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Traveling wave solutions in a two-group SIR epidemic model with constant recruitment

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Abstract Host heterogeneity can be modeled by using multi-group structures in the population. In this paper we investigate the existence and nonexistence of traveling waves of a two-group SIR epidemic model with time delay and constant recruitment and show that the existence of traveling waves is determined by the basic reproduction number R_0. More specifically, we prove that (i) when the basic reproduction number $R_0 > 1$, there exists a minimal wave speed $c^* > 0$, such that for each $c \geq c^*$ the system admits a nontrivial traveling wave solution with wave speed c and for $c < c^*$ there exists no nontrivial traveling wave satisfying the system; (ii) when $R_0 \leq 1$, the system admits no nontrivial traveling waves. Finally, we present some numerical simulations to show the existence of traveling waves of the system.

Keywords Two-group epidemic model · Basic reproduction number · Time delay · Constant recruitment · Traveling wave solutions

Mathematics Subject Classification 35C07 · 35B40 · 35K57 · 92D30

Dedicated to the memory of Professor Karl Hadeler.

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1 Introduction

To better understand the geographic spread of an infectious disease, it is important to take into account the spatial effects in modeling the infectious disease. An epidemic model with spatial effects usually can give rise to a moving zone of transition from a diseases-free state to an infective state which predicts a wave of infection moving out from the initial source of infection. Hence, traveling wave solutions play a key role in studying the spatial spread of infectious diseases (see Hadeler 1994, 1988, 2016; Murray 1989; Rass and Radcliffe 2003; Ruan 2007; Ruan and Wu 2009; Wang and Wu 2010 and the references cited therein).

To study the combined effects of spatial heterogeneity and nonlocal interaction, Li and Zou (2009) derived a SIR epidemic model with non-locality and constant recruitment. They first established the existence, uniqueness and positivity of solutions to the initial-value problem for the system on the whole space $\mathbb{R}$. Then they investigated the existence of traveling wave fronts of the system and obtained a critical value which is a lower bound for the wave speed of the traveling wave fronts. Ducrot and Magal (2011) studied a diffusive epidemic model with age-structure and constant recruitment.[1] They proved an existence and nonexistence result for travelling wave solutions, described the minimal wave speed, and constructed a suitable Lyapunov like functional to discuss their convergence towards equilibrium points at $x = \pm\infty$. Li et al. (2014) investigated the existence, nonexistence and minimal wave speed of traveling waves of a nonlocal dispersal delayed SIR model with constant recruitment and Holling-II incidence rate. It was found that the existence and nonexistence of traveling waves of the system are not only determined by the minimal wave speed c but also by the basic reproduction number R_0 of the corresponding reaction system. Li et al. (2015a) also studied a delayed diffusive SIR epidemic model with Holling-II incidence rate and constant recruitment and established the minimal wave speed by presenting the existence and nonexistence of traveling wave solutions for any positive wave speed. In particular, it was proved that the minimal wave speed decreases when the latency of infection increases. Fu (2016) considered a diffusive SIR model with delay, saturated incidence rate and constant recruitment and studied traveling waves connecting the infection-free equilibrium state and the endemic equilibrium state. With the aid of a pair of upper and lower solutions constructed, he firstly obtained a family of solutions of the truncated problems by applying the Schauder fixed point theorem, and then proved the existence of the traveling waves via a limiting argument. Indeed, it was shown that there exists $c^* > 0$ such that the system admits a traveling wave solution with speed c if and only if $c \geq c^*$. For more studies on traveling waves of various epidemic models, we refer to Bai and Wu (2015), Ducrot and Magal (2009), Hadeler (2016), Li and Yang (2014), Li et al. (2015b), Wu and Weng (2011), Yang et al. (2013), Yang et al. (2011), Zhang and Wang (2014) and the references therein.

[1] The term "external supplies" was first used by Ducrot and Magal (2011) to describe the situation that the host population is recruited at a constant rate from outside of the compartment. Several authors followed them to use this terminology (see, for example, Li et al. 2014). After discussing with the authors of Ducrot and Magal (2011), we all agreed that "constant recruitment" is a more appropriate term to describe the phenomenon.

Recent studies have suggested that host heterogeneity has important effects on the dynamics of infectious diseases at several spatial and temporal scales and that heterogeneity in susceptibility may be of general importance in the ecology of infectious diseases (Dwyer et al. 1997). Epidemiological models with host heterogeneity have been studied extensively, either in terms of a finite number of different susceptibility classes (Andersson and Britton 1998; Bonzi et al. 2011; Hyman and Li 2005, 2006; Rodrigues et al. 2009; Shuai and van den Driessche 2012) or as a continuous distribution of susceptibility (Clancy and Pearce 2013; Dwyer et al. 1997; Katriel 2012; Novozhilov 2008; Veliov 2005). Many ODE models have also been proposed to describe the spread of various infectious diseases with differential susceptibility and differential infectivity, for example, measles, mumps, gonorrhea, HBV, HIV, syphilis and so on, see Cai et al. (2012), Demasse and Ducrot (2013), Guo et al. (2006, 2012), Hadeler and Castillo-Chavez (1995), Yuan and Zou (2010) and references therein. In particular, such models can better reflect the variance of within-group and inter-group transmission rate.

There were also some interesting studies focusing on the traveling wave solutions of diffusive epidemic model with differential susceptibility and differential infectivity. Ai (2010) and Burie et al. (2006) used the different methods to take into account the traveling waves for a model of a fungal disease over a vineyard. Weng and Zhao (2005) investigated the spreading speed and traveling waves for a multi-type SIS epidemic model on a continuous space. Wang et al. (2012) established the existence and nonexistence of traveling waves of a reaction–advection–diffusion epidemic model, which describe the spatio-temporal spread of H5N1 avian influenza in an ecosystem involving the virus in the environment and a wide range of bird species. Ducrot et al. (2010) studied traveling wave solutions of a multi-group age-structured SIR epidemic models and showed that the existence and nonexistence of traveling wave solutions of the system is determined by the basic reproduction number. In addition, their results are applicable to the crisscross transmission of feline immunodeficiency virus and some sexual transmission diseases (Fitzgibbon et al. 1995a, b). Zhao and Wang (2016) established the existence and nonexistence of traveling wave fronts in the diffusive epidemic model with multiple parallel infectious stages and found that the diffusion rate of the infection individuals in each parallel infectious compartment can increase the spreading speed of the disease. Recently, we (Zhao et al. 2017) also studied the existence and nonexistence of traveling wave solutions of a two-group SIR epidemic model, where the latency of disease and the mobility of the individuals in the latent period were incorporated. However, in that paper we did not take into account the constant recruitment for the two-group epidemic model.

In this paper, we continue to investigate the model proposed in Zhao et al. (2017) by introducing the constant recruitment, namely, we consider the following two-group epidemic model with delay

$$\begin{cases} \frac{\partial S_i(t,x)}{\partial t} = d_i \Delta S_i(t,x) + \lambda_i - \delta_i S_i(t,x) - \beta_{i1} S_i(t,x) I_1(t,x) - \beta_{i2} S_i(t,x) I_2(t,x), \\ \frac{\partial I_i(t,x)}{\partial t} = D_i \Delta I_i(t,x) - r_i I_i(t,x) + \epsilon_i S_i(t-\tau,x)\left(\beta_{i1} I_1(t-\tau,x) + \beta_{i2} I_2(t-\tau,x)\right), \\ \frac{\partial R_i(t,x)}{\partial t} = \mathcal{T}_i \Delta R_i(t,x) - \kappa_i R_i(t,x) + \tilde{m}_i I_i(t,x), \end{cases} \tag{1.1}$$

where $t > 0$, $x \in \mathbb{R}$ and $i = 1, 2$, $\tau \geqslant 0$ represents the latency of the infection, $r_i = \tilde{m}_i + \vartheta_i$, $\epsilon_i = e^{-r_i \tau}$ measures the proportion of the infected individuals that can survive the latent period; d_i, D_i and $\mathcal{T}_i$ are the diffusion rates of the susceptible, infectious and recovered individuals, respectively; λ_i is the entering flux of susceptible individuals; δ_i, ϑ_i and κ_i represent the death rates of the susceptible, infectious and recovered individuals, respectively; $\tilde{m}_i$ denotes the recovery rate and β_{ij} denotes the infection contamination rate for $i, j = 1, 2$. In contrast to the model (2.11) of Zhao et al. (2017), in (1.2) we ignore the mobility of individuals during the latent period, while $\epsilon_i = e^{-r_i \tau}$ means that we assume that the latent individuals and the infectious individuals have the same death rates ϑ_i and recovery rates $\widetilde{m}_i$. From (1.1), it is obvious that the equations for $S_i(t, x)$ and $I_i(t, x)$ are fully decoupled from $R_i(t, x)$ $(i = 1, 2)$. Thus, we only need to consider the sub-system as below:

$$\begin{cases} \frac{\partial S_i(t,x)}{\partial t} = d_i \Delta S_i(t, x) + \lambda_i - \delta_i S_i(t, x) - \beta_{i1} S_i(t, x) I_1(t, x) - \beta_{i2} S_i(t, x) I_2(t, x), \\ \frac{\partial I_i(t,x)}{\partial t} = D_i \Delta I_i(t, x) - r_i I_i(t, x) + \epsilon_i S_i(t - \tau, x) \left(\beta_{i1} I_1(t - \tau, x) + \beta_{i2} I_2(t - \tau, x)\right), \end{cases} \tag{1.2}$$

where $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$ and $i = 1, 2$. Here we make an assumption.

(A) $d_i \geqslant D_i > 0$ and $\lambda_i, \beta_{ij}, \delta_i, r_i > 0$ for $i, j = 1, 2$.

The assumption (A) has realistic significance. In fact, the capacity of activity of susceptible individuals should be stronger than the capacity of activity of infected individuals. As reported by Ducrot and Magal (2011), the mathematical analysis of this vital dynamics (constant recruitment) becomes much more difficult to handle, in particular, the convergence of traveling waves at $x \to +\infty$. Fortunately, in this paper we can use the assumption (A) to get an upper bound for the infected components and then use a Lyapunov functional to solve the problem. Note that the joint effects of diffusion and delay on the nonlinear dynamics, such as stability, Hopf bifurcation and Turing instability, of delayed reaction–diffusion equations were studied in Hadeler and Ruan (2007).

The plan of this paper is as follows. In the next section, we investigate traveling waves of (1.2) and provide full information on the existence and nonexistence of traveling wave solutions of (1.2). Namely, when the basic reproduction number $R_0 > 1$, we prove that there exists a positive number c^* such that for each wave speed $c \geq c^*$, system (1.2) admits a nontrivial traveling wave solution with wave speed c. In particular, we use a Lyapunov functional to prove the convergence of traveling waves as $x \to +\infty$. In Sect. 3, we prove the nonexistence of nonnegative traveling wave solutions of system (1.2) when $R_0 \leq 1$ or $R_0 > 1$ and $0 < c < c^*$. In Sect. 4, we numerically simulate the existence of traveling waves of the system.

2 Existence of traveling wave solutions

To investigate traveling wave solutions of (1.2), we need to find constant equilibria of (1.2). It is clear that $(S_1^0, S_2^0, 0, 0) = (\frac{\lambda_1}{\delta_1}, \frac{\lambda_2}{\delta_2}, 0, 0)$ is always an equilibrium of (1.2) which is called the disease-free equilibrium of (1.2). To find a positive equilibrium, it is equivalent to consider the following ODE system

$$\begin{cases} \frac{dS_i(t)}{dt} = \lambda_i - \beta_{i1}S_i(t)I_1(t) - \beta_{i2}S_i(t)I_2(t) - \delta_i S_i(t), \\ \frac{dI_i(t)}{dt} = \epsilon_i\beta_{i1}S_i(t)I_1(t) + \epsilon_i\beta_{i2}S_i(t)I_2(t) - r_i I_i(t), \end{cases} \tag{2.1}$$

where $i = 1, 2$. It is obvious that $(S_1^0, S_2^0, 0, 0)(S_i^0 = \frac{\lambda_i}{\delta_i})$ is also a disease-free equilibrium of system (2.1). From Guo et al. (2006), we obtain that the basic reproduction number of system (2.1) at the disease-free equilibrium $(S_1^0, S_2^0, 0, 0)$, denoted by R_0, can be expressed as

$$R_0 = r(\mathcal{L}),$$

where

$$\mathcal{L} := \begin{pmatrix} \frac{\epsilon_1\beta_{11}S_1^0}{r_1} & \frac{\epsilon_1\beta_{12}S_1^0}{r_1} \\ \frac{\epsilon_2\beta_{21}S_2^0}{r_2} & \frac{\epsilon_2\beta_{22}S_2^0}{r_2} \end{pmatrix}$$

and $r(\mathcal{L})$ denotes the spectral radius of the matrix $\mathcal{L}$. In addition, by (Guo et al. 2006, Proposition 3.1 and Theorem 3.3), we have the following theorem.

Theorem 2.1 *If $R_0 \leqslant 1$, then the only constant equilibrium of system (2.1) is the disease-free equilibrium $(S_1^0, S_2^0, 0, 0)$ with $S_i^0 = \frac{\lambda_i}{\delta_i}$ and it is globally stable. If $R_0 > 1$, then system (2.1) admits two constant equilibria; namely, the disease-free equilibrium $(S_1^0, S_2^0, 0, 0)$ and an endemic equilibrium $(S_1^*, S_2^*, I_1^*, I_2^*)(S_1^*, S_2^*, I_1^*, I_2^* > 0)$. Furthermore, $(S_1^0, S_2^0, 0, 0)$ is unstable and $(S_1^*, S_2^*, I_1^*, I_2^*)$ is globally asymptotically stable.*

In Sect. 2, we always assume that $R_0 > 1$. In this case, system (1.2) admits two equilibria, the disease-free equilibrium $(S_1^0, S_2^0, 0, 0)$ and the endemic equilibrium $(S_1^*, S_2^*, I_1^*, I_2^*)$. In the following we establish the existence of traveling wave solutions of (1.2) connecting these two equilibria $(S_1^0, S_2^0, 0, 0)$ and $(S_1^*, S_2^*, I_1^*, I_2^*)$. A *traveling wave solution* of (1.2) is a special solution with the form as follows

$$(S_1(\xi), S_2(\xi), I_1(\xi), I_2(\xi)), \quad \xi = x + ct \in \mathbb{R}. \tag{2.2}$$

Substituting (2.2) into (1.2), we obtain the wave form equations as follows:

$$\begin{cases} d_1 S_1''(\xi) + \lambda_1 - cS_1'(\xi) - \beta_{11}S_1(\xi)I_1(\xi) - \beta_{12}S_1(\xi)I_2(\xi) - \delta_1 S_1(\xi) = 0, \\ d_2 S_2''(\xi) + \lambda_2 - cS_2'(\xi) - \beta_{21}S_2(\xi)I_1(\xi) - \beta_{22}S_2(\xi)I_2(\xi) - \delta_2 S_2(\xi) = 0, \\ D_1 I_1''(\xi) - cI_1'(\xi) - r_1 I_1(\xi) + \epsilon_1 S_1(\xi - c\tau)\Big(\beta_{11}I_1(\xi - c\tau) + \beta_{12}I_2(\xi - c\tau)\Big) = 0, \\ D_2 I_2''(\xi) - cI_2'(\xi) - r_2 I_2(\xi) + \epsilon_2 S_2(\xi - c\tau)\Big(\beta_{21}I_1(\xi - c\tau) + \beta_{22}I_2(\xi - c\tau)\Big) = 0, \end{cases} \quad \xi \in \mathbb{R}. \tag{2.3}$$

we intend to look for a positive solution $(S_1(\xi), S_2(\xi), I_1(\xi), I_2(\xi))$ of (2.3) with the following boundary conditions

$$S_i(-\infty) = S_i^0, \quad S_i(+\infty) = S_i^*, \quad I_i(-\infty) = 0, \quad I_i(+\infty) = I_i^*, \quad i = 1, 2. \tag{2.4}$$

