsilon_1}{r_2 p_1} \left(r_1 p_2 \int_{-\infty}^{\infty} I_1(x)dx - r_2 p_1 \int_{-\infty}^{\infty} I_2(x)dx \right) &= 0 \end{aligned}$$

and

$$\begin{aligned} \beta_{21}\epsilon_2 \int_{-\infty}^{\infty} \left(S_2^0 - S_2(x)\right) I_1(x)dx + \beta_{22}\epsilon_2 \int_{-\infty}^{\infty} \left(S_2^0 - S_2(x)\right) I_2(x)dx & \\ + \frac{\beta_{21}S_2^0\epsilon_2}{r_1 p_2} \left(r_2 p_1 \int_{-\infty}^{\infty} I_2(x)dx - r_1 p_2 \int_{-\infty}^{\infty} I_1(x)dx \right) &= 0. \end{aligned}$$

Since $S_i(\xi) < S_i^0$ and $I_i(\xi) > 0$ for any $\xi \in \mathbb{R}$, it is impossible that the last two equalities hold at the same time. This completes the proof. □

4.2 $R_0(S_1^0, S_2^0) > 1$ and $c \in (0, c^*)$

Let $R_{i,0} := \frac{\epsilon_1\beta_{1i}S_1^0}{r_i} + \frac{\epsilon_2\beta_{2i}S_2^0}{r_i}$, $i = 1, 2$. The characteristic equation of the matrix $V^{-1}F$ is given by

$$f(\lambda) := \left(\lambda - \frac{\epsilon_1\beta_{11}S_1^0}{r_1}\right)\left(\lambda - \frac{\epsilon_2\beta_{22}S_2^0}{r_2}\right) - \frac{\epsilon_1\epsilon_2\beta_{12}\beta_{21}S_1^0S_2^0}{r_1r_2} = 0.$$

It is easy to see that both roots of the characteristic equation are real. In the following we first show some relationships between $R_0(S_1^0, S_2^0)$ and $R_{i,0}$.

Proposition 4.2 *If $R_0(S_1^0, S_2^0) > 1$, then at least one of $R_{1,0}$ and $R_{2,0}$ is greater than 1. Moreover, we have:* (i) $R_0(S_1^0, S_2^0) = 1$ *if* $R_{1,0} = 1$ *and* $R_{2,0} = 1$*;* (ii) $R_0(S_1^0, S_2^0) < 1$ *if* $R_{1,0} < 1$ *and* $R_{2,0} = 1$*;* (iii) $R_0(S_1^0, S_2^0) < 1$ *if* $R_{1,0} = 1$ *and* $R_{2,0} < 1$*;*

Proof. We prove the first part of the lemma by contradiction. On the contrary, assume that $R_{1,0} \leq 1$ and $R_{2,0} \leq 1$, that is

$$\frac{\epsilon_1\beta_{11}S_1^0 + \epsilon_2\beta_{21}S_2^0}{r_1} \leqslant 1 \quad \text{and} \quad \frac{\epsilon_1\beta_{12}S_1^0 + \epsilon_2\beta_{22}S_2^0}{r_2} \leqslant 1.$$

Using the above inequalities, we have

$$\frac{\epsilon_2\beta_{21}S_2^0}{r_1} < \lambda - \frac{\epsilon_1\beta_{11}S_1^0}{r_1} \quad \text{and} \quad \frac{\epsilon_1\beta_{12}S_1^0}{r_2} < \lambda - \frac{\epsilon_2\beta_{22}S_2^0}{r_2}, \quad \forall\lambda > 1,$$

which imply that

$$f(\lambda) = \left(\lambda - \frac{\epsilon_1\beta_{11}S_1^0}{r_1}\right)\left(\lambda - \frac{\epsilon_2\beta_{22}S_2^0}{r_2}\right) - \frac{\epsilon_1\epsilon_2\beta_{12}\beta_{21}S_1^0S_2^0}{r_1r_2} > 0, \quad \forall\lambda > 1.$$

Therefore, we have $R_0(S_1^0, S_2^0) \leq 1$. This contradicts the fact that $R_0(S_1^0, S_2^0) > 1$. Thus, we completes the proof of the first part of the lemma.

Similarly, we can prove the other cases. This completes the proof. □

Proposition 4.3 *If $R_{1,0} \geq 1$, $R_{2,0} \geq 1$, and $R_{1,0}R_{2,0} > 1$, then $R_0(S_1^0, S_2^0) > 1$.*

Proof. Assume $R_{1,0} > 1$, $R_{2,0} \geq 1$ and $R_{1,0} > R_{2,0}$. Let $l_1 := R_{1,0} - R_{2,0} > 0$. Then we have

$$f(R_{1,0}) = \frac{\epsilon_2\beta_{21}S_2^0}{r_1}\left(\frac{\epsilon_1\beta_{12}S_1^0}{r_2} + l_1\right) - \frac{\epsilon_1\epsilon_2\beta_{12}\beta_{21}S_1^0S_2^0}{r_1r_2} > 0$$

and

$$f(R_{2,0}) = \left(\frac{\epsilon_2\beta_{21}S_2^0}{r_1} - l_1\right)\frac{\epsilon_1\beta_{12}S_1^0}{r_2} - \frac{\epsilon_1\epsilon_2\beta_{12}\beta_{21}S_1^0S_2^0}{r_1r_2} < 0,$$

which imply that there exists $x_0 \in (R_{2,0}, R_{1,0})$ such that $f(x_0) = 0$. Therefore, $R_0(S_1^0, S_2^0) > 1$.

The other cases can be treated similarly. This completes the proof. □

Proposition 4.4 *Assume $R_0(S_1^0, S_2^0) > 1$. If $R_{1,0} \leq 1$ or $R_{2,0} \leq 1$, then $f(1) < 0$.*

Proof. We only consider the case $R_{2,0} \leq 1$. Since $R_0(S_1^0, S_2^0) > 1$, it follows from Proposition 4.2 that in this case there must be $R_{1,0} > 1$. To prove the proposition, we assume $f(1) \geq 0$ on the contrary, which implies that

$$\left(1 - \frac{\epsilon_1\beta_{11}S_1^0}{r_1}\right)\left(1 - \frac{\epsilon_2\beta_{22}S_2^0}{r_2}\right) - \frac{\epsilon_1\epsilon_2\beta_{12}\beta_{21}S_1^0S_2^0}{r_1r_2} \geq 0.$$

Since $R_{2,0} \leq 1$, we have $1 - \frac{\epsilon_2\beta_{22}S_2^0}{r_2} > 0$. Consequently, $1 - \frac{\epsilon_1\beta_{11}S_1^0}{r_1} > 0$. Thus, we have

$$f(\lambda) \quad = \quad \left(\lambda - \frac{\epsilon_1\beta_{11}S_1^0}{r_1}\right)\left(\lambda - \frac{\epsilon_2\beta_{22}S_2^0}{r_2}\right) - \frac{\epsilon_1\epsilon_2\beta_{12}\beta_{21}S_1^0S_2^0}{r_1r_2}$$

$$
\begin{aligned}
&> \left(1-\frac{\epsilon_1\beta_{11}S_1^0}{r_1}\right)\left(1-\frac{\epsilon_2\beta_{22}S_2^0}{r_2}\right)-\frac{\epsilon_1\epsilon_2\beta_{12}\beta_{21}S_1^0S_2^0}{r_1r_2}\\
&\geq 0
\end{aligned}
$$

for all $\lambda > 1$, which implies that $R_0(S_1^0, S_2^0) \leq 1$. This contradicts with $R_0(S_1^0, S_2^0) > 1$ and completes the proof. □

Lemma 4.5 *Assume $R_0(S_1^0, S_2^0) > 1$. For any $c > 0$, if equation* (2.11) *admits a non-trivial traveling wave solution* $(S_1(x+ct), S_2(x+ct), I_1(x+ct), I_2(x+ct))$ *satisfying* (3.22), *then there exists some $\mu_0 > 0$ such that*

$$
\sup_{x\in\mathbb{R}}\{I_i(x)e^{-\mu_0 x}\} < +\infty,\ \ \sup_{x\in\mathbb{R}}\{|I_i'(x)|e^{-\mu_0 x}\} < +\infty,\ \ \sup_{x\in\mathbb{R}}\{|I_i''(x)|e^{-\mu_0 x}\} < +\infty,\ \ i = 1, 2.
$$