Linearizing the third and forth equations of (2.3) at the disease-free equilibrium $(S_1^0, S_2^0, 0, 0)$ yields

$$\begin{cases} D_1 I_1''(\xi) - cI_1'(\xi) - r_1 I_1(\xi) + \epsilon_1 S_1^0 (\beta_{11} I_1(\xi - c\tau) + \beta_{12} I_2(\xi - c\tau)) = 0, \\ D_2 I_2''(\xi) - cI_2'(\xi) - r_2 I_2(\xi) + \epsilon_2 S_2^0 (\beta_{21} I_1(\xi - c\tau) + \beta_{22} I_2(\xi - c\tau)) = 0, \end{cases} \quad \xi \in \mathbb{R}.$$

Letting $\binom{I_1(\xi)}{I_2(\xi)} = e^{\mu\xi} \binom{\eta_1}{\eta_2}$, we obtain the characteristic equations

$$\begin{cases} D_1 \eta_1 \mu^2 - c\eta_1 \mu + \epsilon_1 S_1^0 (\beta_{11} \eta_1 + \beta_{12} \eta_2) e^{-c\mu\tau} - r_1 \eta_1 = 0, \\ D_2 \eta_2 \mu^2 - c\eta_2 \mu + \epsilon_2 S_2^0 (\beta_{21} \eta_1 + \beta_{22} \eta_2) e^{-c\mu\tau} - r_2 \eta_2 = 0. \end{cases} \tag{2.5}$$

Let

$$\tilde{A} = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}, \quad \tilde{D} = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$$

and

$$\tilde{F} = \begin{pmatrix} \epsilon_1 \beta_{11} S_1^0 e^{-\mu c\tau} & \epsilon_1 \beta_{12} S_1^0 e^{-\mu c\tau} \\ \epsilon_2 \beta_{21} S_2^0 e^{-\mu c\tau} & \epsilon_2 \beta_{22} S_2^0 e^{-\mu c\tau} \end{pmatrix}.$$

Denote $\Theta(\mu, c) = \mu^2 \tilde{A} - \mu \tilde{B} - \tilde{D} + \tilde{F}$. Then system (2.5) reduces to

$$\Theta(\mu, c) \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = 0. \tag{2.6}$$

Define $A = \tilde{D}^{-1} \tilde{A}$, $B = \tilde{D}^{-1} \tilde{B}$ and $F = \tilde{D}^{-1} \tilde{F}$. Then (2.6) becomes

$$(-A\mu^2 + B\mu + I)^{-1} F\eta = \eta, \tag{2.7}$$

where $\eta = \binom{\eta_1}{\eta_2}$, $m_i(\mu, c) = -D_i \mu^2 + c\mu + r_i$ $(i = 1, 2)$ and

$$(-A\mu^2 + B\mu + I)^{-1} F = \begin{pmatrix} \frac{\epsilon_1 \beta_{11} S_1^0 e^{-\mu c\tau}}{m_1(\mu,c)} & \frac{\epsilon_1 \beta_{12} S_1^0 e^{-\mu c\tau}}{m_1(\mu,c)} \\ \frac{\epsilon_2 \beta_{21} S_2^0 e^{-\mu c\tau}}{m_2(\mu,c)} & \frac{\epsilon_2 \beta_{22} S_2^0 e^{-\mu c\tau}}{m_2(\mu,c)} \end{pmatrix}.$$

Let $M(\mu, c) = (-A\mu^2 + B\mu + I)^{-1} F$, then (2.7) turns to

$$M(\mu, c)\eta = \eta.$$

Let $\rho(\mu, c)$ be the principal eigenvalue of $M(\mu, c)$ and $\mu_c = \min_{i=1,2} \frac{c+\sqrt{c^2+4D_i r_i}}{2D_i}$. For $c \geq 0$ and $\mu \in [0, \mu_c)$, a straightforward computation gives

$$\rho(\mu, c) = \frac{e^{-\mu c\tau}}{2} \left\{ \left(\frac{\epsilon_1 \beta_{11} S_1^0}{m_1(\mu, c)} + \frac{\epsilon_2 \beta_{22} S_2^0}{m_2(\mu, c)} \right) + \left[\left(\frac{\epsilon_1 \beta_{11} S_1^0}{m_1(\mu, c)} - \frac{\epsilon_2 \beta_{22} S_2^0}{m_2(\mu, c)} \right)^2 + \frac{4\epsilon_1 \epsilon_2 \beta_{12} \beta_{21} S_1^0 S_2^0}{m_1(\mu, c) m_2(\mu, c)} \right]^{\frac{1}{2}} \right\}. \tag{2.8}$$

Proposition 2.2 *The following three statements hold:*

(i) μ_c *is strictly increasing in* $c \in [0, \infty)$ *and* $\lim_{c\to\infty} \mu_c = +\infty$*;*
(ii) $\rho(0, c) = R_0$ *for any* $c \in [0, \infty)$*,* $\rho(\mu, 0)$ *is strictly increasing in* $\mu \in [0, \mu_0)$*, and* $\lim_{\mu\to\mu_c-0} \rho(\mu, c) = +\infty$ *for any* $c \geq 0$*;*
(iii) for any $\mu \in (0, \mu_c)$*,* $\frac{\partial}{\partial c}\rho(\mu, c) < 0$*.*

The proof of Proposition 2.2 is similar to that of Proposition 3.1 in Zhao et al. (2017) and we omit it.

Following Proposition 2.2, we define

$$\tilde{\mu}(c) = \min_{\mu\in[0,\mu_c)} \rho(\mu, c) \quad \text{for } c \geq 0.$$

Then we have $\tilde{\mu}(0) = R_0$, $\lim_{c\to\infty} \tilde{\mu}(c) = 0$ and $\tilde{\mu}(c)$ is continuous and strictly decreasing in $c \in [0, \infty)$. Assume $R_0 > 1$. It follows that there exists a constant $c^* > 0$ such that $\tilde{\mu}(c^*) = 1$, $\tilde{\mu}(c) > 1$ for $c \in [0, c^*)$ and $\tilde{\mu}(c) < 1$ for $c \in (c^*, \infty)$. Let

$$\mu_* = \inf \left\{ \mu \in [0, \mu_{c^*}) : \rho(\mu, c^*) = 1 \right\}.$$

It follows that $\rho(\mu_*, c^*) = 1$ and $\rho(\mu_*, c) < 1$ for any $c > c^*$. Define

$$\mu_1(c) = \sup \left\{ \mu \in (0, \mu_*) : \rho(\mu, c) = 1,\ \rho(\mu', c) \geq 1 \text{ for any } \mu' \in (0, \mu) \right\}.$$

Since $\rho(\mu_*, c) < 1$ for any $c > c^*$, we have the following proposition.

Proposition 2.3 *Assume* $R_0 > 1$*, then there exist* $c^* > 0$ *and* $\mu_* \in (0, \mu_{c^*})$ *such that*

(i) $\rho(\mu, c) > 1$ *for any* $0 \leq c < c^*$ *and* $\mu \in (0, \mu_c)$*;*
(ii) $\rho(\mu_*, c^*) = 1$*,* $\rho(\mu, c^*) > 1$ *for* $\mu \in (0, \mu_*)$ *and* $\rho(\mu, c^*) \geq 1$ *for* $\mu \in (0, \mu_{c^*})$*;*
(iii) for any $c > c^*$*, there exists* $\mu_1(c) \in (0, \mu_*)$ *such that* $\rho(\mu_1(c), c) = 1$*,* $\rho(\mu, c) \geq 1$ *for* $\mu \in (0, \mu_1(c))$*, and* $\rho(\mu_1(c) + \varepsilon_n(c), c) < 1$ *for some decreasing sequence* $\{\varepsilon_n(c)\}$ *satisfying* $\lim_{n\to\infty} \varepsilon_n = 0$ *and* $\varepsilon_n + \mu_1(c) < \mu_*$ *for any* $n \in \mathbb{N}$*. Especially,* $\mu_1(c)$ *is strictly decreasing in* $c \in (c^*, \infty)$*.*

Since the matrix $M(\mu, c)$ is nonnegative and irreducible for $\mu \in [0, \mu_c)$, applying the Perron–Frobenius theorem yields the following proposition.

Proposition 2.4 *Assume that $R_0 > 1$. For $c > c^*$, there exist positive unit vectors $\eta(c) = (\eta_1(c), \eta_2(c))^T$ and $\zeta^n(c) = (\zeta_1^n(c), \zeta_2^n(c))^T$ $(n \in \mathbb{N})$ such that*

$$M(\mu_1(c), c)\eta(c) = \eta(c),$$
$$M(\mu_1(c) + \varepsilon_n(c), c)\zeta^n(c) = \rho(\mu_1(c) + \varepsilon_n(c), c)\zeta^n(c), \quad n \in \mathbb{N}.$$

Next, we fix $c > c^*$. Let $\mu_1(c)$, $\eta(c) = (\eta_1(c), \eta_2(c))^T$, $\varepsilon_n(c)$, and $\zeta^n(c) = (\zeta_1^n(c), \zeta_2^n(c))^T$ $(n \in \mathbb{N})$ be defined in Propositions 2.3 and 2.4. Without loss of generality, we replace $\mu_1(c)$, $\eta(c) = (\eta_1(c), \eta_2(c))^T$, $\varepsilon_n(c)$, and $\zeta^n(c) = (\zeta_1^n(c), \zeta_2^n(c))^T$ $(n \in \mathbb{N})$ with μ_1, $\eta = (\eta_1, \eta_2)^T$, ε_n and $\zeta^n = (\zeta_1^n, \zeta_2^n)^T$ $(n \in \mathbb{N})$. Since $\rho(\mu_1 + \varepsilon_n, c) < 1$, it follows from Proposition 2.4 that

$$\begin{cases} -m_1(\mu_1, c)\eta_1 + \epsilon_1 S_1^0 \left(\beta_{11}\eta_1 + \beta_{12}\eta_2\right) e^{-\mu_1 c\tau} = 0, \\ -m_2(\mu_1, c)\eta_2 + \epsilon_2 S_2^0 \left(\beta_{21}\eta_1 + \beta_{22}\eta_2\right) e^{-\mu_1 c\tau} = 0 \end{cases} \tag{2.9}$$

and

$$\begin{cases} -m_1(\mu_1 + \varepsilon_n, c)\zeta_1^n + \epsilon_1 S_1^0 \left(\beta_{11}\zeta_1^n + \beta_{12}\zeta_2^n\right) e^{-(\mu_1+\varepsilon_n)c\tau} < 0, \\ -m_2(\mu_1 + \varepsilon_n, c)\zeta_2^n + \epsilon_2 S_2^0 \left(\beta_{21}\zeta_1^n + \beta_{22}\zeta_2^n\right) e^{-(\mu_1+\varepsilon_n)c\tau} < 0 \end{cases} \tag{2.10}$$

for any $n \in \mathbb{N}$.

Lemma 2.5 *The vector function $\tilde{P}(\xi) = (p_1(\xi), p_1(\xi))^T$ with $p_i(\xi) = \eta_i e^{\mu_1 \xi}$ satisfies*

$$\begin{cases} D_1 p_1''(\xi) - cp_1'(\xi) + \epsilon_1\beta_{11}S_1^0 p_1(\xi - c\tau) + \epsilon_1\beta_{12}S_1^0 p_2(\xi - c\tau) - r_1 p_1(\xi) = 0, \\ D_2 p_2''(\xi) - cp_2'(\xi) + \epsilon_2\beta_{21}S_2^0 p_1(\xi - c\tau) + \epsilon_2\beta_{22}S_2^0 p_2(\xi - c\tau) - r_2 p_2(\xi) = 0, \end{cases} \quad \xi \in \mathbb{R}.$$

Lemma 2.6 *For each $\omega > 0$ sufficiently small with $\omega < \min\{\mu_1, \frac{c}{d_i}\}$ and $M > 1$ large enough, the vector-value map $S^-(\xi) = (S_1^-(\xi), S_2^-(\xi))^T$ defined by $S_i^-(\xi) = \max\{S_i^0(1 - Me^{\omega\xi}), 0\}(i = 1, 2)$ satisfies*

$$cS_i^{-'}(\xi) \le d_i S_i^{-''}(\xi) + \lambda_i - \delta_i S_i^-(\xi) - \beta_{i1} S_i^-(\xi) p_1(\xi) - \beta_{i2} S_i^-(\xi) p_2(\xi), \quad i = 1, 2, \tag{2.11}$$

with $\xi \neq -\frac{1}{\omega}\ln M$.

Proof We firstly consider S_1^-. When $\xi > -\frac{1}{\omega}\ln M$, we have $S_1^-(\xi) = 0$ and hence the inequality (2.11) holds for S_1^-.

When $\xi < -\frac{1}{\omega}\ln M$, one has $p_i(\xi) = \eta_i e^{\mu_1 \xi} (i = 1, 2)$ and $S_1^- = S_1^0(1 - Me^{\omega\xi})$. Using $\omega < \frac{c}{d_1}$ and $e^{\frac{\mu_1-\omega}{\omega}\ln\frac{1}{M}} \to 0$ as $M \to +\infty$, we deduce that

$$\begin{aligned}
&(-d_1\omega + c)S_1^0\omega Me^{\omega\xi} + \lambda_1 - \delta_1 S_1^0(1 - Me^{\omega\xi}) \\
&\quad -(\beta_{11}\eta_1 + \beta_{12}\eta_2)S_1^0(1 - Me^{\omega\xi})e^{\mu_1\xi} \\
&= (-d_1\omega + c)S_1^0\omega Me^{\omega\xi} - (\beta_{11}\eta_1 + \beta_{12}\eta_2)S_1^0(1 - Me^{\omega\xi})e^{\mu_1\xi} + M\delta_1 S_1^0 e^{\omega\xi} \\
&\geqslant \left[(c - d_1\omega)S_1^0\omega M - (\beta_{11}\eta_1 + \beta_{12}\eta_2)S_1^0 e^{-\frac{\mu_1-\omega}{\omega}\ln M} + M\delta_1 S_1^0\right]e^{\omega\xi} \\
&\geq 0
\end{aligned}$$

for $M > 1$ large enough. Similarly, we can show that (2.11) holds for S_2^-. This completes the proof. □

Lemma 2.7 *Fix $0 < \epsilon < \frac{\omega}{2}$ with $\epsilon = \varepsilon_{n_0}$ for some $n_0 \in \mathbb{N}$. Denote the eigenvector $\zeta^{n_0} = (\zeta_1^{n_0}, \zeta_2^{n_0})^T$ by $\zeta = (\zeta_1, \zeta_2)^T$. Then the function $H(\xi) = (h_1(\xi), h_2(\xi))^T$ with $h_i(\xi) = \max\{(\eta_i e^{\mu_1\xi} - \mathcal{V}\zeta_i e^{(\mu_1+\epsilon)\xi}), 0\}$ satisfies*

$$\begin{aligned}
&ch_1'(\xi) \leq D_1 h_1''(\xi) - r_1 h_1(\xi) + \epsilon_1 S_1^-(\xi - c\tau)\left(\beta_{11}h_1(\xi - c\tau) + \beta_{12}h_2(\xi - c\tau)\right), \\
&\qquad \xi < \frac{1}{\epsilon}\ln\frac{\eta_1}{\mathcal{V}\zeta_1}
\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}
&ch_2'(\xi) \leq D_2 h_2''(\xi) - r_2 h_2(\xi) + \epsilon_2 S_2^-(\xi - c\tau)\left(\beta_{21}h_1(\xi - c\tau) + \beta_{22}h_2(\xi - c\tau)\right), \\
&\qquad \xi < \frac{1}{\epsilon}\ln\frac{\eta_2}{\mathcal{V}\zeta_2},
\end{aligned} \tag{2.13}$$

where $\mathcal{V} > 0$ is large enough so that $\min\left\{\frac{1}{\epsilon}\ln\frac{\mathcal{V}\zeta_1}{\eta_1}, \frac{1}{\epsilon}\ln\frac{\mathcal{V}\zeta_2}{\eta_2}\right\} > \frac{1}{\omega}\ln M$.