Proof. Fix $c > 0$. Assume that $(S_1(x+ct), S_2(x+ct), I_1(x+ct), I_2(x+ct))$ is a non-trivial traveling wave solution of (2.11) satisfying (3.22). Using the fact that $S_i(-\infty) = S_i^0$ and $S_i'(\xi) \leq 0$ for $\xi \in \mathbb{R}$ and the definition of $f_i(x)$, we know that there exists $M > 0$ sufficiently large such that

$$
\int_{-M}^{+\infty} f_i(y)dy > 1-\nu \quad \text{and} \quad S_i(\xi) > S_i^0(1-\nu), \quad \forall\, \xi \in (-\infty, -2M),
$$

where $\nu \in (0, 1)$ is a small constant which will be determined later. For $\xi < -2M$, we have

$$
\begin{aligned}
cI_1'(\xi) &= D_1I_1''(\xi) + \epsilon_1\beta_{11}\int_{-\infty}^{+\infty} f_1(y)I_1(\xi-y-c\tau)S_1(\xi-y-c\tau)dy\\
&\quad +\epsilon_1\beta_{12}\int_{-\infty}^{+\infty} f_1(y)I_1(\xi-y-c\tau)S_1(\xi-y-c\tau)dy - r_1I_1(\xi)\\
&\geq D_1I_1''(\xi) + \epsilon_1\beta_{11}\int_{-M}^{+\infty} f_1(y)S_1(\xi+M-c\tau)I_1(\xi-y-c\tau)dy\\
&\quad +\epsilon_1\beta_{12}\int_{-M}^{+\infty} f_1(y)S_1(\xi+M-c\tau)I_2(\xi-y-c\tau)dy - r_1I_1(\xi)\\
&\quad +\epsilon_1\beta_{11}\int_{-\infty}^{-M} f_1(y)S_1(\xi-y-c\tau)I_1(\xi-y-c\tau)dy\\
&\quad +\epsilon_1\beta_{12}\int_{-\infty}^{-M} f_1(y)S_1(\xi-y-c\tau)I_2(\xi-y-c\tau)dy\\
&> D_1I_1''(\xi) + \epsilon_1\beta_{11}S_1^0(1-\nu)\int_{-M}^{+\infty} f_1(y)I_1(\xi-y-c\tau)dy\\
&\quad +\epsilon_1\beta_{12}S_1^0(1-\nu)\int_{-M}^{+\infty} f_1(y)I_2(\xi-y-c\tau)dy - r_1I_1(\xi)\\
&\geq D_1I_1''(\xi) + \epsilon_1\beta_{11}S_1^0(1-\nu)\int_{-M}^{+\infty} f_1(y)\left(I_1(\xi-y-c\tau)-I_1(\xi)\right)dy
\end{aligned}
$$

$$
\begin{aligned}
&+\epsilon_1\beta_{12}S_1^0(1-\nu)\int_{-M}^{+\infty} f_1(y)\left(I_2(\xi-y-c\tau)-I_2(\xi)\right)dy \\
&+\left(\epsilon_1\beta_{11}S_1^0(1-\nu)^2-r_1\right)I_1(\xi)+\epsilon_1\beta_{12}S_1^0(1-\nu)^2 I_2(\xi). \qquad (4.4)
\end{aligned}
$$

Let $J_i(\xi)=\int_{-\infty}^{\xi} I_i(\eta)d\eta$ for any $\xi\in\mathbb{R}$ and $i=1,2$. It is easy to see that $\int_{-\infty}^{\xi}(f_1*I_i)(\eta)d\eta = (f_1*J_i)(\eta)$. Integrating both sides of inequality (4.4) from $-\infty$ to ξ with $\xi<-2M$, we have

$$
\begin{aligned}
&\left(\epsilon_1\beta_{11}S_1^0(1-\nu)^2-r_1\right)J_1(\xi)+\epsilon_1\beta_{12}S_1^0(1-\nu)^2 J_2(\xi) \\
&\quad< -D_1I_1'(\xi)+cI_1(\xi)-\epsilon_1\beta_{11}S_1^0(1-\nu)\int_{-M}^{+\infty} f_1(y)\left(J_1(\xi-y-c\tau)-J_1(\xi)\right)dy \\
&\quad -\epsilon_1\beta_{12}S_1^0(1-\nu)\int_{-M}^{+\infty} f_1(y)\left(J_2(\xi-y-c\tau)-J_2(\xi)\right)dy. \qquad (4.5)
\end{aligned}
$$

In view of

$$
\begin{aligned}
&\int_{-\infty}^{\xi}\int_{-M}^{+\infty} f_1(y)\left(J_i(\eta-y-c\tau)-J_i(\eta)\right)dyd\eta \\
&\quad=\lim_{z\to-\infty}\int_{z}^{\xi}\int_{-M}^{+\infty} f_1(y)\left(J_i(\eta-y-c\tau)-J_i(\eta)\right)dyd\eta \\
&\quad=\lim_{z\to-\infty}-\int_{z}^{\xi}\int_{-M}^{+\infty}(y+c\tau)f_1(y)\int_0^1 I_i(\eta-\theta(y+c\tau))d\theta dyd\eta \\
&\quad=-\int_{-M}^{+\infty}(y+c\tau)f_1(y)\int_0^1 J_i(\xi-\theta(y+c\tau))d\theta dy
\end{aligned}
$$

for $i=1,2$, integrating both sides of inequality (4.5) from $-\infty$ to ξ ($\xi<-2M$) leads to

$$
\begin{aligned}
&\left(\epsilon_1\beta_{11}S_1^0(1-\nu)^2-r_1\right)\int_{-\infty}^{\xi}J_1(\eta)d\eta+\epsilon_1\beta_{12}S_1^0(1-\nu)^2\int_{-\infty}^{\xi}J_2(\eta)d\eta+D_1I_1(\xi) \\
&\quad< cJ_1(\xi)+\epsilon_1\beta_{11}S_1^0\int_{-M}^{+\infty}(y+c\tau)f_1(y)\int_0^1 J_1(\xi-\theta(y+c\tau))d\theta dy \\
&\quad +\epsilon_1\beta_{12}S_1^0\int_{-M}^{+\infty}(y+c\tau)f_1(y)\int_0^1 J_2(\xi-\theta(y+c\tau))d\theta dy.
\end{aligned}
$$

Since $(y+c\tau)J_i(\xi-\theta(y+c\tau))$ is non-increasing on $\theta\in[0,1]$, we have

$$
\begin{aligned}
&\left(\epsilon_1\beta_{11}S_1^0(1-\nu)^2-r_1\right)\int_{-\infty}^{\xi}J_1(\eta)d\eta+\epsilon_1\beta_{12}S_1^0(1-\nu)^2\int_{-\infty}^{\xi}J_2(\eta)d\eta+D_1I_1(\xi) \\
&\quad< cJ_1(\xi)+(c\tau+\Lambda_1)\epsilon_1\beta_{11}S_1^0J_1(\xi)+(c\tau+\Lambda_1)\epsilon_1\beta_{12}S_1^0J_2(\xi), \qquad (4.6)
\end{aligned}
$$

where $\Lambda_1=\int_0^{+\infty}f_1(y)ydy>\int_{-M}^{+\infty}f_1(y)ydy$. Similarly, for $\xi<-2M$ we have

$$
\epsilon_2\beta_{21}S_2^0(1-\nu)^2\int_{-\infty}^{\xi}J_1(\eta)d\eta+\left(\epsilon_2\beta_{22}S_2^0(1-\nu)^2-r_2\right)\int_{-\infty}^{\xi}J_2(\eta)d\eta+D_2I_2(\xi)
$$

$$< cJ_2(\xi) + (c\tau + \Lambda_2)\epsilon_2\beta_{21}S_2^0 J_1(\xi) + (c\tau + \Lambda_2)\epsilon_2\beta_{22}S_2^0 J_2(\xi), \tag{4.7}$$

where $\Lambda_2 = \int_0^{+\infty} f_2(y)ydy$. In the following we firstly show that there exist positive constants a_1, a_2, b_1, and b_2 such that

$$a_1 \int_{-\infty}^{\xi} J_1(\eta)d\eta + a_2 \int_{-\infty}^{\xi} J_2(\eta)d\eta \leq b_1 J_1(\xi) + b_2 J_2(\xi), \quad \forall\ \xi < -2M. \tag{4.8}$$

Because $R_0(S_1^0, S_2^0) > 1$, it follows from Proposition 4.2 that there must be $\epsilon_1\beta_{11}S_1^0 + \epsilon_2\beta_{21}S_2^0 - r_1 > 0$ or $\epsilon_2\beta_{22}S_2^0 + \epsilon_1\beta_{12}S_1^0 - r_2 > 0$. Therefore, we prove (4.8) by considering the following five cases.

Case 1. $\epsilon_1\beta_{11}S_1^0 - r_1 > 0$. In this case we take $\nu \in (0, 1)$ small enough so that $\epsilon_1\beta_{11}S_1^0(1-\nu)^2 - r_1 > 0$. Due to (4.6), it is sufficient to take $a_1 := \epsilon_1\beta_{11}S_1^0(1-\nu)^2 - r_1$, $a_2 := \epsilon_1\beta_{12}S_1^0(1-\nu)^2$, $b_1 := c + (c\tau + \Lambda_1)\epsilon_1\beta_{11}S_1^0$ and $b_2 := (c\tau + \Lambda_1)\epsilon_1\beta_{12}S_1^0$.

Case 2. $\epsilon_2\beta_{22}S_2^0 - r_2 > 0$. Due to (4.7), the proof is completely similar to Case 1.