Proof The proof is similar to that of Zhao et al. (2017, Lemma 3.6), so we omit the details. □

In the following, set $X > \max\{\frac{1}{\varepsilon}\ln\frac{\mathcal{V}\zeta_1}{\eta_1}, \frac{1}{\varepsilon}\ln\frac{\mathcal{V}\zeta_2}{\eta_2}\}$. Define

$$\Gamma_X = \left\{ (\chi_1(\cdot), \chi_2(\cdot), \varphi_1(\cdot), \varphi_2(\cdot)) \in C([-X, X], \mathbb{R}^4) \;\middle|\; \begin{array}{l} \chi_i(\pm X) = S_i^-(\pm X), \\ \varphi_i(\pm X) = h_i(\pm X), \\ S_i^-(\xi) \leq \chi_i(\xi) \leq S_i^0, \\ h_i(\xi) \leq \varphi_i(\xi) \leq p_i(\xi) \end{array} \right\},$$

where $i = 1, 2$. It is easy to see that Γ_X is closed and convex. Define

$$\hat{\chi}_i(\xi) = \begin{cases} \chi_i(\xi), & |\xi| < X, \\ S_i^-(\xi), & \xi \in (-X - c\tau, -X] \end{cases} \quad \text{and} \quad \hat{\varphi}_i(\xi) = \begin{cases} \varphi_i(\xi), & |\xi| < X, \\ h_i(\xi), & \xi \in (-X - c\tau, -X] \end{cases}$$

for any $(\chi_1(\xi), \chi_2(\xi), \varphi_1(\xi), \varphi_2(\xi)) \in \Gamma_X$ and $i = 1, 2$. For any given

$$(\chi_1(\cdot), \chi_2(\cdot), \varphi_1(\cdot), \varphi_2(\cdot)) \in \Gamma_X,$$

we consider the following boundary-value problem for $\xi \in (-X, X)$,

$$\begin{cases} -d_1 S''_{1,X}(\xi) + cS'_{1,X}(\xi) - \lambda_1 + \beta_{11}\varphi_1(\xi)S_{1,X}(\xi) + \beta_{12}\varphi_2(\xi)S_{1,X}(\xi) + \delta_1 S_{1,X}(\xi) = 0, \\ -d_2 S''_{2,X}(\xi) + cS'_{2,X}(\xi) - \lambda_2 + \beta_{21}\varphi_1(\xi)S_{2,X}(\xi) + \beta_{22}\varphi_2(\xi)S_{2,X}(\xi) + \delta_2 S_{2,X}(\xi) = 0, \\ -D_1 I''_{1,X}(\xi) + cI'_{1,X}(\xi) + r_1 I_{1,X}(\xi) = \epsilon_1 \hat{\chi}_1(\xi - c\tau)\left(\beta_{11}\hat{\varphi}_1(\xi - c\tau) + \beta_{12}\hat{\varphi}_2(\xi - c\tau)\right), \\ -D_2 I''_{2,X}(\xi) + cI'_{2,X}(\xi) + r_2 I_{2,X}(\xi) = \epsilon_2 \hat{\chi}_2(\xi - c\tau)\left(\beta_{21}\hat{\varphi}_1(\xi - c\tau) + \beta_{22}\hat{\varphi}_2(\xi - c\tau)\right) \end{cases} \tag{2.14}$$

with the following boundary conditions

$$S_{i,X}(\pm X) = S_i^-(\pm X), \quad I_{i,X}(\pm X) = h_i(\pm X), \ i = 1, 2. \tag{2.15}$$

Note that the problem (2.14)–(2.15) admits a unique solution

$$(S_{1,X}, \ S_{2,X}, \ I_{1,X}, \ I_{2,X})$$

such that $S_{1,X}, \ S_{2,X}, \ I_{1,X}, \ I_{2,X} \in W^{2,p}((-X, X), \mathbb{R}) \cap C([-X, X], \mathbb{R})$ for any $p > 1$ (see Gilbarg and Trudinger 2001, Corollary 9.18). It then follows from the embedding theorem (see Gilbarg and Trudinger 2001, Theorem 7.26) that $S_{i,X}(\cdot), \ I_{i,X}(\cdot) \in W^{2,p}(-X, X) \hookrightarrow C^{1+\alpha}[-X, X]$ for some $\alpha \in (0, 1)$ and $i = 1, 2$. Define an operator $\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4)$ on Γ_X as

$$\left.\begin{aligned} S_{1,X} &= \mathcal{T}_1(\chi_1, \chi_2, \varphi_1, \varphi_2), \\ S_{2,X} &= \mathcal{T}_2(\chi_1, \chi_2, \varphi_1, \varphi_2), \\ I_{1,X} &= \mathcal{T}_3(\chi_1, \chi_2, \varphi_1, \varphi_2), \\ I_{2,X} &= \mathcal{T}_4(\chi_1, \chi_2, \varphi_1, \varphi_2), \end{aligned}\right. \quad \forall (\chi_1, \chi_2, \varphi_1, \varphi_2) \in \Gamma_X.$$

Theorem 2.8 *The operator $\mathcal{T}$ maps Γ_X into Γ_X.*

Proof It is obvious that 0 is a sub-solution of the first and second equations of (2.14) on $(-X, X)$, respectively, and S_1^0 and S_2^0 are super-solutions of the first and second equations of (2.14) on $(-X, X)$, respectively. Then the maximum principle combining the fact that $0 < S_{i,X}(X) = S_i^-(X) < S_i^0$ and $0 = S_{i,X}(-X) = S_i^-(-X) < S_i^0$ implies that $0 \leq S_{i,X}(\xi) \leq S_i^0$ for any $\xi \in [-X, X]$. It follows from (2.11) that $S_i^-(\xi)$ satisfies

$$\begin{aligned} 0 &\geqslant -d_i S_i^{-''}(\xi) + cS_i^{-'}(\xi) - \lambda_i + \beta_{i1} S_i^-(\xi) p_1(\xi) + \beta_{i2} S_i^-(\xi) p_2(\xi) + \delta_i S_i^-(\xi) \\ &\geqslant -d_i S_i^{-''}(\xi) + cS_i^{-'}(\xi) - \lambda_i + \beta_{i1} S_i^-(\xi)\varphi_1(\xi) + \beta_{i2} S_i^-(\xi)\varphi_2(\xi) + \delta_i S_i^-(\xi) \end{aligned}$$

for $[-X, X']$ with $X' = -\frac{1}{\omega} \ln M$. Since $S_{i,X}(-X) = S_i^-(-X)$ and $S_{i,X}(X') \geq S_i^-(X') = 0$, we obtain $S_i^-(\xi) \leqslant S_{i,X}(\xi)$ for $\xi \in [-X, X']$ by the maximum principle. Thus, one has $S_i^-(\xi) \leqslant S_{i,X}(\xi) \leq S_i^0$ for $\xi \in [-X, X]$.

Next, we consider $I_{1,X}(\xi)$ and $I_{2,X}(\xi)$. Firstly, using the maximum principle, we obtain that $I_{1,X}(\xi) \geqslant 0$ and $I_{2,X}(\xi) \geqslant 0$ for any $\xi \in [-X, X]$. According to $S_i^-(\xi) \leq \hat{\chi}_i(\xi) \leqslant S_i^0$ and $h_i(\xi) \leq \hat{\varphi}_i(\xi) \leqslant p_i(\xi)$ for any $\xi \in (-X - c\tau, X)$, one has

$$\begin{aligned}&\epsilon_i\Big(\beta_{i1}\hat{\chi}_i(\xi - c\tau)\hat{\varphi}_1(\xi - c\tau) + \beta_{i2}\hat{\chi}_i(\xi - c\tau)\hat{\varphi}_2(\xi - c\tau)\Big)\\ &\quad\leq \epsilon_i\Big(\beta_{i1}S_i^0 p_1(\xi - c\tau) + \beta_{i2}S_i^0 p_2(\xi - c\tau)\Big)\end{aligned}$$

for $\xi \in (-X, X)$ and $i = 1, 2$, which combining Lemma 2.5 implies that $p_i(\xi)(i = 1, 2)$ are super-solutions of the last two equations of (2.14) on $[-X, X]$, respectively. On the other hand, there holds

$$\begin{aligned}&\epsilon_i\left(\beta_{i1}\hat{\chi}_i(\xi - c\tau)\hat{\varphi}_1(\xi - c\tau) + \beta_{i2}\hat{\chi}_i(\xi - c\tau)\hat{\varphi}_2(\xi - c\tau)\right)\\ &\quad\geq \epsilon_i\left(\beta_{i1}S_i^-(\xi - c\tau)h_1(\xi - c\tau) + \beta_{i2}S_i^-(\xi - c\tau)h_2(\xi - c\tau)\right)\end{aligned}$$

for $\xi \in (-X, X)$ and $i = 1, 2$, which combining (2.12) and (2.13) implies that $h_i(\xi)(i = 1, 2)$ are sub-solutions of the last two equations of (2.14) on $[-X, X_i^{'}]$ with $X_i^{'} = \frac{1}{\varepsilon}\ln\frac{\eta_i}{\mathcal{V}\zeta_i}(i = 1, 2)$, respectively. By the comparison principle, we have $I_{1,X}(\xi) \geqslant h_1(\xi)$ and $I_{2,X}(\xi) \geq h_2(\xi)$ for all $\xi \in [-X, X]$. Thus, we obtain that

$$p_1(\xi) \geqslant I_{1,X}(\xi) \geqslant h_1(\xi), \quad p_2(\xi) \geqslant I_{2,X}(\xi) \geqslant h_2(\xi), \quad \forall\xi \in [-X, X].$$

This completes the proof. □

Making use of the classic embedding theorem, we know that $\mathcal{T}$ is a compact operator from Γ_X to Γ_X. Using the globally elliptic estimates (Gilbarg and Trudinger 2001) and the embedding theorem, it is easy to show that $\mathcal{T} : \Gamma_X \to \Gamma_X$ is completely continuous. Thus, the Schauder's fixed point theorem implies that there exists a vector function $(S_{1,X}, S_{2,X}, I_{1,X}, I_{2,X}) \in \Gamma_X$ satisfying

$$(S_{1,X}, S_{2,X}, I_{1,X}, I_{2,X}) = \mathcal{T}(S_{1,X}, S_{2,X}, I_{1,X}, I_{2,X})$$

for $\xi \in [-X, X]$. In particular, $(S_{1,X}, S_{2,X}, I_{1,X}, I_{2,X})$ satisfies

$$\begin{cases} -d_1S''_{1,X}(\xi) + cS'_{1,X}(\xi) - \lambda_1 + \Big(\beta_{11}I_{1,X}(\xi) + \beta_{12}I_{2,X}(\xi)\Big)S_{1,X}(\xi) + \delta_1S_{1,X}(\xi) = 0,\\ -d_2S''_{2,X}(\xi) + cS'_{2,X}(\xi) - \lambda_2 + \Big(\beta_{21}I_{1,X}(\xi) + \beta_{22}I_{2,X}(\xi)\Big)S_{2,X}(\xi) + \delta_2S_{2,X}(\xi) = 0,\\ -D_1I''_{1,X}(\xi) + cI'_{1,X}(\xi) + r_1I_{1,X}(\xi) = \epsilon_1\hat{S}_{1,X}(\xi - c\tau)\left(\beta_{11}\hat{I}_{1,X} + \beta_{12}\hat{I}_{2,X}\right)(\xi - c\tau),\\ -D_2I''_{2,X}(\xi) + cI'_{2,X}(\xi) + r_2I_{2,X}(\xi) = \epsilon_2\hat{S}_{2,X}(\xi - c\tau)\left(\beta_{21}\hat{I}_{1,X} + \beta_{22}\hat{I}_{2,X}\right)(\xi - c\tau),\\ S_i(\pm X) = S_i^-(\pm X), \quad I_i(\pm X) = I_i^-(\pm X)\end{cases} \tag{2.16}$$

for every $\xi \in (-X, X)$, where

$$\left(\beta_{i1}\hat{I}_{1,X} + \beta_{i2}\hat{I}_{2,X}\right)(\xi - c\tau) = \beta_{i1}\hat{I}_{1,X}(\xi - c\tau) + \beta_{i2}\hat{I}_{2,X}(\xi - c\tau), \quad i = 1, 2,$$

$$\hat{S}_{i,X}(\xi) = \begin{cases} S_{i,X}(\xi), & |\xi| < X, \\ S_i^-(\xi), & \xi \in (-X - c\tau, -X], \end{cases} \qquad i = 1, 2$$

and

$$\hat{I}_{i,X}(\xi) = \begin{cases} I_{i,X}(\xi), & |\xi| < X, \\ h_i(\xi), & \xi \in (-X - c\tau, -X], \end{cases} \qquad i = 1, 2.$$

Theorem 2.9 *For given $Y > 0$, there exists a constant $M(Y) > 0$, which is independent of*

$$X > \max\left\{Y, \frac{1}{\varepsilon}\ln\frac{\eta_1}{\mathcal{V}\zeta_1}, \frac{1}{\varepsilon}\ln\frac{\eta_2}{\mathcal{V}\zeta_2}, \frac{1}{\omega}\ln\frac{1}{M}\right\}$$

such that

$$\| S_{i,X} \|_{C^3[-Y,Y]}, \| I_{i,X} \|_{C^{2,1}[-Y,Y]} \leq M(Y), \quad i = 1, 2. \tag{2.17}$$

Proof We consider the Eq. (2.16). It is clear that $S_{i,X}(\xi) \leqslant S_i^0$ and $I_i(\xi) \leqslant \eta_i e^{\mu_1 Y} \leqslant M_i(Y)$ for $\xi \in [-Y, Y]$ and $i = 1, 2$. Applying the $L^p (p \geqslant 2)$ estimates of linear elliptic differential equations to $S_{i,X}$ gives

$$||S_{i,X}||_{W^{2,p}(-Y,Y)} \leqslant \mathcal{O}\left(\lambda_i + \sum_{j=1}^{2}\beta_{ij} S_i^0 \eta_i e^{\mu_1 Y} + ||\chi_i||_{W^{2,p}(-Y,Y)}\right),$$

where $\mathcal{O}$ is a constant depending upon Y, and χ_i is taken to be a linear function connecting the points $(-Y, S_{i,X}(-Y))$ and $(Y, S_{i,X}(Y))$. As a consequence, we can choose a positive constant $\bar{\mathcal{M}}(Y)$ which is depending on Y such that $||S_{i,X}||_{W^{2,p}(-Y,Y)} \leqslant \bar{\mathcal{M}}(Y)$ for any $X > Y$. Since $W^{2,p}(-Y, Y) \hookrightarrow C^{1,\alpha}[-Y, Y]$ for $\alpha = 1 - \frac{1}{p}$, we have that there exists a constant $\mathcal{P}(Y)$ depending on Y such that $||S_{i,X}||_{C^{1,\alpha}[-Y,Y]} \leqslant \mathcal{P}(Y)||S_{i,X}||_{W^{2,p}(-Y,Y)}$. We further conclude that $||S_{i,X}||_{C^{1,\alpha}[-Y,Y]} \leqslant \mathcal{P}(Y)\bar{\mathcal{M}}(Y)$. According to the $S_{i,X}$ equation, we obtain $||S_{i,X}||_{C^2[-Y,Y]} \leqslant M(Y)$, where $M(Y)$ is a positive constant depending upon Y. By a similar argument, we have $||I_{i,X}||_{C^2[-Y,Y]} \leqslant M(Y)(i = 1, 2)$. Differentiating both sides of the $S_{i,X}$ equations of (2.16), we have that $||S_{i,X}||_{C^3[-Y,Y]} \leqslant M(Y)$, where $M(Y)$ is a positive constant depending upon Y. By the definitions of $\hat{S}_{i,X}$ and $\hat{I}_{i,X}(i = 1, 2)$, we have that $||I_{i,X}||_{C^{2,1}[-Y,Y]} \leqslant M(Y)$ for some positive constant $M(Y)$ depending on Y. This completes the proof. □

Take a sequence of positive numbers $\{X_m\}_{m>0}$ such that $X_m \to +\infty$ when $m \to +\infty$. Then by Theorem 2.9, there exists a solution $(S_1, S_2, I_1, I_2) \in C^2(\mathbb{R}, \mathbb{R}^4)$ of (2.3) such that

$$S_i^-(\xi) \leqslant S_i(\xi) \leqslant S_i^0, \quad h_i(\xi) \leqslant I_i(\xi) \leqslant p_i(\xi), \quad \forall \xi \in \mathbb{R}, \quad i = 1, 2. \tag{2.18}$$

By (2.18), we have

$$\lim_{\xi\to-\infty} S_i(\xi) = S_i^0, \quad \lim_{\xi\to-\infty} I_i(\xi) = 0. \tag{2.19}$$

In the following, we show some properties of solutions (S_1, S_2, I_1, I_2).

Lemma 2.10 *Let* $\gamma = \min\{\delta_1, \delta_2, r_1, r_2\}$. *Then we have*

$$0 < I_i(\xi) \leqslant \frac{\lambda_i \epsilon_i \sqrt{d_i}}{\gamma\sqrt{D_i}}, \quad \frac{\lambda_i}{\delta_i + \beta_{i1}\frac{\lambda_1\epsilon_1\sqrt{d_1}}{\gamma\sqrt{D_1}} + \beta_{i2}\frac{\lambda_2\epsilon_2\sqrt{d_2}}{\gamma\sqrt{D_2}}} \leqslant S_i(\xi) \leqslant \frac{\lambda_i}{\gamma}, \tag{2.20}$$

where $i = 1, 2$ *and* $\xi \in \mathbb{R}$.

Proof Since I_i $(i = 1, 2)$ are nonnegative and not identically zero, the strong maximum principle implies that $I_i(\xi) > 0$ for any $\xi \in \mathbb{R}$. It then follows that

$$\begin{cases} -d_1 S_1''(\xi) + cS_1'(\xi) + \gamma S_1(\xi) \leqslant \lambda_1 - \beta_{11}S_1(\xi)I_1(\xi) - \beta_{12}S_1(\xi)I_2(\xi), \\ -d_{S2} S_2''(\xi) + cS_2'(\xi) + \gamma S_2(\xi) \leqslant \lambda_2 - \beta_{21}S_2(\xi)I_1(\xi) - \beta_{22}S_2(\xi)I_2(\xi), \\ -D_1 I_1''(\xi) + cI_1'(\xi) + \gamma I_1(\xi) \leqslant \epsilon_1 S_1(\xi - c\tau)(\beta_{11}I_1 + \beta_{12}I_2)(\xi - c\tau), \\ -D_2 I_2''(\xi) + cI_2'(\xi) + \gamma I_2(\xi) \leqslant \epsilon_2 S_2(\xi - c\tau)(\beta_{21}I_1 + \beta_{22}I_2)(\xi - c\tau), \end{cases} \quad \forall \xi \in \mathbb{R}, \tag{2.21}$$

where $(\beta_{21}I_1 + \beta_{22}I_2)(\xi - c\tau) := \beta_{21}I_1(\xi - c\tau) + \beta_{22}I_2(\xi - c\tau)$, $\forall \xi \in \mathbb{R}$. Set

$$m_i(\xi) := \beta_{i1}S_i(\xi)I_1(\xi) + \beta_{i2}S_i(\xi)I_2(\xi)$$

and

$$n_i(\xi) = \epsilon_i S_i(\xi - c\tau)(\beta_{i1}I_1(\xi - c\tau) + \beta_{i2}I_2(\xi - c\tau))$$

for each $\xi \in \mathbb{R}$ and $i = 1, 2$. Consider the following Cauchy problems

$$\begin{cases} \frac{\partial}{\partial t}u_i(t,\xi) - d_i\frac{\partial}{\partial \xi^2}u_i(t,\xi) + c\frac{\partial}{\partial \xi}u_i(t,\xi) + \gamma u_i(t,\xi) = \lambda_i - m_i(\xi), \\ u_i(0,\xi) = S_i(\xi), \end{cases} \quad \forall t > 0, \ \xi \in \mathbb{R}, \tag{2.22}$$

and

$$\begin{cases} \frac{\partial}{\partial t}v_i(t,\xi) - D_i\frac{\partial}{\partial \xi^2}v_i(t,\xi) + c\frac{\partial}{\partial \xi}v_i(t,\xi) + \gamma v_i(t,\xi) = n_i(\xi), \\ v_i(0,\xi) = I_i(\xi), \end{cases} \quad \forall t > 0, \ \xi \in \mathbb{R}, \tag{2.23}$$

we have (see Friedman 1964, Chapter 1, Theorems 12 and 16)

$$u_i(t,\xi) = e^{-\gamma t}\int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_i t}} e^{-\frac{(\xi - ct - y)^2}{4d_i t}} S_i(y)dy$$

$$+\int_0^t \int_{\mathbb{R}} \frac{e^{-\gamma s}}{\sqrt{4\pi d_i s}} e^{-\frac{(\xi-cs-y)^2}{4d_i s}} (\lambda_i - m_i(y))\, dyds$$

and

$$v_i(t,\xi) = e^{-\gamma t} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi D_i t}} e^{-\frac{(\xi-ct-y)^2}{4D_i t}} I_i(y) dy$$
$$+\int_0^t \int_{\mathbb{R}} \frac{e^{-\gamma s}}{\sqrt{4\pi D_i s}} e^{-\frac{(\xi-cs-y)^2}{4D_i s}} n_i(y) dyds$$

for any $t > 0$, $\xi \in \mathbb{R}$ and $i = 1, 2$. Then, by the comparison principle (see the Phragmèn-Lindelöf principle in Protter and Weinberger 1983, Chapter 3, Theorem 10), we have

$$S_i(\xi) \leqslant u_i(t,\xi) \quad \text{and} \quad I_i(\xi) \leqslant v_i(t,\xi)$$

for any $t > 0$, $\xi \in \mathbb{R}$ and $i = 1, 2$. Letting $t \to \infty$ yields

$$S_i(\xi) \leqslant \frac{\lambda_i}{\gamma} - f_{d_i}(\xi) \quad \text{and} \quad I_i(\xi) \leq g_{D_i}(\xi), \quad \forall \xi \in \mathbb{R}, \quad i = 1, 2,$$

where

$$f_{d_i}(\xi) = \int_0^{+\infty} \frac{e^{-\gamma t}}{\sqrt{4\pi d_i t}} \int_{-\infty}^{+\infty} m_i(\xi - y - ct) e^{-\frac{y^2}{4d_i t}} dydt$$

and

$$g_{D_i}(\xi) = \int_0^{+\infty} \frac{e^{-\gamma t}}{\sqrt{4\pi D_i t}} \int_{-\infty}^{+\infty} n_i(\xi - y - ct) e^{-\frac{y^2}{4D_i t}} dydt$$
$$= \epsilon_i f_{D_i}(\xi - c\tau), \ \forall \xi \in \mathbb{R}.$$

By the expression of $f_{d_i}(\xi)$, one has

$$\sqrt{d_i} f_{d_i}(\xi) = \int_0^{+\infty} \frac{e^{-\gamma t}}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} m_i(\xi - y - ct) e^{-\frac{y^2}{4d_i t}} dydt \ \forall \xi \in \mathbb{R}, \ i = 1, 2. \tag{2.24}$$

Furthermore, due to (2.24) and $d_i \geqslant D_i$, it is easy to see that

$$\sqrt{d_i} f_{d_i}(\xi) \geqslant \sqrt{D_i} f_{D_i}(\xi), \ \forall \xi \in \mathbb{R}, \quad i = 1, 2,$$

which leads to

$$\sqrt{D_i} I_i(\xi) \leqslant \sqrt{D_i} g_{D_i}(\xi)$$
$$= \epsilon_i \sqrt{D_i} f_{D_i}(\xi - c\tau)$$

$$\begin{aligned}&\leqslant \epsilon_i \sqrt{d_i} f_{d_i}(\xi - c\tau)\\ &\leqslant \epsilon_i \sqrt{d_i} \frac{\lambda_i}{\gamma}, \ \forall \xi \in \mathbb{R}, \ \ i = 1, 2.\end{aligned}$$

This completes the proof of inequalities for I_i $(i = 1, 2)$.

Let

$$\inf_{\xi \in \mathbb{R}} S_i(\xi) := \varrho_i \geqslant 0 \ \text{ and } \ \sigma_i := \frac{\lambda_i}{\delta_i + \frac{\beta_{i1}\lambda_1\epsilon_1\sqrt{d_1}}{\gamma\sqrt{D_1}} + \frac{\beta_{i2}\lambda_2\epsilon_2\sqrt{d_2}}{\gamma\sqrt{D_2}}}, \ \ i = 1, 2.$$

In the following we show that

$$\varrho_i \geqslant \sigma_i, \ \ i = 1, 2.$$

Without loss of generality, we assume on the contrary that $\varrho_1 < \sigma_1$. Since $S_1(\xi)$ satisfies

$$d_1 S_1''(\xi) - cS_1'(\xi) + \lambda_1 - \left(\delta_1 + \beta_{11}\frac{\lambda_1\epsilon_1\sqrt{d_1}}{\sqrt{D_1}\gamma} + \beta_{12}\frac{\lambda_2\epsilon_2\sqrt{d_2}}{\sqrt{D_2}\gamma}\right) S_1(\xi) \leqslant 0, \ \forall \xi \in \mathbb{R}, \tag{2.25}$$

it is impossible that there exists a local minimum point $\xi_0 \in \mathbb{R}$ satisfying $S_1(\xi_0) < \sigma_1$. Then we conclude that there exists $\xi_1 > 0$ such that $S_1(\xi)$ is nonincreasing in $[\xi_1, +\infty)$ and

$$\lim_{\xi \to \infty} S_1(\xi) = \varrho_1 < \sigma_1.$$

Since $S_1(\xi)$, $S_2(\xi)$, $I_1(\xi)$ and $I_2(\xi)$ are bounded in $\xi \in \mathbb{R}$, then by the interior L^p estimate for elliptic equations (Gilbarg and Trudinger 2001) and the embedding theorem, we have that there exists a constant $C_0 > 0$ such that

$$\|S_i(\cdot)\|_{C^3(\mathbb{R})}, \|I_i(\cdot)\|_{C^3(\mathbb{R})} < C_0, \ \ i = 1, 2.$$

Applying the inequality $\|\cdot\|^2_{C^1([x,+\infty)} \leqslant 4\|\cdot\|_{C([x,+\infty)}\|\cdot\|_{C^2([x,+\infty)}$ to the function $S_1(\xi) - \varrho_1$, we have that

$$\lim_{\xi \to \infty} S_1'(\xi) = \lim_{\xi \to \infty} S_1''(\xi) = 0.$$

Letting $\xi \to +\infty$ in (2.25) yields $\lambda_1 - \frac{\lambda_1\varrho_1}{\sigma_1} \leqslant 0$. This is a contradiction due to $\varrho_1 < \sigma_1$. This completes the proof. □

Proposition 2.11 *Assume that $(S_1(\xi), S_2(\xi), I_1(\xi), I_2(\xi))$ is a positive bounded solution of (2.3) and (2.19). Then there exist two positive constants M_1 and $\bar{M}$ such that*

$$\max\left\{\max_{x\in[\xi-1,\xi+1]} I_1(x), \max_{x\in[\xi-1,\xi+1]} I_2(x)\right\} \le M_1 \min\left\{\min_{x\in[\xi-1,\xi+1]} I_1(x), \min_{x\in[\xi-1,\xi+1]} I_2(x)\right\} \tag{2.26}$$

and

$$\left|\frac{I_1'(\xi)}{I_1(\xi)}\right| + \left|\frac{I_2'(\xi)}{I_2(\xi)}\right| \le \bar{M} \tag{2.27}$$

for any $\xi \in \mathbb{R}$.

Proof Using

$$I_i(\xi) = \frac{1}{\rho_i}\int_{-\infty}^{\xi} e^{\bar{\Lambda}_{i1}(\xi-x)}\epsilon_i S_i(x-c\tau)\left(\beta_{i1}I_1(x-c\tau)+\beta_{i2}I_2(x-c\tau)\right)dx + \frac{1}{\rho_i}\int_{\xi}^{+\infty} e^{\bar{\Lambda}_{i2}(\xi-x)}\epsilon_i S_i(x-c\tau)\left(\beta_{i1}I_1(x-c\tau)+\beta_{i2}I_2(x-c\tau)\right)dx \tag{2.28}$$

for any $\xi \in \mathbb{R}$, where $i = 1, 2$, $\bar{\Lambda}_{i1} = \frac{c-\sqrt{c^2+4D_ir_i}}{2D_i}$, $\bar{\Lambda}_{i2} = \frac{c+\sqrt{c^2+4D_ir_i}}{2D_i}$, $\rho_i = D_i(\bar{\Lambda}_{i2} - \bar{\Lambda}_{i1})$ and $\bar{\Lambda} := \max\{-\bar{\Lambda}_{11}, \bar{\Lambda}_{12}, -\bar{\Lambda}_{21}, \bar{\Lambda}_{22}\}$, we can show that for each $\xi \in \mathbb{R}$, $I_i(\xi+y)e^{-\bar{\Lambda}y}$ is decreasing in $y \in \mathbb{R}$ and $I_i(\xi+y)e^{\bar{\Lambda}y}$ is increasing in $y \in \mathbb{R}$. Furthermore, we have

$$-D_iI_i''(\xi) + cI_i'(\xi) + r_iI_i(\xi) > \epsilon_i e^{-\bar{\Lambda}c\tau}\Big(\beta_{i1}S_i(\xi-c\tau)I_1(\xi) + \epsilon_i\beta_{i2}S_i(\xi-c\tau)I_2(\xi)\Big) \tag{2.29}$$

and

$$-D_iI_i''(\xi) + cI_i'(\xi) + r_iI_i(\xi) < \epsilon_i e^{\bar{\Lambda}c\tau}\Big(\beta_{i1}S_i(\xi-c\tau)I_1(\xi) + \epsilon_i\beta_{i2}S_i(\xi-c\tau)I_2(\xi)\Big) \tag{2.30}$$

for any $\xi \in \mathbb{R}$ and $i = 1, 2$. By (Földes and Poláčik 2009, Theorem 3.9 and Lemma 3.10), there exist two positive constants $\mathcal{K}$ and $\mathcal{C}$ such that for any $\xi \in \mathbb{R}$

$$\inf\left\{\inf_{x\in(\xi-2,\xi+2)} I_1(x), \inf_{x\in(\xi-2,\xi+2)} I_2(x)\right\} \geqslant \mathcal{K}\max\left\{\|I_1\|_{L^p(\xi-2,\xi+2)}, \|I_2\|_{L^p(\xi-2,\xi+2)}\right\}$$

and

$$\max\left\{\|I_1\|_{L^p(\xi-2,\xi+2)}, \|I_2\|_{L^p(\xi-2,\xi+2)}\right\}$$
$$\geqslant \mathcal{C} \max\left\{\sup_{x\in(\xi-1,\xi+1)} I_1(x), \sup_{x\in(\xi-1,\xi+1)} I_2(x)\right\},$$

where $p > 1$ is a constant. According to

$$\inf_{x\in(\xi-1,\xi+1)} I_i(x) \geqslant \inf_{x\in(\xi-2,\xi+2)} I_i(x), \quad \forall \xi \in \mathbb{R}, \; i = 1, 2,$$

then we know that (2.26) holds.