Case 3. $\epsilon_1\beta_{11}S_1^0 - r_1 \leq 0$, $\epsilon_2\beta_{22}S_2^0 - r_2 \leq 0$, $\epsilon_1\beta_{11}S_1^0 + \epsilon_2\beta_{21}S_2^0 - r_1 > 0$ and $\epsilon_1\beta_{12}S_1^0 + \epsilon_2\beta_{22}S_2^0 - r_2 > 0$. In this case we take $\nu \in (0, 1)$ satisfying $\epsilon_1\beta_{11}S_1^0(1-\nu)^2 + \epsilon_2\beta_{21}S_2^0(1-\nu)^2 - r_1 > 0$ and $\epsilon_1\beta_{12}S_1^0(1-\nu)^2 + \epsilon_2\beta_{22}S_2^0(1-\nu)^2 - r_2 > 0$. By adding two sides of inequalities (4.6) and (4.7) respectively, we have that

$$\begin{aligned}
&\left[\epsilon_1\beta_{11}S_1^0(1-\nu)^2 + \epsilon_2\beta_{21}S_2^0(1-\nu)^2 - r_1\right] \int_{-\infty}^{\xi} J_1(\eta)d\eta \\
&\quad + \left[\epsilon_2\beta_{22}S_2^0(1-\nu)^2 + \epsilon_1\beta_{12}S_1^0(1-\nu)^2 - r_2\right] \int_{-\infty}^{\xi} J_2(\eta)d\eta \\
&\quad < c\left(J_1(\xi) + J_2(\xi)\right) + \left[(c\tau + \Lambda_1)\epsilon_1\beta_{11}S_1^0 + (c\tau + \Lambda_2)\epsilon_2\beta_{21}S_2^0\right] J_1(\xi) \\
&\qquad + \left[(c\tau + \Lambda_1)\epsilon_1\beta_{12}S_1^0 + (c\tau + \Lambda_2)\epsilon_2\beta_{22}S_2^0\right] J_2(\xi)
\end{aligned}$$

for any $\xi < -2M$. Then it is sufficient to take $a_1 := \epsilon_1\beta_{11}S_1^0(1-\nu)^2 + \epsilon_2\beta_{21}S_2^0(1-\nu)^2 - r_1$, $a_2 := \epsilon_2\beta_{22}S_2^0(1-\nu)^2 + \epsilon_1\beta_{12}S_1^0(1-\nu)^2 - r_2$, $b_1 := c + (c\tau + \Lambda_1)\epsilon_1\beta_{11}S_1^0 + (c\tau + \Lambda_2)\epsilon_2\beta_{21}S_2^0$ and $b_2 := c + (c\tau + \Lambda_1)\epsilon_1\beta_{12}S_1^0 + (c\tau + \Lambda_2)\epsilon_2\beta_{22}S_2^0$.

Case 4. $\epsilon_1\beta_{11}S_1^0 - r_1 \leq 0$, $\epsilon_1\beta_{11}S_1^0 + \epsilon_2\beta_{21}S_2^0 - r_1 > 0$ and $\epsilon_1\beta_{12}S_1^0 + \epsilon_2\beta_{22}S_2^0 - r_2 \leq 0$. Following Proposition 4.4, in this case we have $f(1) < 0$, that is

$$1 - \left(\frac{\epsilon_1\beta_{11}S_1^0}{r_1} + \frac{\epsilon_2\beta_{22}S_2^0}{r_2}\right) + \frac{\epsilon_1\epsilon_2\left(\beta_{11}\beta_{22} - \beta_{12}\beta_{21}\right)S_1^0S_2^0}{r_1r_2} < 0.$$

We take $\nu \in (0, 1)$ such that $\epsilon_1\beta_{11}S_1^0(1-\nu)^2 + \epsilon_2\beta_{21}S_2^0(1-\nu)^2 - r_1 > 0$ and

$$1 - \left(\frac{\epsilon_1\beta_{11}S_1^0}{r_1} + \frac{\epsilon_2\beta_{22}S_2^0}{r_2}\right)(1-\nu)^2 + \frac{\epsilon_1\epsilon_2\left(\beta_{11}\beta_{22} - \beta_{12}\beta_{21}\right)S_1^0S_2^0}{r_1r_2}(1-\nu)^4 < 0. \tag{4.9}$$

Set

$$\mathcal{A} = \begin{pmatrix} \epsilon_1\beta_{11}S_1^0(1-\nu)^2 - r_1 & \epsilon_1\beta_{12}S_1^0(1-\nu)^2 \\ \epsilon_2\beta_{21}S_2^0(1-\nu)^2 & \epsilon_2\beta_{22}S_2^0(1-\nu)^2 - r_2 \end{pmatrix}.$$

It is obvious that the inequality (4.9) implies $|\mathcal{A}| < 0$.

Note that $\epsilon_1\beta_{11}S_1^0(1-\nu)^2 - r_1 < 0$ and $\epsilon_2\beta_{22}S_2^0(1-\nu)^2 - r_2 < 0$. Multiplying two sides of (4.6) and (4.7) by $\epsilon_2\beta_{22}S_2^0(1-\nu)^2 - r_2$ and $-\epsilon_1\beta_{12}S_1^0(1-\nu)^2$ respectively, and adding up the corresponding terms, we obtain

$$-|\mathcal{A}| \int_{-\infty}^{\xi} J_1(\eta)d\eta \leq (C_2B_3 - C_4B_1)\, J_1(\xi) + (C_2B_4 - C_4B_2)\, J_2(\xi), \quad \forall\, \xi < -2M,$$

where $B_1 := c + (c\tau + \Lambda_1)\epsilon_1\beta_{11}S_1^0$, $B_2 := (c\tau + \Lambda_1)\epsilon_1\beta_{12}S_1^0$, $B_3 := (c\tau + \Lambda_2)\epsilon_2\beta_{21}S_2^0$, $B_4 := c + (c\tau + \Lambda_2)\epsilon_2\beta_{22}S_2^0$, $C_2 := \epsilon_1\beta_{12}S_1^0(1-\nu)^2$ and $C_4 := \epsilon_2\beta_{22}S_2^0(1-\nu)^2 - r_2$. Similarly, we have

$$-|\mathcal{A}| \int_{-\infty}^{\xi} J_2(\eta)d\eta \leq (C_3B_1 - C_1B_3)\, J_1(\xi) + (C_3B_2 - C_1B_4)\, J_2(\xi), \quad \forall\, \xi < -2M,$$

where $C_1 := \epsilon_1\beta_{11}S_1^0(1-\nu)^2 - r_1$ and $C_3 := \epsilon_2\beta_{21}S_2^0(1-\nu)^2$. Now let $a_1 := -|\mathcal{A}|$, $a_2 := -|\mathcal{A}|$, $b_1 := C_2B_3 + C_3B_1 - C_1B_3 - C_4B_1$, $b_2 := C_2B_4 + C_3B_2 - C_1B_4 - C_4B_2$. Then (4.8) is proved.

Case 5: $\epsilon_2\beta_{22}S_2^0 - r_2 \leq 0$, $\epsilon_1\beta_{11}S_1^0 + \epsilon_2\beta_{21}S_2^0 - r_1 \leq 0$ and $\epsilon_1\beta_{12}S_1^0 + \epsilon_2\beta_{22}S_2^0 - r_2 > 0$. This case can be treated by a similar argument to Case 4. We omit the details.

Now we are in the position to prove the main result of the lemma. Let $J(\xi) = J_1(\xi) + J_2(\xi)$. Then inequality (4.8) implies that there exist constants $a > 0$ and $b > 0$ such that

$$a \int_{-\infty}^{\xi} J(\eta)d\eta \leq bJ(\xi), \quad \forall\, \xi < -2M.$$

Consequently, we obtain that

$$a \int_{-\infty}^{0} J(\xi + \eta)d\eta \leq bJ(\xi), \quad \forall\, \xi < -2M.$$

Since $J(\cdot)$ is increasing, we have that $a\eta J(\xi - \eta) \leq bJ(\xi)$ for any $\xi < -2M$ and any $\eta > 0$. Therefore, there exist $\eta_0 > 0$ large enough and $\omega_0 \in (0, 1)$ satisfying

$$J(\xi - \eta_0) \leqslant \omega_0 J(\xi), \quad \forall\, \xi < -2M.$$

Let $w(x) = J(x)e^{-\mu_0 x}$ with $\mu_0 = \frac{1}{\eta_0} \ln \frac{1}{\omega_0} > 0$. Then, we have

$$w(\xi - \eta_0) = J(\xi - \eta_0)e^{-\mu_0(\xi - \eta_0)} \leq \omega J(\xi)e^{-\mu_0(\xi - \eta_0)} = w(\xi), \qquad \xi < -2M.$$

Since $w(\xi) \to 0$ as $\xi \to +\infty$, we have that there exists a constant κ_0 satisfying

$$w(x) \leq \kappa_0, \quad \forall x \in \mathbb{R},$$

which implies that $J(x) \leq \kappa_0 e^{\mu_0 x}$ for any $x \in \mathbb{R}$. Consequently, there exists $q_0 > 0$ satisfying $\int_{-\infty}^{x} J_i(\eta)d\eta \leq q_0 e^{\mu_0 x}$ for any $x < 0$, $i = 1, 2$. It follows from inequalities (4.6) and (4.7) that there exists $p_0 > 0$ such that

$$I_i(x) \leq p_0 e^{\mu_0 x}, \quad \forall\, x \in \mathbb{R}.$$

Finally, using (4.4) and (4.5), we obtain that

$$\sup_{x\in\mathbb{R}}\{I_1(x)e^{-\mu_0 x}\} < +\infty, \qquad \sup_{x\in\mathbb{R}}\{|I_1'(x)|e^{-\mu_0 x}\} < +\infty, \qquad \sup_{x\in\mathbb{R}}\{|I_1''(x)|e^{-\mu_0 x}\} < +\infty.$$

Similarly, we have

$$\sup_{x\in\mathbb{R}}\{I_2(x)e^{-\mu_0 x}\} < +\infty, \qquad \sup_{x\in\mathbb{R}}\{|I_2'(x)|e^{-\mu_0 x}\} < +\infty, \qquad \sup_{x\in\mathbb{R}}\{|I_2''(x)|e^{-\mu_0 x}\} < +\infty.$$

This completes this proof. □

In the following we prove the main result of this subsection.