Furthermore, we show that (2.27) holds. In fact, applying the L^p interior estimates (Gilbarg and Trudinger 2001, Theorem 9.11) to the equations satisfied by I_1 and I_2, we have that there exists a positive constant $M_2 > 0$ such that for any $\xi \in \mathbb{R}$,

$$\max\left\{\|I_1\|_{W^{2,p}\left(\xi-\frac{1}{2},\xi+\frac{1}{2}\right)}, \|I_2\|_{W^{2,p}\left(\xi-\frac{1}{2},\xi+\frac{1}{2}\right)}\right\}$$
$$\leq M_2\left(\|\tilde{I}_1\|_{L^p\left(\xi-1,\xi+1\right)} + \|\tilde{I}_2\|_{L^p\left(\xi-1,\xi+1\right)}\right),$$

where

$$\|\tilde{I}_i\|_{L^p\left(\xi-1,\xi+1\right)} := \|I_i(\cdot - c\tau)\|_{L^p(\xi-1,\xi+1)}, \quad i = 1, 2.$$

Since $I_i(\xi + y)e^{\bar{\Lambda} y}$ is increasing in $y \in \mathbb{R}$, one has

$$\left(\|\tilde{I}_1\|_{L^p\left(\xi-1,\xi+1\right)} + \|\tilde{I}_2\|_{L^p\left(\xi-1,\xi+1\right)}\right)$$
$$\leqslant e^{\bar{\Lambda} c\tau}\left(\|I_1\|_{L^p\left(\xi-1,\xi+1\right)} + \|I_2\|_{L^p\left(\xi-1,\xi+1\right)}\right).$$

Using the embedding theorem, we have that there exists $M_3 > 0$ such that

$$\max\left\{\|I_1'\|_{C\left([\xi-\frac{1}{2},\xi+\frac{1}{2}]\right)}, \|I_2'\|_{C\left([\xi-\frac{1}{2},\xi+\frac{1}{2}]\right)}\right\}$$
$$\leq M_3 \max\left\{\max_{[\xi-1,\xi+1]} I_1, \max_{[\xi-1,\xi+1]} I_2\right\} \tag{2.31}$$

for any $\xi \in \mathbb{R}$. Let $\bar{M} = M_1 M_3$. It follows from (2.26) and (2.31) that the inequality (2.27) holds. This completes the proof. □

Now we prove the convergence of traveling waves as $x \to +\infty$, which is a consequence of some suitable Lyapunov functional. Let $g(x) = x - 1 - \ln x$. Define

$$\bar{\mathcal{C}} = \left\{ \begin{array}{l} (S_1(\cdot), S_2(\cdot), I_1(\cdot), I_2(\cdot)) \in (C^1(\mathbb{R}, (0, +\infty)) \times \cdots \times C^1(\mathbb{R}, (0, +\infty))) \\ \exists \bar{M} > 0, \quad \left|\frac{I_1'(\xi)}{I_1(\xi)}\right| + \left|\frac{I_2'(\xi)}{I_2(\xi)}\right| \leqslant \bar{M}, \quad \forall \xi \in \mathbb{R}, \end{array} \right\}.$$

For each $(S_1, S_2, I_1, I_2) \in \bar{\mathcal{C}}$, define

$$
\begin{aligned}
V(\xi) =& \frac{1}{\epsilon_1 \beta_{12} S_1^* I_2^*} \left(\epsilon_1 V_{S1}(\xi) + V_{I1}(\xi) + \epsilon_1 \beta_{11} S_1^* I_1^* W_{11}(\xi) + \epsilon_1 \beta_{12} S_1^* I_2^* W_{12}(\xi)\right) \\
&+ \frac{1}{\epsilon_2 \beta_{21} S_2^* I_1^*} \left(\epsilon_2 V_{S2}(\xi) + V_{I2}(\xi) + \epsilon_2 \beta_{21} S_2^* I_1^* W_{21}(\xi)\right. \\
&\qquad \left. + \epsilon_2 \beta_{22} S_2^* I_2^* W_{22}(\xi)\right), \ \forall \xi \in \mathbb{R},
\end{aligned} \tag{2.32}
$$

where

$$
\begin{aligned}
V_{Si}(\xi) =& S_i^* \left(d_i S_i'(\xi) \left(\frac{1}{S_i(\xi)} - \frac{1}{S_i^*} \right) + cg \left(\frac{S_i(\xi)}{S_i^*} \right) \right), \\
V_{Ii}(\xi) =& I_i^* \left(D_i I_i'(\xi) \left(\frac{1}{I_i(\xi)} - \frac{1}{I_i^*} \right) + cg \left(\frac{I_i(\xi)}{I_i^*} \right) \right)
\end{aligned}
$$

and

$$
W_{ij}(\xi) = \int_0^{c\tau} g \left(\frac{S_i(\xi - \sigma) I_j(\xi - \sigma)}{S_i^* I_j^*} \right) d\sigma, \ \forall \xi \in \mathbb{R}, \ i = 1, 2.
$$

Then we shall show the following result.

Theorem 2.12 *Let* (A) *be satisfied and the solution* (S_1, S_2, I_1, I_2) *be a positive solution of system* (2.3) *such that for some constant* $\mathcal{H} > 1$,

$$
\frac{1}{\mathcal{H}} \leqslant S_i(\xi) \leqslant \frac{\lambda_i}{\gamma}, \tag{2.33}
$$

$$
I_i(\xi) \leqslant \mathcal{H} I_i^*, \tag{2.34}
$$

and

$$
\left| \frac{I_1'(\xi)}{I_1(\xi)} \right| + \left| \frac{I_2'(\xi)}{I_2(\xi)} \right| \leqslant \mathcal{H} \tag{2.35}
$$

for each $\xi \in \mathbb{R}$ *and* $i = 1, 2$. *Then there exists a constant* $m > 0$ *(only depending upon* $\mathcal{H}$*) such that*

$$
-m \leqslant V(\xi) < \infty, \quad \forall \xi \in \mathbb{R} \tag{2.36}
$$

and the map $\xi \to V(\xi)$ *is non-increasing. Moreover, if* $\xi \to V(\xi)$ *is a constant then* $S_i = S_i^*, \ I_i = I_i^*, \ i = 1, 2.$

Proof Note that S_i is bounded in $C^2(\mathbb{R})$. Due to inequalities (2.33)–(2.35), we obtain for any $\xi \in \mathbb{R}$ that

$$
\begin{aligned}
&\left| \epsilon_i S_i^* d_i S_i'(\xi) \left(\frac{1}{S_i(\xi)} - \frac{1}{S_i^*} \right) + D_i I_i^* I_i'(\xi) \left(\frac{1}{I_i(\xi)} - \frac{1}{I_i^*} \right) \right| \\
&\leqslant \epsilon_i S_i^* d_i ||S_i'||_\infty (\mathcal{H} + \frac{1}{S_i^*}) + D_i I_i^* \left(\left| \frac{I_i'(\xi)}{I_i} \right| + \left| \frac{I_i'(\xi)}{I_i^*} \right| \right) \\
&\leqslant \epsilon_i S_i^* d_i ||S_i'||_\infty \left(\mathcal{H} + \frac{1}{S_i^*} \right) + D_i I_i^* \left| \frac{I_i'(\xi)}{I_i(\xi)} \right| + D_i I_i^* \mathcal{H} \left| \frac{I_i'(\xi)}{I_i(\xi)} \right|
\end{aligned}
$$

$$\leqslant \epsilon_i S_i^* d_i ||S_i'||_\infty \left(\mathcal{H} + \frac{1}{S_i^*}\right) + D_i I_i^* \mathcal{H}(1+\mathcal{H}), \quad i = 1, 2. \tag{2.37}$$

Let

$$\omega_i(\xi) = c\left(\epsilon_i S_i^* g\left(\frac{S_i(\xi)}{S_i^*}\right) + I_i^* g\left(\frac{I_i(\xi)}{I_i^*}\right) + \sum_{j=1}^{2} \epsilon_i \beta_{ij} S_i^* I_j^* W_{ij}(\xi)\right),$$
$$\xi \in \mathbb{R}, \quad i = 1, 2$$

and

$$\omega(\xi) = \frac{1}{\epsilon_1 \beta_{12} S_1^* I_2^*}\omega_1(\xi) + \frac{1}{\epsilon_2 \beta_{21} S_2^* I_1^*}\omega_2(\xi), \quad \forall \xi \in \mathbb{R}. \tag{2.38}$$

It is obvious that

$$0 \leqslant \omega(\xi) < \infty, \quad \forall \xi \in \mathbb{R}.$$

As a consequence, the inequality (2.36) holds.

Let us now show that the map $\xi \to V(\xi)$ is non-increasing. By a direct calculation, we have for any $\xi \in \mathbb{R}$ that

$$\begin{aligned}
V_{S1}'(\xi) &= -\delta_1 \frac{(S_1(\xi) - S_1^*)^2}{S_1(\xi)} + \beta_{11} S_1^* I_1^* \left(1 - \frac{S_1^*}{S_1(\xi)} - \frac{S_1(\xi) I_1(\xi)}{S_1^* I_1^*} + \frac{I_1(\xi)}{I_1^*}\right) \\
&\quad + \beta_{12} S_1^* I_2^* \left(1 - \frac{S_1^*}{S_1(\xi)} - \frac{S_1(\xi) I_2(\xi)}{S_1^* I_2^*} + \frac{I_2(\xi)}{I_2^*}\right) - d_1 S_1^* \left(\frac{S_1'(\xi)}{S_1(\xi)}\right)^2, \\
V_{S2}'(\xi) &= -\delta_2 \frac{(S_2(\xi) - S_2^*)^2}{S_2(\xi)} + \beta_{21} S_2^* I_1^* \left(1 - \frac{S_2^*}{S_2(\xi)} - \frac{S_2(\xi) I_1(\xi)}{S_2^* I_1^*} + \frac{I_1(\xi)}{I_1^*}\right) \\
&\quad + \beta_{22} S_2^* I_2^* \left(1 - \frac{S_2^*}{S_2(\xi)} - \frac{S_2(\xi) I_2(\xi)}{S_2^* I_2^*} + \frac{I_2(\xi)}{I_2^*}\right) - d_2 S_2^* \left(\frac{S_2'(\xi)}{S_2(\xi)}\right)^2, \\
W_{ij}'(\xi) &= g\left(\frac{S_i(\xi) I_j(\xi)}{S_i^* I_j^*}\right) - g\left(\frac{S_i(\xi - c\tau) I_j(\xi - c\tau)}{S_i^* I_j^*}\right), \quad i, j = 1, 2, \\
V_{I1}'(\xi) &= -D_1 I_1^* \left(\frac{I_1'(\xi)}{I_1(\xi)}\right)^2 + \epsilon_1 \beta_{11} S_1^* I_1^* \left(\frac{S_1(\xi - c\tau) I_1(\xi - c\tau)}{S_1^* I_1^*} - \frac{I_1(\xi)}{I_1^*}\right. \\
&\quad \left. - \frac{S_1(\xi - c\tau) I_1(\xi - c\tau)}{S_1^* I_1(\xi)} + 1\right) \\
&\quad + \epsilon_1 \beta_{12} S_1^* I_2^* \left(\frac{S_1(\xi - c\tau) I_2(\xi - c\tau)}{S_1^* I_2^*}\right. \\
&\quad \left. - \frac{I_1(\xi)}{I_1^*} - \frac{S_1(\xi - c\tau) I_2(\xi - c\tau) I_1^*}{S_1^* I_2^* I_1(\xi)} + 1\right)
\end{aligned}$$

and

$$
\begin{aligned}
V_{I2}'(\xi) = & -D_2 I_2^* \left(\frac{I_2'(\xi)}{I_2}\right)^2 + \epsilon_2 \beta_{21} S_2^* I_1^* \left(\frac{S_2(\xi - c\tau) I_1(\xi - c\tau)}{S_2^* I_1^*} - \frac{I_2(\xi)}{I_2^*}\right. \\
& \left. - \frac{S_2(\xi - c\tau) I_1(\xi - c\tau) I_2^*}{S_2^* I_1^* I_2(\xi)} + 1\right) \\
& + \epsilon_2 \beta_{22} S_2^* I_2^* \left(\frac{S_2(\xi - c\tau) I_2(\xi - c\tau)}{S_2^* I_2^*}\right. \\
& \left. - \frac{I_2(\xi)}{I_2^*} - \frac{S_2(\xi - c\tau) I_2(\xi - c\tau)}{S_2^* I_2(\xi)} + 1\right).
\end{aligned}
$$

Let $\bar{V}_i(\xi) = \epsilon_i V_{Si}'(\xi) + V_{Ii}'(\xi) + \epsilon_i \beta_{i1} S_i^* I_1^* W_{i1}'(\xi) + \epsilon_i \beta_{i2} S_i^* I_2^* W_{12}'(\xi)$ for any $\xi \in \mathbb{R}$ and $i = 1, 2$. Then we have for any $\xi \in \mathbb{R}$ that

$$
\begin{aligned}
\bar{V}_1(\xi) = & \epsilon_1 \beta_{11} S_1^* I_1^* \left(2 - \frac{S_1^*}{S_1(\xi)} - \frac{S_1(\xi - c\tau) I_1(\xi - c\tau)}{S_1^* I_1(\xi)}\right. \\
& \left. + \ln \frac{S_1(\xi - c\tau) I_1(\xi - c\tau)}{S_1(\xi) I_1(\xi)}\right) \\
& + \epsilon_1 \beta_{12} S_1^* I_2^* \left(2 - \frac{S_1^*}{S_1(\xi)} + \frac{I_2(\xi)}{I_2^*} - \frac{I_1(\xi)}{I_1^*} - \frac{S_1(\xi - c\tau) I_2(\xi - c\tau) I_1^*}{S_1^* I_2^* I_1(\xi)}\right. \\
& \left. + \ln \frac{S_1(\xi - c\tau) I_2(\xi - c\tau)}{S_1(\xi) I_2(\xi)}\right) - \epsilon_1 \delta_1 \frac{(S_1(\xi) - S_1^*)^2}{S_1(\xi)} - \epsilon_1 d_1 S_1^* \left(\frac{S_1'(\xi)}{S_1(\xi)}\right)^2 \\
& - D_1 I_1^* \left(\frac{I_1'(\xi)}{I_1(\xi)}\right)^2
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{V}_2(\xi) = & \epsilon_2 \beta_{22} S_2^* I_2^* \left(2 - \frac{S_2^*}{S_2(\xi)} - \frac{S_2(\xi - c\tau) I_2(\xi - c\tau)}{S_2^* I_2(\xi)}\right. \\
& \left. + \ln \frac{S_2(\xi - c\tau) I_2(\xi - c\tau)}{S_2(\xi) I_2(\xi)}\right) dy \\
& + \epsilon_2 \beta_{21} S_2^* I_1^* \left(2 - \frac{S_2^*}{S_2(\xi)} + \frac{I_1(\xi)}{I_1^*} - \frac{I_2(\xi)}{I_2^*} - \frac{S_2(\xi - c\tau) I_1(\xi - c\tau) I_2^*}{S_2^* I_1^* I_2(\xi)}\right. \\
& \left. + \ln \frac{S_2(\xi - c\tau) I_1(\xi - c\tau)}{S_2(\xi) I_1(\xi)}\right) - \epsilon_2 \delta_2 \frac{(S_2(\xi) - S_2^*)^2}{S_2(\xi)} - \epsilon_2 d_2 S_2^* \left(\frac{S_2'(\xi)}{S_2(\xi)}\right)^2 \\
& - D_2 I_2^* \left(\frac{I_2'(\xi)}{I_2(\xi)}\right)^2.
\end{aligned}
$$

It follows that

$$
V'(\xi) = -\frac{\beta_{11} I_1^*}{\beta_{12} I_2^*} \left\{ g\left(\frac{S_1^*}{S_1(\xi)}\right) + g\left(\frac{S_1(\xi - c\tau) I_1(\xi - c\tau)}{S_1^* I_1(\xi)}\right) \right\}
$$

$$
\begin{aligned}
&-\left\{g\left(\frac{S_1^*}{S_1(\xi)}\right)+g\left(\frac{S_1(\xi)I_2(\xi-c\tau)I_1^*}{S_1^*I_2^*I_1(\xi)}\right)\right\}\\
&-\frac{\beta_{22}I_2^*}{\beta_{21}I_1^*}\left\{g\left(\frac{S_2^*}{S_2(\xi)}\right)+g\left(\frac{S_2(\xi-c\tau)I_2(\xi-c\tau)}{S_2^*I_2(\xi)}\right)\right\}\\
&-\left\{g\left(\frac{S_2^*}{S_2(\xi)}\right)+g\left(\frac{S_2(\xi)I_1(\xi-c\tau)I_2^*}{S_2^*I_1^*I_2(\xi)}\right)\right\}\\
&-\frac{\delta_1}{\beta_{12}S_1^*I_2^*}\frac{(S_1(\xi)-S_1^*)^2}{S_1(\xi)}-\frac{d_1}{\beta_{12}I_2^*}\left(\frac{S_1'(\xi)}{S_1(\xi)}\right)^2-\frac{\delta_2}{\beta_{21}S_2^*I_1^*}\frac{(S_2(\xi)-S_2^*)^2}{S_2(\xi)}\\
&-\frac{d_2}{\beta_{21}I_1^*}\left(\frac{S_2'(\xi)}{S_2(\xi)}\right)^2-\frac{D_1I_1^*}{\epsilon_1\beta_{12}S_1^*I_2^*}\left(\frac{I_1'(\xi)}{I_1(\xi)}\right)^2-\frac{D_2I_2^*}{\epsilon_2\beta_{21}S_2^*I_1^*}\left(\frac{I_2'(\xi)}{I_2(\xi)}\right)^2\\
\leqslant\ & 0,\quad \forall\xi\in\mathbb{R}.
\end{aligned}
$$

When $V(\xi)$ is a constant, we have $\frac{dV(\xi)}{d\xi}\equiv 0$ for all $\xi\in\mathbb{R}$, which implies that

$$
S_i^{'}(\xi)\equiv 0,\quad I_i^{'}(\xi)\equiv 0,\quad S_i(\xi)\equiv S_i^*,\quad I_i(\xi)\equiv I_i^*,\quad \xi\in\mathbb{R}.
$$

The proof is completed. □

Theorem 2.13 *Assume $R_0>1$ and (A) holds. Then for each $c>c^*$, system (1.2) has a traveling wave solution satisfying (2.3) and (2.4).*

Proof Using the previous argument, we know that there exist positive functions

$$
(S_1(\cdot), S_1(\cdot), I_1(\cdot), I_2(\cdot))
$$

satisfying (2.3), (2.20) and

$$
\lim_{\xi\to-\infty} S_i(\xi)=S_i^0,\quad \lim_{\xi\to-\infty} I_i(\xi)=0,\quad i=1,2.
$$

The reminder is to show that

$$
\lim_{\xi\to+\infty} S_i(\xi)=S_i^*,\quad \lim_{\xi\to+\infty} I_i(\xi)=I_i^*,\quad i=1,2.
$$

Consider an arbitrary increasing sequence $\{\xi_m\}_{m\geqslant 0}$ with $\xi_m>0$ such that $\xi_m\to+\infty$ when $m\to+\infty$ as well as the sequences of

$$
S_{i,m}(\xi)=S_i(\xi+\xi_m),\quad I_{i,m}(\xi)=I_i(\xi+\xi_m),\quad i=1,2.
$$

Due to elliptic estimates, up to a subsequence, one may assume that the sequences $(S_{1,m}, S_{2,m}, I_{1,m}, I_{2,m})$ converge towards some functions $(S_{1,\infty}, S_{2,\infty}, I_{1,\infty}, I_{2,\infty})$ in $C_{loc}^1(\mathbb{R})\times\cdots\times C_{loc}^1(\mathbb{R})$. As a consequence, $(S_{1,\infty}, S_{2,\infty}, I_{1,\infty}, I_{2,\infty})$ is a solution of the system (2.3). Moreover, since the map $\xi \to V(\xi)$ is non-increasing, we have for each $m \geqslant 0$ that

$$V(S_{1,m}, S_{2,m}, I_{1,m}, I_{2,m})(\xi) = V(S_1, S_2, I_1, I_2)(\xi + \xi_m) \leqslant V(S_1, S_2, I_1, I_2)(\xi), \quad \forall \xi \in \mathbb{R}.$$

Since it is bounded from below, there exists $l \in \mathbb{R}$ such that

$$\lim_{m\to+\infty} V(S_{1,m}, S_{2,m}, I_{1,m}, I_{2,m})(\xi) = l, \quad \forall \xi \in \mathbb{R}.$$

Since

$$\lim_{m\to+\infty} V(S_{1,m}, S_{2,m}, I_{1,m}, I_{2,m})(\xi) = V(S_{1,\infty}, S_{2,\infty}, I_{1,\infty}, I_{2,\infty})(\xi)$$

in $C^1_{loc}(\mathbb{R})$, we have $V(S_{1,\infty}, S_{2,\infty}, I_{1,\infty}, I_{2,\infty})(\xi) \equiv l$, which combining with Theorem 2.12 implies that

$$S_{i,\infty}(\cdot) = S_i^*, \quad I_{i,\infty}(\cdot) = I_i^*, \quad S'_{i,\infty}(\cdot) = 0, \quad I'_{i,\infty}(\cdot) = 0, \quad i = 1, 2.$$

By the arbitrariness of the sequence $\{\xi_m\}$, we obtain

$$\lim_{\xi\to+\infty} S_i(\xi) = S_i^*, \quad \lim_{\xi\to+\infty} I_i(\xi) = I_i^*, \quad i = 1, 2.$$

This completes the proof. □

Theorem 2.14 *Let* (*A*) *be satisfied and* $R_0 > 1$. *Then for* $c = c^*$, *system* (1.2) *admits a traveling wave solution* $(S_1(\cdot), S_2(\cdot), I_1(\cdot), I_2(\cdot))$ *satisfying* (2.3) *and* (2.4).

Proof Assume that $\{c_m\} \in (c^*, c^* + 1)$ is a decreasing sequence satisfying $\lim_{m\to\infty} c_m = c^*$. Following Theorem 2.13, for each c_m there exists a solution

$$(S_{1,m}, S_{2,m}, I_{1,m}, I_{2,m})$$

of (2.3) such that (2.4), (2.20), (2.26) and (2.27) hold. Since $(S_{1,m}(\cdot + a), S_{2,m}(\cdot + a), I_{1,m}(\cdot + a), I_{2,m}(\cdot + a))$ are also solutions of (2.3) and (2.4) for any $a \in \mathbb{R}$, we can assume that

$$S_{1,m}(0) = \frac{S_1^0 + S_1^*}{2}.$$

Using the interior elliptic estimates, Arzela-Ascoli theorem and a diagonalization argument, one has that there exists a subsequence of $\{(S_{1,m}, S_{2,m}, I_{1,m}, I_{2,m})\}_{m\in\mathbb{N}}$, again denoted by $\{(S_{1,m}, S_{2,m}, I_{1,m}, I_{2,m})\}_{m\in\mathbb{N}}$, satisfying $(S_{1,m}, S_{2,m}, I_{1,m}, I_{2,m}) \to$

(S_1, S_2, I_1, I_2) as $m \to \infty$ in $C^2_{loc}(\mathbb{R}, \mathbb{R}^4)$. It is clear that (S_1, S_2, I_1, I_2) also satisfies (2.3) and

$$S_1(0) = \frac{S_1^0 + S_1^*}{2}. \tag{2.39}$$

Using (2.39), we further obtain $(S_1, S_2, I_1, I_2) \not\equiv (S_1^0, S_2^0, 0, 0)$. Since S_i and I_i are nonnegative, we have $S_i(\xi) > 0$ and $I_i(\xi) > 0$ on $\xi \in \mathbb{R}$. Using these observation, we know that (S_1, S_2, I_1, I_2) satisfies (2.20) (2.26) and (2.27). Similar to the argument of Theorem 2.12, we have

$$S_i(+\infty) = S^*, \quad I_i(+\infty) = I_i^*, \quad i = 1, 2.$$

In order to complete the proof of Theorem 2.14, we need to prove

$$S_i(-\infty) = S_i^0, \quad I_i(-\infty) = 0, \quad i = 1, 2. \tag{2.40}$$

Theorem 2.12 implies that $-V'(\xi) \geqslant 0$. Then we can obtain either

$$\lim_{\xi \to -\infty} V(\xi) = L < +\infty \tag{2.41}$$

or

$$\lim_{\xi \to -\infty} V(\xi) = +\infty. \tag{2.42}$$

If (2.41) holds, by the argument similar to that in Theorem 2.12, we can obtain

$$S_i(-\infty) = S_i^*, \quad I_i(-\infty) = I_i^*, \quad i = 1, 2.$$

Because of $-V'(\xi) \geqslant 0$, one has $V(\xi) \equiv 0$ in $\mathbb{R}$. As a consequence, Theorem 2.12 implies that

$$S_i(\xi) \equiv S^*, \quad I_i(\xi) \equiv I_i^*, \quad \xi \in \mathbb{R}, \quad i = 1, 2$$

which contradicts with (2.39). Therefore, it is impossible that the inequality (2.41) holds. Thus, there must be

$$\lim_{\xi \to -\infty} V(\xi) = +\infty. \tag{2.43}$$

We firstly show that

$$\liminf_{\xi \to -\infty} I_1(\xi) = 0.$$

Otherwise, one has that $\liminf_{\xi \to -\infty} I_1(\xi) > 0$ which, combining $\lim_{\xi \to +\infty} I_1(\xi) = I_1^*$ and $I_1(\xi) > 0$ for any $\xi \in \mathbb{R}$, yields that there exists $\delta > 0$ such that $I_1(\xi) > \delta$ for $\xi \in \mathbb{R}$. Since $I_2(\xi)$ can be expressed as

$$I_2(\xi) = \int_0^{+\infty} \frac{e^{-r_2 t}}{\sqrt{4\pi D_2 t}} \int_{-\infty}^{+\infty} n_i(\xi - y - ct) e^{-\frac{y^2}{4D_2 t}} dy dt,$$

then there exists $\bar{\delta} > 0$ such that

$$I_2(\xi) > \bar{\delta}, \quad \xi \in \mathbb{R}.$$

As a consequence, we have

$$\limsup_{\xi \to -\infty} \omega(\xi) < +\infty, \tag{2.44}$$

which leads to a contradiction with (2.43), where $\omega(\xi)$ is defined by (2.38). Thus, one has $\liminf_{\xi \to -\infty} I_1(\xi) = 0$.

Secondly, we prove that

$$\lim_{\xi \to -\infty} I_1(\xi) = 0.$$

If

$$\limsup_{\xi \to -\infty} I_1(\xi) = \delta > 0$$

for some $\delta > 0$, then there exists a sequence $\{\xi_k\}$ such that $\xi_k \to -\infty$ as $k \to +\infty$ and

$$\lim_{k \to +\infty} I_1(\xi_k) = \delta.$$

Let $N_0 \in \mathbb{N}$ with $c^*\tau \in [N_0, N_0 + 1)$. Due to (2.26), one has

$$\begin{aligned}
\min_{i=1,2} \min_{\xi \in (\xi_k - N - 1, \xi_k - N + 1)} I_i(\xi) &\geqslant \frac{1}{M_1} \max_{i=1,2} \max_{\xi \in (\xi_k - N - 1, \xi_k - N + 1)} I_i(\xi) \\
&\geqslant \frac{1}{M_1} \min_{i=1,2} \min_{\xi \in (\xi_k - N, \xi_k - N + 2)} I_i(\xi) \\
&\geqslant \frac{1}{M_1^2} \max_{i=1,2} \max_{\xi \in (\xi_k - N, \xi_k - N + 2)} I_i(\xi) \\
&\geqslant \cdots \\
&\geqslant \frac{1}{M_1^N} \max_{i=1,2} \max_{\xi \in (\xi_k - 1, \xi_k + 1)} I_i(\xi) \\
&\geqslant \frac{\delta}{2M_1^N}
\end{aligned}$$

for any $N \in \{1, \cdots, N_0\}$ and $k > K$, where $K \in \mathbb{N}$ satisfies $I_1(\xi_k) > \frac{\delta}{2}$ for $k > K$. Consequently, we have

$$\min_{i=1,2} \min_{y \in [0,\, c^*\tau]} I_i(\xi_k - y) \geqslant \frac{\delta}{2M_1^{N_0}}, \quad \forall k > K.$$

It follows that

$$\limsup_{k \to \infty} V(\xi_k) < \infty,$$

which leads to a contradiction with (2.43). Thus, we have $\lim_{\xi \to -\infty} I_1(\xi) = 0$. By a similar argument, we have $\lim_{\xi \to -\infty} I_2(\xi) = 0$.

Finally, we show that

$$\lim_{\xi\to-\infty} S_1(\xi) = S_1^0 = \frac{\lambda_1}{\delta_1}.$$

We divide the proof into two steps.

Step 1. We prove that $\lim_{\xi\to-\infty} S_1(\xi)$ exists. On the contrary, we assume that $\lim_{\xi\to-\infty} S_1(\xi)$ does not exist. Since $S_1(\xi)$ satisfies (2.20), then we have

$$\liminf_{\xi\to-\infty} S_1(\xi) < \limsup_{\xi\to-\infty} S_1(\xi) \leqslant \frac{\lambda_1}{\delta_1}.$$

Let $\{\xi_l\}$ such that $\xi_l \to -\infty$ as $l \to +\infty$ and

$$\begin{aligned} &\lim_{l\to+\infty} S_1(\xi_l) = \liminf_{\xi\to-\infty} S_1(\xi) < \frac{\lambda_1}{\delta_1}, \\ &\frac{d}{d\xi} S_1(\xi_l) = 0, \quad \frac{d^2}{d\xi^2} S_1(\xi_l) \geqslant 0. \end{aligned} \tag{2.45}$$

Since

$$\lim_{l\to+\infty} I_i(\xi_l) = 0, \quad i = 1, 2,$$

then the S_1-th equation with $c = c^*$ implies that

$$\lim_{l\to+\infty} S_1(\xi_l) \geqslant \frac{\lambda_1}{\delta_1},$$

which leads to a contradiction with the inequality (2.45).