Theorem 4.6 *Assume that $R_0(S_1^0, S_2^0) > 1$. For $c \in (0, c^*)$, there exists no non-trivial traveling wave solution $(S_1(x + ct), S_2(x + ct), I_1(x + ct), I_2(x + ct))$ of system* (2.11) *satisfying* (3.22)*.*

Proof. We prove the theorem by contradiction. Fix $c \in (0, c^*)$. Suppose on the contrary that there exists a non-trivial traveling wave solution $(S_1(x+ct), S_2(x+ct), I_1(x+ct), I_2(x+ct))$ of system (2.11) so that (3.22) holds. By Lemma 4.5, there exists $\mu_0 > 0$ such that

$$\sup_{x\in\mathbb{R}}\{I_i(x)e^{-\mu_0 x}\} < +\infty, \quad \sup_{x\in\mathbb{R}}\{|I_i'(x)|e^{-\mu_0 x}\} < +\infty, \quad \sup_{x\in\mathbb{R}}\{|I_i''(x)|e^{-\mu_0 x}\} < +\infty, \quad i = 1, 2.$$

Consider $R_1(\xi) := S_1^0 - S_1(\xi)$. Integrating

$$cS_1'(\xi) = d_1 S_1''(\xi) - \beta_{11} S_1(\xi) I_1(\xi) - \beta_{12} S_1(\xi) I_2(\xi)$$

from $-\infty$ to ξ yields

$$c(S_1(\xi) - S_1^0) = d_1 S_1'(\xi) - \beta_{11} \int_{-\infty}^{\xi} S_1(\eta) I_1(\eta) d\eta - \beta_{12} \int_{-\infty}^{\xi} S_1(\eta) I_2(\eta) d\eta.$$

Let $E_1(\xi) = \beta_{11} \int_{-\infty}^{\xi} S_1(\eta) I_1(\eta) d\eta + \beta_{12} \int_{-\infty}^{\xi} S_1(\eta) I_2(\eta) d\eta$ for any $\xi \in \mathbb{R}$. It is not difficult to show that $E_1(\xi) \leq C_M e^{\mu_0 \xi}$ for any $\xi \in \mathbb{R}$, where $C_M > 0$ is a constant. Due to the definition of $R_1(\xi)$, we have

$$d_1 R_1'(\xi) - cR_1(\xi) = -E_1(\xi), \quad \xi \in \mathbb{R}.$$

Solving the last equation yields

$$R_1(\xi) = \hat{C}_M e^{\frac{c}{d_1}\xi} + \frac{1}{d_1} e^{\frac{c}{d_1}\xi} \int_\xi^0 e^{-\frac{c}{d_1}\eta} E_1(\eta) d\eta$$

where $\hat{C}_M = R_1(0)$. Since $E_1(\xi) = O(e^{\mu_0 \xi})$ as $\xi \to -\infty$, it is easy to see that $R_1(\xi) = O(e^{\mu_0' \xi})$ as $\xi \to -\infty$, where $\mu_0' = \min\{\mu_0, \frac{c}{d_1}, \frac{c}{d_2}\}$. In view of $0 \le R_1(\xi) \le S_1^0$, we have

$$\sup_{x \in \mathbb{R}} \{R_1(x) e^{-\mu_0' x}\} < +\infty.$$

Let $R_2(x) := S_2^0 - S_2(x)$, $x \in \mathbb{R}$. Similarly, we have

$$\sup_{x \in \mathbb{R}} \{R_2(x) e^{-\mu_0' x}\} < +\infty.$$

In view of $\sup_{x\in\mathbb{R}}\{I_i(x) e^{-\mu_0 x}\} < +\infty$, we define the one-sided Laplace transform of I_i by

$$\mathrm{L}_i(\lambda) = \int_{-\infty}^0 e^{-\lambda \xi} I_i(\xi) d\xi, \qquad i = 1, 2.$$

Here we only consider $\lambda \in \mathbb{R}_+$. Since $I_i(\xi) > 0$ for any $\xi \in \mathbb{R}$ and $\mathrm{L}_i(\cdot)$ is increasing in its domain, for each $i = 1, 2$, either there exists a positive constant $\nu_i > \mu_0$ such that $\mathrm{L}_i(\lambda) < +\infty$ for any $0 \le \lambda < \nu_i$ and $\lim_{\lambda \to \nu_i - 0} \mathrm{L}_i(\lambda) = +\infty$, or $\mathrm{L}_i(\lambda) < +\infty$ for any $\lambda \ge 0$. Now we further define the two-sided Laplace transform of I_i by

$$\mathcal{L}_i(\lambda) = \int_{-\infty}^{+\infty} e^{-\lambda \xi} I_i(\xi) d\xi, \qquad i = 1, 2.$$

We only consider $\lambda \in \mathbb{R}_+$. Since $I_i(\xi)$ is bounded in $\mathbb{R}$, we have $\int_0^{+\infty} e^{-\lambda\xi} I_i(\xi) d\xi < +\infty$ for any $\lambda \ge 0$. Thus, $\mathcal{L}_i(\lambda)$ shares the same property with $\mathrm{L}_i(\lambda)$ in $\lambda \ge 0$, that is, for each $i = 1, 2$, either there exists a positive constant $\nu_i > \mu_0$ such that $\mathcal{L}_i(\lambda) < +\infty$ for any $0 \le \lambda < \nu_i$ and $\lim_{\lambda \to \nu_i - 0} \mathcal{L}_i(\lambda) = +\infty$, or $\mathcal{L}_i(\lambda) < +\infty$ for any $\lambda \ge 0$.

For each $i = 1, 2$, we denote $\nu_i = +\infty$ if $\mathcal{L}_i(\lambda) < +\infty$ for any $\lambda \ge 0$. In the following we first show that indeed $\nu_1 = +\infty$ and $\nu_2 = +\infty$, namely, for both $i = 1, 2$, $\mathcal{L}_i(\lambda) < +\infty$ for any $\lambda \ge 0$. We prove this claim by a contradiction argument. Without loss of generality, we suppose $0 < \nu_1 < +\infty$ and $\nu_1 \le \nu_2 \le +\infty$ on the contrary. We consider two cases: 1) $0 < \nu_1 < \nu_2 \le +\infty$; 2) $0 < \nu_1 = \nu_2 < +\infty$. We first consider the first case. Assume $0 < \nu_1 < \nu_2 \le +\infty$. In view of

$$\begin{aligned} &D_1 I_1''(\xi) - c I_1'(\xi) - r_1 I_1(\xi) + \beta_{11} S_1^0 (f_1 * I_1)(\xi - c\tau) + \beta_{12} S_1^0 (f_1 * I_2)(\xi - c\tau) \\ &\quad = \beta_{11} \left[f_1 * \left((S_1^0 - S_1) I_1 \right) \right] (\xi - c\tau) + \beta_{12} \left[f_1 * \left((S_1^0 - S_1) I_2 \right) \right] (\xi - c\tau) \end{aligned}$$

and

$$\int_{-\infty}^{+\infty} e^{-\lambda\xi}(f_1 * I_i)(\xi - c\tau)d\xi = \mathcal{L}_i(\lambda)J_1(\lambda, c),$$

we have

$$\begin{aligned}
&\mathcal{L}_1(\lambda)\left(D_1\lambda^2 - c\lambda - r_1 + \beta_{11}S_1^0 J_1(\lambda, c)\right) + \mathcal{L}_2(\lambda)\beta_{12}S_1^0 J_1(\lambda, c) \\
&\quad = \int_{-\infty}^{+\infty} e^{-\lambda\xi}\beta_{11}\left[f_1 * \left((S_1^0 - S_1)I_1\right)\right](\xi - c\tau)d\xi \\
&\qquad + \int_{-\infty}^{+\infty} e^{-\lambda\xi}\beta_{12}\left[f_1 * \left((S_1^0 - S_1)I_2\right)\right](\xi - c\tau)d\xi.
\end{aligned} \tag{4.10}$$

Similarly, we have

$$\begin{aligned}
&\mathcal{L}_1(\lambda)\beta_{21}S_2^0 J_2(\lambda, c) + \mathcal{L}_2(\lambda)\left(D_2\lambda^2 - c\lambda - r_2 + \beta_{22}S_2^0 J_2(\lambda, c)\right) \\
&\quad = \int_{-\infty}^{+\infty} e^{-\lambda\xi}\beta_{21}\left[f_2 * \left((S_2^0 - S_2)I_1\right)\right](\xi - c\tau)d\xi \\
&\qquad + \int_{-\infty}^{+\infty} e^{-\lambda\xi}\beta_{22}\left[f_2 * \left((S_2^0 - S_2)I_2\right)\right](\xi - c\tau)d\xi.
\end{aligned} \tag{4.11}$$

Since $0 < S_i^0 - S_i(\xi) \leq S_i^0$ for any $\xi \in \mathbb{R}$ and $\sup_{x\in\mathbb{R}}\left\{\left(S_i^0 - S_i(\xi)\right)e^{-\mu_0' x}\right\} < +\infty$, we have that

$$\int_{-\infty}^{+\infty} e^{-\lambda\xi}\beta_{i1}\left[f_i * \left((S_i^0 - S_i)I_1\right)\right](\xi - c\tau)d\xi < +\infty, \quad \forall\ \lambda \in \left(0, \nu_1 + \mu_0'\right)$$

and

$$\int_{-\infty}^{+\infty} e^{-\lambda\xi}\beta_{i2}\left[f_i * \left((S_i^0 - S_i)I_2\right)\right](\xi - c\tau)d\xi < +\infty, \quad \forall\ \lambda \in \left(0, \nu_2 + \mu_0'\right).$$

In view of $\nu_1 < \nu_2$, letting $\lambda \to \nu_1 - 0$ in (4.11) yields a contradiction because the first term tends to infinity and the other terms have bounded limits as $\lambda \to \nu_1 - 0$. This implies that the assumption $0 < \nu_1 < \nu_2 \leq +\infty$ is impossible.