Step 2. We prove that $\lim_{\xi\to-\infty} S_1(\xi) = \frac{\lambda_1}{\delta_1}$. Let $\lim_{\xi\to-\infty} S_1(\xi) = k_1$. In view of

$$-d_1 S_1''(\xi) + cS_1'(\xi) + \delta_1 S_1(\xi) = \lambda_1 - \beta_{11} S_1(\xi) I_1(\xi) - \beta_{12} S_1(\xi) I_2(\xi),$$

one has that

$$\begin{aligned} S_1(\xi) =& \frac{1}{\rho} \int_{-\infty}^{\xi} e^{\Lambda_1(\xi-x)} \Big(\lambda_1 - \beta_{11} S_1(x) I_1(x) - \beta_{12} S_1(x) I_2(x)\Big) dx \\ &+ \frac{1}{\rho} \int_{\xi}^{+\infty} e^{\Lambda_2(\xi-x)} \Big(\lambda_1 - \beta_{11} S_1(x) I_1(x) - \beta_{12} S_1(x) I_2(x)\Big) dx \\ =& \frac{1}{\rho} \int_{0}^{+\infty} e^{\Lambda_1 x} \Big(\lambda_1 - \beta_{11} S_1(\xi - x) I_1(\xi - x) - \beta_{12} S_1(\xi - x) I_2(\xi - x)\Big) dx \\ &+ \frac{1}{\rho} \int_{-\infty}^{0} e^{\Lambda_2 x} \Big(\lambda_1 - \beta_{11} S_1(\xi - x) I_1(\xi - x) \\ &\qquad - \beta_{12} S_1(\xi - x) I_2(\xi - x)\Big) dx, \end{aligned}$$

where

$$\rho = d_1(\Lambda_2 - \Lambda_1), \quad \Lambda_1 = \frac{c - \sqrt{c^2 + 4d_1\delta_1}}{2d_1}, \quad \Lambda_2 = \frac{c + \sqrt{c^2 + 4d_1\delta_1}}{2d_1}.$$

Let $\xi \to -\infty$, the Lebesgue's dominated convergence theorem implies that

$$k_1 = \frac{\lambda_1}{\rho}\int_0^{+\infty} e^{\Lambda_1 x}dx + \frac{\lambda_1}{\rho}\int_{-\infty}^{0} e^{\Lambda_2 x}dx.$$

By a straightforward computation, we have that

$$k_1 = \frac{\lambda_1}{\rho}\left(\frac{1}{\Lambda_1} - \frac{1}{\Lambda_2}\right) = \frac{\lambda_1}{\delta_1}.$$

Thus, one has

$$\lim_{\xi \to -\infty} S_1(\xi) = \frac{\lambda_1}{\delta_1}.$$

By the same way, we can obtain

$$\lim_{\xi \to -\infty} S_2(\xi) = \frac{\lambda_2}{\delta_2}.$$

This completes the proof. □

3 Nonexistence of traveling waves

In the section, we show the nonexistence of traveling wave solutions of system (2.3) for the following three cases: (I) $R_0 < 1$; (II) $R_0 = 1$; (III) $R_0 > 1$ and $c < c^*$.

3.1 Case I: $R_0 < 1$

Theorem 3.1 *Assume that* $R_0 < 1$. *There exists no nonnegative bounded solution* $(S_1(\xi), S_2(\xi), I_1(\xi), I_2(\xi))$ *of* (2.3) *satisfying* (2.4).

Proof We prove Theorem 3.1 by contradiction. Assume that there exists a solution $(S_1(\xi), S_2(\xi), I_1(\xi), I_2(\xi))$ satisfying (2.3) and (2.4). Let $I_{i,sup} := \sup_{\xi\in\mathbb{R}} I_i(\xi)$, $i = 1, 2$. By (2.3), we have

$$\begin{cases} D_1 I_1''(\xi) - cI_1'(\xi) + \epsilon_1\left(\beta_{11}S_1^0 I_{1,sup} + \beta_{12}S_1^0 I_{2,sup}\right) - r_1 I_1(\xi) \geqslant 0, \\ D_2 I_2''(\xi) - cI_2'(\xi) + \epsilon_2\left(\beta_{21}S_2^0 I_{1,sup} + \beta_{22}S_2^0 I_{2,sup}\right) - r_2 I_2(\xi) \geqslant 0, \end{cases} \quad \forall \xi \in \mathbb{R}.$$

The comparison principle implies that

$$\begin{pmatrix} I_1(\xi) \\ I_2(\xi) \end{pmatrix} \leqslant \mathcal{L}\begin{pmatrix} I_{1,sup} \\ I_{2,sup} \end{pmatrix}, \ \forall \xi \in \mathbb{R},$$

which in turn implies that

$$\begin{pmatrix} I_{1,sup} \\ I_{2,sup} \end{pmatrix} \leqslant \mathcal{L} \begin{pmatrix} I_{1,sup} \\ I_{2,sup} \end{pmatrix}.$$

See Sect. 2 for the definitions of $\mathcal{L}$ and $r(\mathcal{L})$. It is easy to see that the matrix $\mathcal{L}$ is nonnegative and irreducible, and $r(\mathcal{L}) = R_0$. The Perron-Frobenius theorem yields that there exists a vector $\mathcal{Q} = (q_1, q_2)^T \in \mathbb{R}^2$ with $q_1 > 0$ and $q_2 > 0$ such that $\mathcal{L}\mathcal{Q} = R_0\mathcal{Q}$. Note that there exists a large constant $\varrho > 0$ satisfying

$$\begin{pmatrix} I_{1,sup} \\ I_{2,sup} \end{pmatrix} \leq \varrho\mathcal{Q}.$$

Consequently, we have

$$\begin{pmatrix} I_{1,sup} \\ I_{2,sup} \end{pmatrix} \leqslant \mathcal{L}^n \begin{pmatrix} I_{1,sup} \\ I_{2,sup} \end{pmatrix} \leqslant \varrho\mathcal{L}^n\mathcal{Q} = \varrho R_0^n\mathcal{Q} \to 0$$

as $n \to \infty$, which contradicts to the fact that $I_1(\xi) > 0$ and $I_2(\xi) > 0$ for any $\xi \in \mathbb{R}$. This completes the proof. □

3.2 Case II: $R_0 = 1$

Theorem 3.2 *Assume that* $R_0 = 1$. *There exists no nonnegative bounded solution* $(S_1(\xi), S_2(\xi), I_1(\xi), I_2(\xi))$ *of* (2.3) *satisfying* (2.4).

Proof Since $R_0 = 1$, there exists $\mathcal{K} = (k_1, k_2)^T \in \mathbb{R}^2$ with $k_1 > 0$ and $k_2 > 0$ such that $\mathcal{L}\mathcal{K} = \mathcal{K}$. It follows that

$$r_1 = \beta_{11}S_1^0\epsilon_1 + \frac{\beta_{12}S_1^0\epsilon_1 k_2 r_1}{k_1 r_2}, \quad r_2 = \beta_{22}S_2^0\epsilon_2 + \frac{\beta_{21}S_2^0\epsilon_2 k_1 r_2}{k_2 r_1}. \tag{3.1}$$

Take a sequence $\{\xi_m\}_{m\in\mathbb{N}} \subset \mathbb{R}$ such that

$$\lim_{m\to+\infty} I_1(\xi_m) = \tilde{B} := \sup_{\xi\in\mathbb{R}} I_1(\xi). \tag{3.2}$$

Next we shall show that $\tilde{B} = 0$. To do so, let us argue by contradiction. Assume $\tilde{B} > 0$. Consider the function sequence $(S_{1,m}(\xi), S_{2,m}(\xi), I_{1,m}(\xi), I_{2,m}(\xi)) = (S_1(\xi + \xi_m), S_2(\xi+\xi_m), I_1(\xi+\xi_m), I_2(\xi+\xi_m))$ for $m \in \mathbb{N}$. Using the elliptic estimates, we can assume, possibly along a subsequence, that $(S_{1,m}(\xi), S_{2,m}(\xi), I_{1,m}(\xi), I_{2,m}(\xi)) \to (\hat{S}_1, \hat{S}_2, \hat{I}_1, \hat{I}_2)$ as $m \to +\infty$ in $C^2_{loc}(\mathbb{R})$ and $(\hat{S}_1, \hat{S}_2, \hat{I}_1, \hat{I}_2)$ satisfies

$$
\begin{aligned}
&d_1\hat{S}_1''(\xi) - c\hat{S}_1'(\xi) + \lambda_1 - \delta_1\hat{S}_1(\xi) - \beta_{11}\hat{S}_1(\xi)\hat{I}_1(\xi) - \beta_{12}\hat{S}_1(\xi)\hat{I}_2(\xi) = 0,\\
&d_{S2}\hat{S}_2''(\xi) - c\hat{S}_2'(\xi) + \lambda_2 - \delta_2\hat{S}_2(\xi) - \beta_{21}\hat{S}_2(\xi)\hat{I}_1(\xi) - \beta_{22}\hat{S}_2(\xi)\hat{I}_2(\xi) = 0,\\
&D_1\hat{I}_1''(\xi) - c\hat{I}_1'(\xi) - r_1\hat{I}_1(\xi) + \epsilon_1\hat{S}_1(\xi - c\tau)\left(\beta_{11}\hat{I}_1(\xi - c\tau) + \beta_{12}\hat{I}_2(\xi - c\tau)\right) = 0,\\
&D_2\hat{I}_2''(\xi) - c\hat{I}_2'(\xi) - r_2\hat{I}_2(\xi) + \epsilon_2\hat{S}_2(\xi - c\tau)\left(\beta_{21}\hat{I}_1(\xi - c\tau) + \beta_{22}\hat{I}_2(\xi - c\tau)\right) = 0,\\
&\hat{I}_1(0) = \tilde{B}, \ \ \hat{I}_1(\xi) \leqslant \tilde{B},\\
&0 \leqslant \hat{S}_i(\xi) \leqslant \frac{\lambda_i}{\delta_i} := S_i^0, \ \ i = 1, 2
\end{aligned}
$$

for any $\xi \in \mathbb{R}$. Using the comparison principle and the second equality of (3.1), we have

$$\epsilon_2\beta_{21}S_2^0\tilde{B} \geqslant \left(r_2 - \epsilon_2\beta_{22}S_2^0\right) I_{2,sup} = \frac{\epsilon_2\beta_{21}S_2^0 k_1 r_2}{k_2 r_1} I_{2,sup},$$

which implies that

$$I_{2,sup} \leqslant \frac{k_2 r_1}{k_1 r_2}\tilde{B},$$

Thus, by the first equality of (3.1), plugging the above inequality into the $\hat{I}_1$-th equation yields that

$$
\begin{aligned}
0 &= D_1\hat{I}_1''(0) - r_1\hat{I}_1(0) + \epsilon_1\left(\beta_{11}\hat{S}_1(-c\tau)\hat{I}_1(-c\tau) + \beta_{12}\hat{S}_1(-c\tau)\hat{I}_2(-c\tau)\right)\\
&\leqslant D_1\hat{I}_1''(0) + \epsilon_1\hat{S}_1(-c\tau)\left(\beta_{11}\tilde{B} + \beta_{12}\frac{k_2 r_1}{k_1 r_2}\tilde{B}\right) - r_1\tilde{B}\\
&= D_1\hat{I}_1''(0) + \epsilon_1\hat{S}_1(-c\tau)\frac{r_1}{S_1^0\varepsilon_1}\tilde{B} - r_1\tilde{B}.
\end{aligned}
$$

Due to $\hat{I}_1''(0) \leqslant 0$, one has $\epsilon_1\hat{S}_1(-c\tau)\frac{r_1}{S_1^0\varepsilon_1}\tilde{B} - r_1\tilde{B} \geqslant 0$, which implies that

$$\hat{S}_1(-c\tau) - S_1^0 \geqslant 0.$$

When $\hat{S}_1(-c\tau) - S_1^0 > 0$, it leads to a contradiction with $\hat{S}_1(\xi) \leqslant S_1^0$ for all $\xi \in \mathbb{R}$. Thus, one has $\hat{S}_1(-c\tau) - S_1^0 = 0$, which leads to $\hat{S}_1(\xi) \equiv S_1^0$ for each $\xi \in \mathbb{R}$. Plugging $\hat{S}_1(\xi) \equiv S_1^0$ into the S_1-th equation, we get $\hat{I}_1(\xi) \equiv 0$, $\forall\xi \in \mathbb{R}$. As a consequence, one has $\tilde{B} = 0$, which leads to a contradiction with $\tilde{B} > 0$. The proof is completed. □

3.3 Case III: $R_0 > 1$ and $c \in (0, c^*)$

Let $R_{i,0} := \frac{\epsilon_1\beta_{1i}S_1^0}{r_1} + \frac{\epsilon_2\beta_{2i}S_2^0}{r_2}$, $i = 1, 2$. The characteristic equation of the matrix $\mathcal{L}$ is given by

$$f(\lambda) := \left(\lambda - \frac{\epsilon_1\beta_{11}S_1^0}{r_1}\right)\left(\lambda - \frac{\epsilon_2\beta_{22}S_2^0}{r_2}\right) - \frac{\epsilon_1\epsilon_2\beta_{12}\beta_{21}S_1^0 S_2^0}{r_1 r_2} = 0.$$

From Zhao et al. (2017, Propositions 4.2, 4.3 and 4.4), we can get some relationships between R_0 and $R_{i,0}(i = 1, 2)$ as follows:

Proposition 3.3 *If* $R_0 > 1$*, then at least one of* $R_{1,0}$ *and* $R_{2,0}$ *is greater than 1. Moreover, we have:* (i) $R_0 = 1$ *if* $R_{1,0} = 1$ *and* $R_{2,0} = 1$*;* (ii) $R_0 < 1$ *if* $R_{1,0} < 1$ *and* $R_{2,0} = 1$*;* (iii) $R_0 < 1$ *if* $R_{1,0} = 1$ *and* $R_{2,0} < 1$.

Proposition 3.4 *If* $R_{1,0} \geq 1$*,* $R_{2,0} \geq 1$ *and* $R_{1,0}R_{2,0} > 1$*, then* $R_0 > 1$.

Proposition 3.5 *Assume* $R_0 > 1$*. If* $R_{1,0} \leq 1$ *or* $R_{2,0} \leq 1$*, then* $f(1) < 0$.

Lemma 3.6 *Assume* $R_0 > 1$*. For any* $c > 0$*, if system* (1.2) *admits a positive traveling wave solution* $(S_1(x + ct), S_2(x + ct), I_1(x + ct), I_2(x + ct))$ *satisfying* (2.3) *and* (2.4)*, then there exist some constants* $\mathcal{J} > 0$ *and* $M > 0$ *large enough such that*

$$\int_{-\infty}^{x} I_i(\xi)d\xi \leqslant \mathcal{J}, \quad x < -2M + c\tau, \quad i = 1, 2.$$

Proof Fix $c > 0$. Assume that $(S_1(x + ct), S_2(x + ct), I_1(x + ct), I_2(x + ct))$ is a nonnegative traveling wave solution of (1.2) satisfying (2.3) and (2.4). Since $S_i(-\infty) = S_i^0$, there exists $M > 0$ sufficiently large such that

$$S_i(\xi) > S_i^0(1 - \nu), \quad \forall\, \xi \in (-\infty, -2M + c\tau), \quad i = 1, 2,$$

where $\nu \in (0, 1)$ is a small constant which will be determined later.