Consider the second case, namely, $0 < \nu_1 = \nu_2 =: \nu_0 < +\infty$. If one of inequalities $D_1\nu_0^2 - c\nu_0 - r_1 + \beta_{11}S_1^0 J_1(\nu_0, c) \geq 0$ and $D_2\nu_0^2 - c\nu_0 - r_2 + \beta_{22}S_2^0 J_2(\nu_0, c) \geq 0$ holds, then letting $\lambda \to \nu_1 - 0$ in (4.10) or (4.11) yields a contradiction. If both inequalities

$$D_1\nu_0^2 - c\nu_0 - r_1 + \beta_{11}S_1^0 J_1(\nu_0, c) < 0 \quad \text{and} \quad D_2\nu_0^2 - c\nu_0 - r_2 + \beta_{22}S_2^0 J_2(\nu_0, c) < 0 \tag{4.12}$$

hold, then we can rewrite (4.10) and (4.11) into

$$M(\lambda, c)\begin{pmatrix} \mathcal{L}_1(\lambda) \\ \mathcal{L}_2(\lambda) \end{pmatrix} - \begin{pmatrix} \mathcal{L}_1(\lambda) \\ \mathcal{L}_2(\lambda) \end{pmatrix} = \begin{pmatrix} \frac{h_1(\lambda)}{m_1(\lambda,c)} \\ \frac{h_2(\lambda)}{m_2(\lambda,c)} \end{pmatrix}, \quad \lambda \in (0, \nu_0),$$

where $h_i(\lambda) := \sum_{j=1,2}\int_{-\infty}^{+\infty} e^{-\lambda\xi}\beta_{ij}\left(f_i * \left(\left(S_i^0 - S_i\right)I_j\right)\right)(\xi - c\tau)d\xi$. It is obvious that $\nu_0 < \lambda(c)$ due to (4.12). Since $c \in (0, c^*)$ and $R_0 > 1$, it follows from Proposition 3.2 that

$\inf_{\lambda\in[0,\nu_0]}\rho(\lambda,c)>1$. Since the matrix $M(\lambda,c)$ is positive, it is not difficult to show that either

$$\frac{\beta_{11}S_1^0J_1(\lambda,c)}{m_1(\lambda,c)}\mathcal{L}_1(\lambda)+\frac{\beta_{12}S_1^0J_1(\lambda,c)}{m_1(\lambda,c)}\mathcal{L}_2(\lambda)\ge\rho(\lambda,c)\mathcal{L}_1(\lambda),\quad \lambda\in(0,\nu_0)$$

holds, or

$$\frac{\beta_{21}S_2^0J_2(\lambda,c)}{m_2(\lambda,c)}\mathcal{L}_1(\lambda)+\frac{\beta_{22}S_2^0J_2(\lambda,c)}{m_2(\lambda,c)}\mathcal{L}_2(\lambda)\ge\rho(\lambda,c)\mathcal{L}_2(\lambda),\quad \lambda\in(0,\nu_0)$$

holds. Hence, for any $\lambda\in(0,\nu_0)$, there holds either

$$(\rho(\lambda,c)-1)\,\mathcal{L}_1(\lambda)\le\frac{h_1(\lambda)}{m_1(\lambda,c)} \tag{4.13}$$

or

$$(\rho(\lambda,c)-1)\,\mathcal{L}_2(\lambda)\le\frac{h_2(\lambda)}{m_2(\lambda,c)}. \tag{4.14}$$

Since $\inf_{\lambda\in[0,\nu_0]}m_i(\lambda,c)>0$ and $h_i(\lambda)$ is well defined in $[0,\nu_0+\mu_0')$, letting $\lambda\to\nu_0-0$ in (4.13) and (4.14) yields a contradiction due to $\lim_{\lambda\to\nu_0-0}\mathcal{L}_i(\lambda)=+\infty$. Thus, we have proved that the assumption $0<\nu_1=\nu_2=:\nu_0<+\infty$ is also impossible.

Now we complete the proof of the theorem. Note that we have proved that for each $i=1,2$, $\mathcal{L}_i(\lambda)<+\infty$ for any $\lambda>0$. It follows from (4.2) that

$$\begin{aligned}I_i(\xi)\;=\;&\frac{1}{\rho_i}\int_{-\infty}^{\xi}e^{\Lambda_{i1}(\xi-x)}\left(f_i*(\beta_{i1}S_iI_1+\beta_{i2}S_iI_2)\right)(x-c\tau)dx\\&+\frac{1}{\rho_i}\int_{\xi}^{+\infty}e^{\Lambda_{i2}(\xi-x)}\left(f_i*(\beta_{i1}S_iI_1+\beta_{i2}S_iI_2)\right)(x-c\tau)dx,\quad\forall\,\xi\in\mathbb{R},\end{aligned}\tag{4.15}$$

where $i=1,2$, $\Lambda_{i1}=\frac{c-\sqrt{c^2+4D_ir_i}}{2D_i}$, $\Lambda_{i2}=\frac{c+\sqrt{c^2+4D_ir_i}}{2D_i}$ and $\rho_i=D_i(\Lambda_{i2}-\Lambda_{i1})$. Let $\Lambda:=\max\{-\Lambda_{11},\Lambda_{12},-\Lambda_{21},\Lambda_{22}\}$. Using (4.15) we can show that for each $\xi\in\mathbb{R}$, $I_i(\xi+y)e^{-\Lambda y}$ is decreasing in $y\in\mathbb{R}$ and $I_i(\xi+y)e^{\Lambda y}$ is increasing in $y\in\mathbb{R}$. Consequently, we have

$$\begin{aligned}&D_iI_i''(\xi)-cI_i'(\xi)-r_iI_i(\xi)+\beta_{i1}S_i^0(f_i*I_1)(\xi-c\tau)+\beta_{i2}S_i^0(f_i*I_2)(\xi-c\tau)\\&\quad+\beta_{i1}\left(f_i*\left(\left(S_i^0-S_i\right)I_1\right)\right)(\xi-c\tau)+\beta_{i2}\left(f_i*\left(\left(S_i^0-S_i\right)I_2\right)\right)(\xi-c\tau)\\&\quad<S_i^0\int_{-c\tau}^{+\infty}f_i(y)\left(\beta_{i1}I_1(\xi-c\tau-y)+\beta_{i2}I_2(\xi-c\tau-y)\right)dy\\&\quad\;\;+S_i^0\int_{-\infty}^{-c\tau}f_i(y)\left(\beta_{i1}I_1(\xi-c\tau-y)+\beta_{i2}I_2(\xi-c\tau-y)\right)dy\\&\quad=S_i^0\int_{-c\tau}^{+\infty}f_i(y)e^{\Lambda(c\tau+y)}e^{-\Lambda(c\tau+y)}\left(\beta_{i1}I_1(\xi-(c\tau+y))+\beta_{i2}I_2(\xi-(c\tau+y))\right)dy\\&\quad\;\;+S_i^0\int_{-\infty}^{-c\tau}f_i(y)e^{-\Lambda(c\tau+y)}e^{\Lambda(c\tau+y)}\left(\beta_{i1}I_1(\xi-(c\tau+y))+\beta_{i2}I_2(\xi-(c\tau+y))\right)dy\end{aligned}$$

$$\begin{aligned}
&\leq S_i^0 \int_{-c\tau}^{+\infty} f_i(y) e^{\Lambda(c\tau+y)} dy \, (\beta_{i1} I_1(\xi) + \beta_{i2} I_2(\xi)) \\
&\quad + S_i^0 \int_{-\infty}^{-c\tau} f_i(y) e^{-\Lambda(c\tau+y)} dy \, (\beta_{i1} I_1(\xi) + \beta_{i2} I_2(\xi)) \\
&\leq 2 S_i^0 \varrho_i e^{c\tau\Lambda} \, (\beta_{i1} I_1(\xi) + \beta_{i2} I_2(\xi))
\end{aligned} \tag{4.16}$$

for any $\xi \in \mathbb{R}$, where $\varrho_i = \int_{-\infty}^{+\infty} f_i(y) e^{\Lambda y} dy$, $i = 1, 2$. Using inequality (4.16) we obtain

$$\begin{aligned}
&\int_{-\infty}^{+\infty} e^{-\lambda\xi} I_1(\xi) \left(-m_1(\lambda, c) + \beta_{11} S_1^0 J_1(\lambda, c) - 2 S_1^0 \varrho_1 \beta_{11} e^{c\tau\Lambda}\right) d\xi \\
&\qquad + \int_{-\infty}^{+\infty} e^{-\lambda\xi} I_2(\xi) \left(\beta_{12} S_1^0 J_1(\lambda, c) - 2 S_1^0 \varrho_1 \beta_{12} e^{c\tau\Lambda}\right) d\xi \leq 0
\end{aligned}$$

and

$$\begin{aligned}
&\int_{-\infty}^{+\infty} e^{-\lambda\xi} I_1(\xi) \left(\beta_{21} S_2^0 J_2(\lambda, c) - 2 S_2^0 \varrho_2 \beta_{21} e^{c\tau\Lambda}\right) d\xi \\
&\qquad + \int_{-\infty}^{+\infty} e^{-\lambda\xi} I_2(\xi) \left(-m_2(\lambda, c) + \beta_{22} S_2^0 J_2(\lambda, c) - 2 S_2^0 \varrho_2 \beta_{22} e^{c\tau\Lambda}\right) d\xi \leq 0.
\end{aligned}$$

Adding up the last two inequalities, we obtain

$$\int_{-\infty}^{+\infty} e^{-\lambda\xi} I_1(\xi) \chi_1(\lambda) d\xi + \int_{-\infty}^{+\infty} e^{-\lambda\xi} I_2(\xi) \chi_2(\lambda) d\xi \leq 0, \tag{4.17}$$

where

$$\chi_1(\lambda) := -m_1(\lambda, c) + \beta_{11} S_1^0 J_1(\lambda, c) + \beta_{21} S_2^0 J_2(\lambda, c) - 2 S_1^0 \varrho_1 \beta_{11} e^{c\tau\Lambda} - 2 S_2^0 \varrho_2 \beta_{21} e^{c\tau\Lambda},$$

$$\chi_2(\lambda) := -m_2(\lambda, c) + \beta_{12} S_1^0 J_1(\lambda, c) + \beta_{22} S_2^0 J_2(\lambda, c) - 2 S_1^0 \varrho_1 \beta_{12} e^{c\tau\Lambda} - 2 S_2^0 \varrho_2 \beta_{22} e^{c\tau\Lambda}.$$

However, letting $\lambda \to +\infty$ in (4.17) yields a contradiction because $\lim_{\lambda\to+\infty} \chi_i(\lambda) = +\infty$. This completes the proof. □

5 Dependence of the minimal speed c^* on the model parameters

In this section, we focus on the dependence of the minimal wave speed c^* on the parameters of system (2.11). By virtue of (3.7), it is easy to see that the minimal wave speed c^* which is defined by Proposition 3.1 depends on the diffusion rates of the infectious individuals D_i, the transmission rates β_{ij}, the removed rates of the infectious individuals r_i, the diffusion rates of the latent groups D_{L_i}, the removed rates of the latent groups M_i

and the time delay τ, where $i, j = 1, 2$. For the sake of convenience, we denote $\rho(\lambda, c)$ by ρ, where $\rho(\lambda, c)$ is defined in (3.7). In addition, we always assume $R_0 > 1$ in the following.

(A) We consider the continuous dependence of the minimal wave speed c^* on the diffusion rates of the infectious individuals $D_i (i = 1, 2)$. For $c \geq 0$ and $\lambda \in (0, \lambda(c))$, $i \neq j$ and $i, j = 1, 2$, a straightforward calculation yields

$$\frac{\partial \rho}{\partial D_i} = \frac{1}{2}\left(\frac{(p_1(\lambda,c) + (-1)^{i+1} n(\lambda,c))\beta_{ii}}{p_1(\lambda,c)} + \frac{2\beta_{12}\beta_{21} S_j^0 J_j(\lambda,c)}{p_1(\lambda,c) m_j(\lambda,c)}\right) \frac{\lambda^2 S_i^0 J_i(\lambda,c)}{m_i^2(\lambda,c)} > 0,$$

where $J_i(\lambda, c) = e^{(D_{Li}\lambda^2 - c\lambda - M_i)\tau}$, $\lambda(c)$, $p_1(\lambda, c)$, $n(\lambda, c)$ and $m_i(\lambda, c)(i = 1, 2)$ are defined in Proposition 3.1. Using this observation and Proposition 3.1 (iii), we have that $c^* = c^*(D_i)$ is increasing on $D_i > 0 (i = 1, 2)$, which means that the diffusion of the infection individuals can increase the spread speed of the disease, see Figure 1(a).

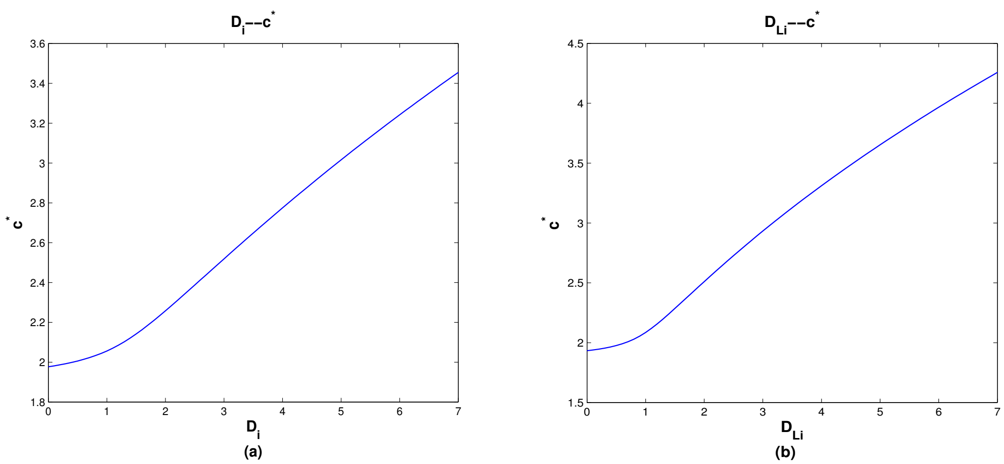

Figure 1: Numerical simulations of the continuous dependence of the minimal wave speed c^* on D_i and D_{L_i} for $i = 1, 2$. Here the parameters are as follows: $S_1^0 = S_2^0 = 50$, $\beta_{11} = \beta_{22} = 0.24$, $\beta_{12} = \beta_{21} = 0.08$, $r_1 = r_2 = 1.1$, $M_1 = M_2 = 1$ and $\tau = 1$. In addition, $D_j = 1.2$ and $D_{L_1} = D_{L_2} = 1$ in (a), and $D_{L_j} = 1$ and $D_1 = D_2 = 1.2$ in (b) for $i \neq j$ and $i, j = 1, 2$, respectively.

(B) Similar to (A), for $c \geq 0$ and $\lambda \in (0, \lambda(c))$, we can easily get

$$\frac{\partial \rho}{\partial D_{L_i}} > 0, \ i = 1, 2; \qquad \frac{\partial \rho}{\partial \beta_{ij}} > 0, \ i, j = 1, 2;$$
$$\frac{\partial \rho}{\partial r_i} < 0, \ i = 1, 2; \qquad \frac{\partial \rho}{\partial M_i} < 0, \ i = 1, 2,$$

which together with Proposition 3.1 (iii) imply that c^* is increasing on β_{ij} and D_{L_i}, and is decreasing on r_i and M_i, see Figure 1(b), Figure 2 and Figure 3.

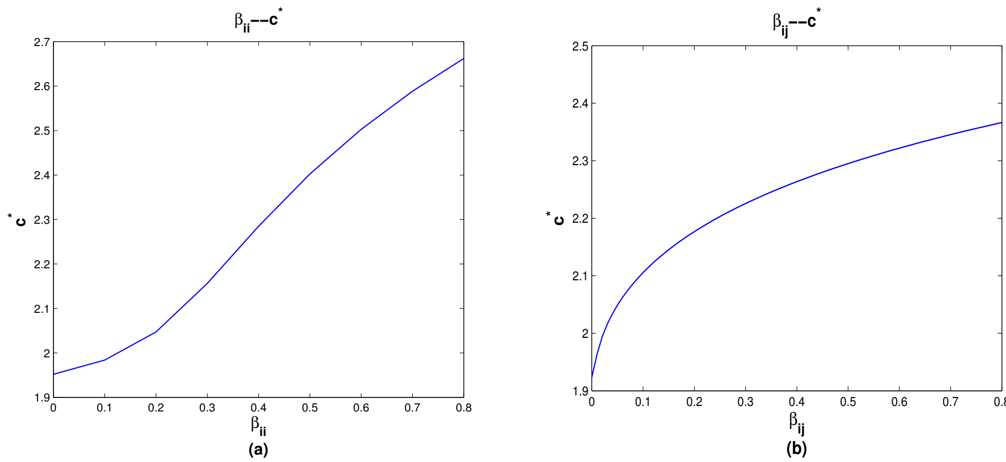

Figure 2: Numerical simulations of the continuous dependence of the minimal wave speed c^* on $\beta_{ij}(i, j = 1, 2)$. The parameter values are as follows: $S_1^0 = S_2^0 = 50$, $D_1 = D_2 = 1.2$, $r_1 = r_2 = 1.1$, $D_{L_1} = D_{L_2} = 1$, $M_1 = M_2 = 1$ and $\tau = 1$. In addition, $\beta_{12} = \beta_{21} = 0.08$ and $\beta_{jj} = 0.24$ in (a) and $\beta_{ji} = 0.08$ and $\beta_{11} = \beta_{22} = 0.24$ in (b), where $i, j = 1, 2$, $i \neq j$, respectively.

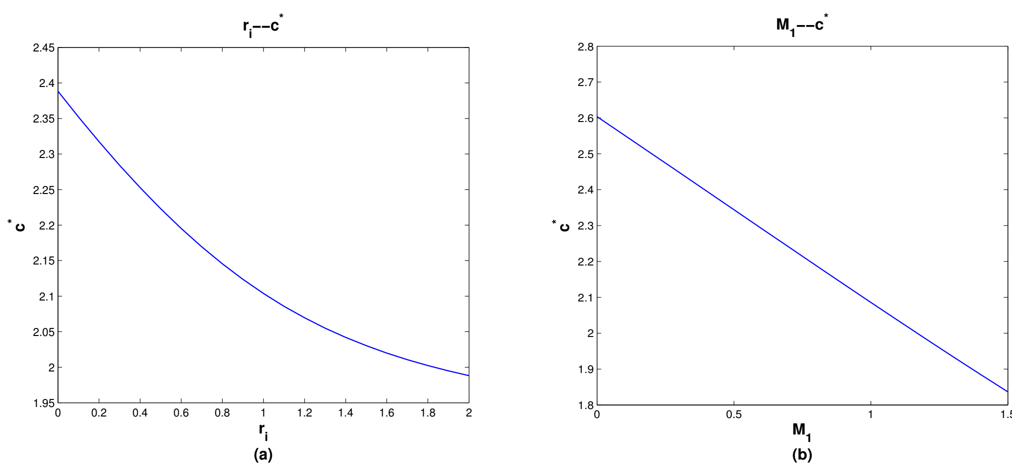

Figure 3: Numerical simulations of the continuous dependence of the minimal wave speed c^* on r_i and M_i for $i = 1, 2$. For a convenience, the parameter values are as follows: $S_1^0 = S_2^0 = 50$, $D_1 = D_2 = 1.2$, $D_{L_1} = D_{L_2} = 1$ and $\tau = 1$, $\beta_{11} = \beta_{22} = 0.24$ and $\beta_{12} = \beta_{21} = 0.08$. In addition, $r_j = 1.1$ and $M_1 = M_2 = 1$ in (a) and $M_j = 1$ and $r_1 = r_2 = 1.1$ in (b), $i, j = 1, 2$, $i \neq j$, respectively.

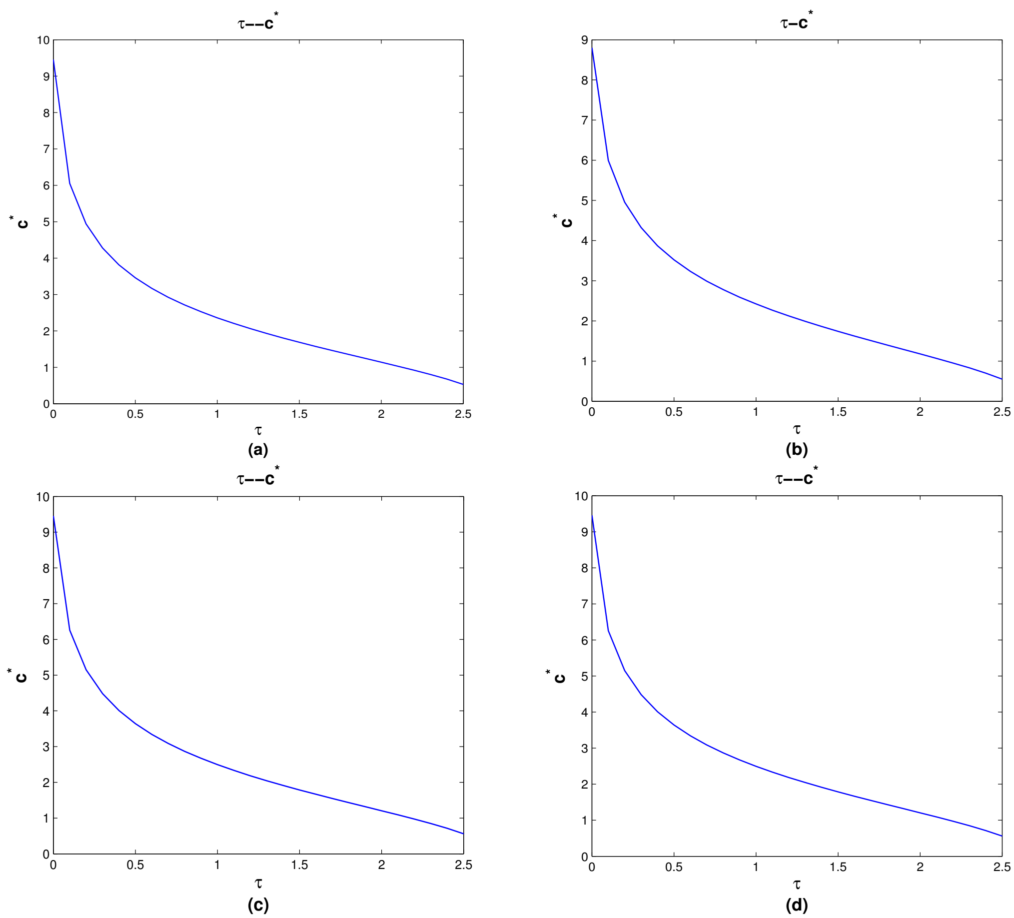

Figure 4: Numerical simulations of the continuous dependence of the minimal wave speed c^* on τ. The parameter values are as follows: $S_1^0 = S_2^0 = 50$, $\beta_{11} = \beta_{22} = 0.24$, $\beta_{12} = \beta_{21} = 0.08$, $r_1 = r_2 = 1.1$ and $M_1 = M_2 = 1$. In addition, $1.5 = D_1 > D_{L_1} = 1.3$ and $1.5 = D_2 > D_{L_2} = 1.3$ in (a), $1.3 = D_1 < D_{L_1} = 1.5$ and $1.3 = D_2 < D_{L_2} = 1.5$ in (b), $1.5 = D_1 < D_{L_1} = 1.7$ and $1.5 = D_2 > D_{L_2} = 1.3$ in (c), and $1.5 = D_1 > D_{L_1} = 1.3$ and $1.5 = D_2 < D_{L_2} = 1.7$ in (d), respectively.

(C) We consider the continuous dependence of the minimal wave speed c^* on the time delay $\tau > 0$. For the sake of convenience, take

$$S_1^0 = S_2^0 = S_0, \quad D_1 = D_2 = D, \quad D_{L_1} = D_{L_2} = D_L \ (D_L \leq D), \tag{5.1}$$

$$\beta_{11} = \beta_{22} = \beta^+, \quad \beta_{12} = \beta_{21} = \beta^-, \quad r_1 = r_2 = r, \quad M_1 = M_2 = M. \tag{5.2}$$

Let $m(\lambda, c) = D\lambda^2 - c\lambda - r$. Fix $\tau_0 > 0$, then there exists a unique pair of $\lambda_*(\tau_0) > 0$ and $c^*(\tau_0)$ such that $\rho(\lambda_*, c^*) = 1$ and $\frac{\partial \rho}{\partial \lambda}\Big|_{(\lambda_*, c^*)} = 0$, from which we have $D_L\lambda_*^2(\tau_0) - c^*(\tau_0)\lambda_*(\tau_0) < 0$. Rewrite $\rho(\lambda, c)$ as $\rho(\lambda, c, \tau)$. Then we have

$$\begin{aligned}
& \frac{\partial \rho(\lambda_*(\tau_0), c^*(\tau_0), \tau_0 + \nu)}{\partial \nu}\bigg|_{\nu=0} \\
= \ & \frac{(\beta^+ + \beta^-)S^0(D_L\lambda_*^2(\tau_0) - c^*(\tau_0)\lambda_*(\tau_0) - M)e^{(D_L\lambda_*^2(\tau_0) - c^*(\tau_0)\lambda_*(\tau_0) - M)\tau_0}}{m(\lambda_*(\tau_0), c^*(\tau_0))} \\
< \ & 0,
\end{aligned}$$

which implies that c^* is a decreasing function of $\tau > 0$. Due to the mathematical difficulties, here we only handle the special case as (5.1) and (5.2). For the general case, we can find that c^* is also decreasing on $\tau > 0$, see Figure 4.

6 Discussion

In this paper we have constructed a reaction-diffusion system (2.10) with non-locality and time delay to describe the spatial spread of an infectious disease in two groups/sub-populations. It is assumed that the susceptibility of individuals for infection and the infectivity of individuals are distinct in these two groups/sub-populations, the infectious disease has a fixed latent period, and the latent individuals of the populations diffuse in the spatial domian. Our results indicate that the existence of traveling wave solutions is determined by the basic reproduction number $R_0(S_1^0, S_2^0)$, a threshold evalued at the disease-free equilibrium. When $R_0(S_1^0, S_2^0) > 1$, there exists a positive number c^* such that for each $c > c^*$, the system admits a nontrivial traveling wave solution with wave speed c; when $R_0(S_1^0, S_2^0) \leq 1$ or $R_0(S_1^0, S_2^0) > 1$ and $c < c^*$, there is no nontrivial traveling wave solution. Thus, once one subpopulation is infected by the infectious disease, it will be spread geographically to the other subpopulation.

Here we would like to emphasize that when $R_0(S_1^0, S_2^0) > 1$, the existence of traveling wave solutions of the system with speed $c = c^*$ is not established. Following from Theorem 3.10 and its proof, we know that for the traveling wave solution $(S_1(x+ct), S_2(x+ct), I_1(x+ct), I_2(x+ct))$ of the system with speed $c > c^*$, the susceptible components $S_1(\cdot)$ and $S_2(\cdot)$

are decreasing (front type), but the infective components $I_1(\cdot)$ and $I_2(\cdot)$ are not monotone (pulse type), which make it very difficult to establish the existence of traveling wave solutions of the system with speed $c = c^*$ by taking a limit for a sequence of traveling wave solutions with speeds c_n ($c_n > c^*$ and $c_n \to c^*$ as $n \to \infty$). It needs to mention that Wu [63] recently established the existence of traveling wave with critical speed for a discrete diffusive epidemic model of the Kermack–Mckendrick type by a delicate analysis of traveling waves with super-critical speeds and the limiting argument. We expect that the argument of Wu [63] can be applied to our model and leave it as a future work. In addition, considering the spreading speed for solutions of our model that the initial values of infective components have compact support is very meaningful. Recently, there were some studies focusing on such topic in infection models for one group without vital dynamics (births and deaths), and with or without latent period, see Beaumont et al. [5], Ducrot and Giletti [14], and Jones et al. [34–36]. In [5, 14, 34–36], it was always assumed that the susceptible individuals are immobile, which lead to that the systems studied by [5, 14, 34–36] can be induced to a single scalar equation and hence, the theory on the spreading speed developed by Liang and Zhao LZ2007, Thieme [52, 53], Thieme and Zhao [54] can be used. However, since our model has two groups and the diffusion rates d_1 and d_2 are positive, it seems to be impossible to induce system (2.10) to a cooperative system with two components, and hence, the method used in [5, 14, 34–36] could not be applied to the current system. Therefore, the existence of the spreading speed for system (2.10) (even for a system with one group) is a challenging and open problem.

Note that Fitzgibbon et al. [20] used the following reaction-diffusion system with nonlocality and time delay on a bounded domain to model the spread of Feline Immunodeficiency Virus (FIV)

$$\begin{cases} \frac{\partial u_i}{\partial t} = d_i \Delta u_i - \beta_{i1} u_i v_1 - \beta_{i2} u_i v_2, \\ \frac{\partial v_i}{\partial t} = D_i \Delta v_i - \lambda_i v_i + \omega_i(\tau) * \left(u_i(t-\tau,\cdot)\left(\beta_{i1} v_1(t-\tau,\cdot) + \beta_{i2} v_2(t-\tau,\cdot)\right)\right), \\ \frac{\partial u_i}{\partial n} = \frac{\partial v_i}{\partial n} = 0, \ x \in \partial\Omega, t > 0, \end{cases} \tag{6.1}$$

where $i = 1, 2$, $\omega_i(\tau, x) * y(t-\tau, x) = \int_\Omega \omega_i(\tau, x-\xi) y(t-\tau, \xi) d\xi$ and $\omega_i(t, x)$ is the fundamental solution associated with the partial differential operator $\partial_t - d_i \Delta - \lambda_i$ and no flux boundary condition for $i = 1, 2$, see [20, fomulas (3.2a), (3.2c) and (3.11a)-(3.11d)] and [21]. If we let $\Omega = \mathbb{R}$ in (6.1), then system (6.1) becomes

$$\begin{cases} \frac{\partial u_i}{\partial t} = d_i \Delta u_i - \beta_{i1} u_i v_1 - \beta_{i2} u_i v_2, \\ \frac{\partial v_i}{\partial t} = D_i \Delta v_i - \lambda_i v_i + \int_{\mathbb{R}} g_i(\tau, x-y)\left(u_i(t-\tau, y)\left(\beta_{i1} v_i(t-\tau, y) + \beta_{i2} v_i(t-\tau, y)\right)\right) dy, \end{cases} \tag{6.2}$$

which is a special case of system (2.10), where $g_i(t,x) = e^{-\lambda_i t}\frac{1}{4\pi d_i t}e^{\frac{-x^2}{4d_i t}}$, $i = 1, 2$. According to [9, 29], we take $u_1^0 = u_2^0 = 25$, $d_1 = d_2 = d_3 = d_4 = 1$, $\beta_{11} = 0.06$, $\beta_{12} = \beta_{21} = \beta_{22} = 0.01$, $\lambda_1 = \lambda_2 = 0.2$, and $\tau = 2$. Using these parameters, we obtain

$$R_0 = \rho\begin{pmatrix} 5.0272 & 0.8378 \\ 0.8378 & 0.8378 \end{pmatrix} > 1$$

and $c^* = 1.1543$, which implies that the disease will outbreak. It follows from the results of Sections 3-4 that a best strategy to control the disease is to decrease the transmission rates β_{ij} and increase the removed rates λ_i so that $R_0 \le 1$. Otherwise, it follows from the result of Section 5 that one can decrease the diffusion rates D_1 and D_2 so that the spread speed of the disease become slower. Of course, the results of this paper can also be applied to other sexual transmission diseases.

As mentioned in Section 1, Ducrot, Magal and Ruan [19] have studied the following multigroup age-structured epidemic model

$$\begin{cases} \frac{\partial \rho_i}{\partial t} = d_{ui}\Delta\rho_i - \rho_i \sum_{j=1}^n \int_0^{+\infty} \psi_j(t,a,x)da, \\ \frac{\partial \psi_i}{\partial t} + \frac{\partial \psi_i}{\partial a} = d_{vi}\Delta\psi_i - \mu_i\psi_i, \\ \psi_i(t,0,x) = \rho_i \sum_{j=1}^n \int_0^{+\infty} K_{i,j}(a)\psi_j(t,a,x)da. \end{cases} \tag{6.3}$$

When $K_{i,j}(a) = \widetilde{K}_{i,j}1_{[\tau_{i,j},\infty)}(a)$, $\mu_i(a) = \widetilde{\mu}_i$, where $1_{[\tau_{i,j},\infty)}(a)$ denotes the characteristic function on $[\tau_{i,j},\infty)$, $\widetilde{K}_{i,j} \ge 0$, $\tau_{i,j} \ge 0$ and $\widetilde{\mu}_i > 0$, system (6.3) reduces to

$$\begin{cases} \frac{\partial \rho_i}{\partial t} = d_{ui}\Delta\rho_i - \rho_i \sum_{j=1}^n e^{\tau_{i,j}\widetilde{\mu}_i}\widetilde{K}_{i,j}T_{d_{vj}\Delta}(\tau_{i,j})\psi_j(t-\tau_{i,j},\cdot), \\ \frac{\partial \psi_i}{\partial t} = d_{vi}\Delta\psi_i - \widetilde{\mu}_i\psi_i + \rho_i \sum_{j=1}^n e^{\tau_{i,j}\widetilde{\mu}_i}\widetilde{K}_{i,j}T_{d_{vj}\Delta}(\tau_{i,j})\psi_j(t-\tau_{i,j},\cdot), \end{cases} \tag{6.4}$$

where $T_{d\Delta}(t)\varphi(x) = \frac{1}{4\pi t d}\int_{-\infty}^{\infty}\varphi(x-y)e^{-\frac{y^2}{4dt}}dy$. It is obvious that system (6.4) is different from our system (2.11). In contrast to Theorems 3.10 and 4.6 of this paper, the results of [19] showed that when the basic reproduction number $R_0 > 1$, there exists a number $c^* > 0$ such that for any $c > c^*$, system (6.3) admits a traveling wave solution with wave speed c. In addition, it is different from Theorem 4.1 of this paper, where the nonexistence of traveling wave solutions of (2.11) for both $R_0 < 1$ and $R_0 = 1$ has been proved, while in [19] the authors only proved the nonexistence of traveling wave solutions of (6.3) for $R_0 < 1$ but the case when $R_0 = 1$ remains open. Here we would like to mention that the methodology used in Section 4 to prove the nonexistence of traveling waves of (2.11) for $R_0 > 1$ and $0 < c < c^*$ can be easily applied to system (6.4). However, to apply the method to (6.3), it seems difficult due to the presence of the age variable a and needs some elaborate analysis for the traveling wave solutions of (6.3).

Acknowledgements. The authors would like to thank the two anonymous reviewers and the handling editor for their helpful comments and suggestions which helped us in improving the paper.

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