For $\xi < -2M + c\tau$, we have

$$\begin{aligned}
&\epsilon_1\Big(\beta_{11}I_1(\xi - c\tau)S_1(\xi - c\tau) + \beta_{12}S_1(\xi - c\tau)I_2(\xi - c\tau)\Big) - r_1I_1(\xi)\\
&\quad\geqslant \epsilon_1\beta_{11}S_1^0(1 - \nu)[I_1(\xi - c\tau) - I_1(\xi)]\\
&\qquad+\epsilon_1\beta_{12}S_1^0(1 - \nu)[I_2(\xi - c\tau) - I_2(\xi)]\\
&\qquad+\Big(\epsilon_1\beta_{11}S_1^0(1 - \nu) - r_1\Big)I_1(\xi) + \epsilon_1\beta_{12}S_1^0(1 - \nu)I_2(\xi).
\end{aligned} \tag{3.3}$$

Similarly, for $\xi < -2M + c\tau$, one has

$$\begin{aligned}
&\epsilon_2\Big(\beta_{21}I_1(\xi - c\tau)S_2(\xi - c\tau) + \beta_{22}S_2(\xi - c\tau)I_2(\xi - c\tau)\Big) - r_2I_2(\xi)\\
&\quad\geqslant \epsilon_2\beta_{21}S_2^0(1 - \nu)[I_1(\xi - c\tau) - I_1(\xi)]\\
&\qquad+\epsilon_2\beta_{22}S_2^0(1 - \nu)[I_2(\xi - c\tau) - I_2(\xi)]\\
&\qquad+\Big(\epsilon_2\beta_{22}S_1^0(1 - \nu) - r_2\Big)I_2(\xi) + \epsilon_2\beta_{21}S_2^0(1 - \nu)I_1(\xi).
\end{aligned} \tag{3.4}$$

For $y < x < -2M + c\tau$, let $\tilde{J}_i(x, y) = \int_y^x I_i(\xi)d\xi$. Integrating both sides of (3.3) from y to $x(y < x < -2M + c\tau)$ yields

$$\Big(\epsilon_1\beta_{11}S_1^0(1 - \nu) - r_1\Big)\tilde{J}_1(x, y) + \epsilon_1\beta_{12}S_1^0(1 - \nu)\tilde{J}_2(x, y)$$

$$
\begin{aligned}
&\leqslant \int_y^x (\epsilon_1\beta_{11}I_1(\xi-c\tau)S_1(\xi-c\tau)+\epsilon_1\beta_{12}I_2(\xi-c\tau)S_1(\xi-c\tau)-r_1I_1(\xi))\,d\xi\\
&\quad -\epsilon_1 S_1^0(1-\nu)\left(\beta_{11}\int_y^x (I_1(\xi-c\tau)-I_1(\xi))\,d\xi\right.\\
&\qquad\qquad \left.+\beta_{12}\int_y^x (I_2(\xi-c\tau)-I_2(\xi))\,d\xi\right).
\end{aligned}
\tag{3.5}
$$

Similarly, for any $y < x < -2M + c\tau$, one has

$$
\begin{aligned}
&\left(\epsilon_2\beta_{22}S_2^0(1-\nu)-r_2\right)\tilde{J}_2(x,y)+\epsilon_2\beta_{21}S_2^0(1-\nu)\tilde{J}_1(x,y)\\
&\leqslant \int_y^x (\epsilon_2\beta_{21}I_1(\xi-c\tau)S_2(\xi-c\tau)+\epsilon_2\beta_{22}I_2(\xi-c\tau)S_2(\xi-c\tau)-r_2I_2(\xi))\,d\xi\\
&\quad -\epsilon_2 S_2^0(1-\nu)\left(\beta_{22}\int_y^x (I_2(\xi-c\tau)-I_2(\xi))\,d\xi\right.\\
&\qquad\qquad \left.+\beta_{21}\int_y^x (I_1(\xi-c\tau)-I_1(\xi))\,d\xi\right).
\end{aligned}
\tag{3.6}
$$

In the following, we show that there exist constants $\mathcal{J} > 0$ such that

$$
\int_{-\infty}^x I_i(\xi)d\xi \leqslant \mathcal{J}, \quad x < -2M + c\tau, \quad i = 1, 2. \tag{3.7}
$$

In order to prove (3.7), we take into account the following five cases:

Case 1: $\epsilon_1\beta_{11}S_1^0 - r_1 > 0$.

In this case we take $\nu \in (0, 1)$ small enough so that $\epsilon_1\beta_{11}S_1^0(1-\nu) - r_1 > 0$. Due to Lemma 2.10 and (2.27), we have

$$
\|I_i(x)\|_{C^2(\mathbb{R})} \leqslant \overline{\mathcal{P}}, \quad \lim_{x\to-\infty} I_i'(x) = 0, \quad x \in \mathbb{R}, \quad i = 1, 2, \tag{3.8}
$$

where $\overline{\mathcal{P}}$ is a positive constant. It follows that

$$
\begin{aligned}
&\int_{-\infty}^x \{\epsilon_i\beta_{i1}I_1(\xi-c\tau)S_i(\xi-c\tau)+\epsilon_i\beta_{i2}I_2(\xi-c\tau)S_i(\xi-c\tau)-r_iI_i(\xi)\}\,d\xi\\
&= \lim_{y\to-\infty}\int_y^x \{\epsilon_i\beta_{i1}I_1(\xi-c\tau)S_i(\xi-c\tau)\\
&\qquad +\epsilon_i\beta_{i2}I_2(\xi-c\tau)S_i(\xi-c\tau)-r_iI_i(\xi)\}\,d\xi\\
&= -D_iI_i'(x)+cI_i(x), \quad \forall x \in (-\infty,+\infty), \quad i = 1, 2.
\end{aligned}
$$

Furthermore, we have

$$
\int_{-\infty}^x (I_i(\xi-c\tau)-I_i(\xi))\,dyd\xi = \lim_{z\to-\infty}\int_z^x (I_i(\xi-c\tau)-I_i(\xi))\,d\xi
$$

$$
\begin{aligned}
&= \lim_{z\to-\infty} -c\tau \int_z^x \int_0^1 I_i'(\xi-\theta(c\tau))d\theta d\xi \\
&= -c\tau \int_0^1 I_i(x-\theta c\tau)d\theta, \quad x<-2M+c\tau
\end{aligned}
$$

for $i=1,2$. Letting $y\to-\infty$ in (3.5) yields

$$
\begin{aligned}
&\left(\epsilon_1\beta_{11}S_1^0(1-\nu)-r_1\right)\int_{-\infty}^x I_1(\xi)d\xi+\epsilon_1\beta_{12}S_1^0(1-\nu)\int_{-\infty}^x I_2(\xi)d\xi \\
&= -D_iI_i'(x)+cI_i(x)+\epsilon_1c\tau S_1^0(1-\nu) \\
&\quad\left(\beta_{11}\int_0^1 I_1(x-\theta c\tau)d\theta+\beta_{12}\int_0^1 I_2(x-\theta c\tau)d\theta\right) \\
&\leqslant -D_i\overline{\mathcal{P}}+c\overline{\mathcal{P}}+\epsilon_1c\tau S_1^0(1-\nu)\left(\beta_{11}+\beta_{12}\right)\overline{\mathcal{P}},
\end{aligned}
$$

which implies that the inequality (3.7) holds for any $x<-2M+c\tau$.

Case 2: $\epsilon_2\beta_{22}S_2^0-r_2>0$.
This case is similar to case 1 and we omit the details.

Case 3: $\epsilon_1\beta_{11}S_1^0-r_1\le 0$, $\epsilon_2\beta_{22}S_2^0-r_2\le 0$, $\epsilon_1\beta_{11}S_1^0+\epsilon_2\beta_{21}S_2^0-r_1>0$ and $\epsilon_1\beta_{12}S_1^0+\epsilon_2\beta_{22}S_2^0-r_2>0$.

In this case we take $\nu\in(0,1)$ satisfying $\epsilon_1\beta_{11}S_1^0(1-\nu)+\epsilon_2\beta_{21}S_2^0(1-\nu)-r_1>0$ and $\epsilon_1\beta_{12}S_1^0(1-\nu)+\epsilon_2\beta_{22}S_2^0(1-\nu)-r_2>0$. By adding both sides of inequalities (3.5) and (3.6) respectively, we have that

$$
\begin{aligned}
&\left[\epsilon_1\beta_{11}S_1^0(1-\nu)+\epsilon_2\beta_{21}S_2^0(1-\nu)-r_1\right]\tilde{J}_1(x,y) \\
&\quad+\left[\epsilon_2\beta_{22}S_2^0(1-\nu)+\epsilon_1\beta_{12}S_1^0(1-\nu)-r_2\right]\tilde{J}_2(x,y) \qquad (3.9)\\
&\leqslant \int_y^x\{\epsilon_1\beta_{11}I_1(\xi-c\tau)S_1(\xi-c\tau)+\epsilon_1\beta_{12}I_2(\xi-c\tau)S_1(\xi-c\tau)-r_1I_1(\xi)\}d\xi \\
&\quad+\int_y^x\{\epsilon_2\beta_{21}I_1(\xi-c\tau)S_2(\xi-c\tau)+\epsilon_2\beta_{22}I_2(\xi-c\tau)S_2(\xi-c\tau)-r_2I_2(\xi)\}d\xi \\
&\quad-\left(\epsilon_1\beta_{12}S_1^0(1-\nu)+\epsilon_2\beta_{22}S_2^0(1-\nu)\right)\int_y^x\left(I_2(\xi-c\tau)-I_2(\xi)\right)d\xi \\
&\quad-\left(\epsilon_2\beta_{21}S_2^0(1-\nu)+\epsilon_1\beta_{11}S_1^0(1-\nu)\right)\int_y^x\left(I_1(\xi-c\tau)-I_1(\xi)\right)d\xi \qquad (3.10)
\end{aligned}
$$

for any $y<x<-2M+c\tau$. Similarly, letting $y\to-\infty$ on both sides of (3.9), we obtain inequality (3.7).

Case 4: $\epsilon_1\beta_{11}S_1^0-r_1\le 0$, $\epsilon_2\beta_{22}S_2^0-r_2\le 0$, $\epsilon_1\beta_{11}S_1^0+\epsilon_2\beta_{21}S_2^0-r_1>0$ and $\epsilon_1\beta_{12}S_1^0+\epsilon_2\beta_{22}S_2^0-r_2\le 0$.

Following Proposition 3.5, in this case there is $f(1) < 0$, that is

$$1 - \left(\frac{\epsilon_1 \beta_{11} S_1^0}{r_1} + \frac{\epsilon_2 \beta_{22} S_2^0}{r_2} \right) + \frac{\epsilon_1 \epsilon_2 \left(\beta_{11} \beta_{22} - \beta_{12} \beta_{21} \right) S_1^0 S_2^0}{r_1 r_2} < 0.$$

We take $\nu \in (0, 1)$ such that $\epsilon_1 \beta_{11} S_1^0 (1 - \nu) + \epsilon_2 \beta_{21} S_2^0 (1 - \nu) - r_1 > 0$ and

$$1 - \left(\frac{\epsilon_1 \beta_{11} S_1^0}{r_1} + \frac{\epsilon_2 \beta_{22} S_2^0}{r_2} \right) (1 - \nu) + \frac{\epsilon_1 \epsilon_2 \left(\beta_{11} \beta_{22} - \beta_{12} \beta_{21} \right) S_1^0 S_2^0}{r_1 r_2} (1 - \nu)^2 < 0. \tag{3.11}$$

Set

$$\mathcal{A} = \begin{pmatrix} \epsilon_1 \beta_{11} S_1^0 (1 - \nu) - r_1 & \epsilon_1 \beta_{12} S_1^0 (1 - \nu) \\ \epsilon_2 \beta_{21} S_2^0 (1 - \nu) & \epsilon_2 \beta_{22} S_2^0 (1 - \nu) - r_2 \end{pmatrix}.$$

It is obvious that inequality (3.11) implies $|\mathcal{A}| < 0$. Note that $\epsilon_1 \beta_{11} S_1^0 (1 - \nu) - r_1 < 0$ and $\epsilon_2 \beta_{22} S_2^0 (1 - \nu) - r_2 < 0$. Multiplying (3.5) and (3.6) by $\epsilon_2 \beta_{22} S_2^0 (1 - \nu) - r_2$ and $-\epsilon_1 \beta_{12} S_1^0 (1 - \nu)$ respectively, and adding them up, we obtain that there exist two constants $\mathcal{S}_i < 0 (i = 1, 2)$ such that

$$\begin{aligned} & - |\mathcal{A}| \tilde{J}_1(x, y) \\ & \quad \leq \left(r_2 - \epsilon_2 \beta_{22} S_2^0 (1 - \nu) \right) \\ & \quad \int_y^x \left\{ \epsilon_1 S_1(\xi - c\tau) \left(\beta_{11} I_1(\xi - c\tau) + \beta_{12} I_2(\xi - c\tau) \right) - r_1 I_1(\xi) \right\} d\xi \\ & \quad + \epsilon_1 \beta_{12} S_1^0 (1 - \nu) \\ & \quad \int_y^x \left\{ \epsilon_2 S_2(\xi - c\tau) \left(\beta_{21} I_1(\xi - c\tau) + \beta_{22} I_2(\xi - c\tau) \right) - r_2 I_2(\xi) \right\} d\xi \\ & \quad + \mathcal{S}_1 \int_y^x \left(I_1(\xi - c\tau) - I_1(\xi) \right) d\xi + \mathcal{S}_2 \int_y^x \left(I_2(\xi - c\tau) - I_2(\xi) \right) d\xi, \end{aligned}$$

where $y < x < -2M + c\tau$. Letting $y \to -\infty$, we have the inequality (3.7) for $I_1(x)$. Similarly, we can show that (3.7) holds for $I_2(x)$.

Case 5: $\epsilon_2 \beta_{22} S_2^0 - r_2 \leq 0, \epsilon_1 \beta_{11} S_1^0 + \epsilon_2 \beta_{21} S_2^0 - r_1 \leq 0$ and $\epsilon_1 \beta_{12} S_1^0 + \epsilon_2 \beta_{22} S_2^0 - r_2 > 0$.

This case can be treated by a similar argument to that for Case 4. We omit the details. The proof is completed. □

In the following, we let $J_i(x) = \int_{-\infty}^x I_i(\xi) d\xi$ for any $x < -2M$.

Lemma 3.7 *Assume $R_0 > 1$. For any $c > 0$, if system (1.2) admits a positive traveling wave solution $(S_1(x + ct), S_2(x + ct), I_1(x + ct), I_2(x + ct))$ satisfying (2.3) and (2.4), then there exists some $\mu_0 > 0$ such that*

$$\sup_{\xi \in \mathbb{R}} \{ I_i(\xi) e^{-\mu_0 \xi} \} < +\infty, \quad \sup_{\xi \in \mathbb{R}} \{ |I_i'(\xi)| e^{-\mu_0 \xi} \} < +\infty,$$

$$\sup_{\xi\in\mathbb{R}}\{|I_i''(\xi)|e^{-\mu_0\xi}\} < +\infty,\ i = 1, 2.$$

Proof Fix $c > 0$. Assume that $(S_1(x + ct), S_2(x + ct), I_1(x + ct), I_2(x + ct))$ is a positive traveling wave solution of (1.2) satisfying (2.3) and (2.4). Let $M > 0$ and $\nu \in (0, 1)$ be defined as in Lemma 3.6. Then one has

$$S_i(\xi) > S_i^0(1 - \nu), \quad \forall\, \xi \in (-\infty, -2M).$$

For $\xi < -2M$, one has

$$\begin{aligned} cI_1'(\xi) &= D_1 I_1''(\xi) + \epsilon_1\beta_{11} I_1(\xi - c\tau) S_1(\xi - c\tau) \\ &\quad + \epsilon_1\beta_{12} I_1(\xi - c\tau) S_1(\xi - c\tau) - r_1 I_1(\xi) \\ &\geq D_1 I_1''(\xi) + \epsilon_1\beta_{11} S_1^0(1 - \nu) I_1(\xi - c\tau) \\ &\quad + \epsilon_1\beta_{12} S_1^0(1 - \nu) I_2(\xi - c\tau) - r_1 I_1(\xi) \\ &= D_1 I_1''(\xi) + \epsilon_1\beta_{11} S_1^0(1 - \nu)\Big(I_1(\xi - c\tau) - I_1(\xi)\Big) \\ &\quad + \epsilon_1\beta_{12} S_1^0(1 - \nu)\Big(I_2(\xi - c\tau) - I_2(\xi)\Big) \\ &\quad + \Big(\epsilon_1\beta_{11} S_1^0(1 - \nu) - r_1\Big) I_1(\xi) + \epsilon_1\beta_{12} S_1^0(1 - \nu) I_2(\xi). \end{aligned} \tag{3.12}$$

Due to (3.8), integrating both sides of inequality (3.12) from $-\infty$ to x with $x < -2M$ yields

$$\begin{aligned} &\Big(\epsilon_1\beta_{11} S_1^0(1 - \nu) - r_1\Big) J_1(x) + \epsilon_1\beta_{12} S_1^0(1 - \nu) J_2(x) \\ &\quad \leqslant -D_1 I_1'(x) + cI_1(x) - \epsilon_1\beta_{11} S_1^0(1 - \nu)\left(J_1(x - c\tau) - J_1(x)\right) \\ &\quad\quad - \epsilon_1\beta_{12} S_1^0(1 - \nu)\left(J_2(x - c\tau) - J_2(x)\right). \end{aligned} \tag{3.13}$$

In addition, integrating both sides of inequality (3.13) from y to ξ $(y < \xi < -2M)$ leads to

$$\begin{aligned} &\Big(\epsilon_1\beta_{11} S_1^0(1 - \nu) - r_1\Big) \int_y^\xi J_1(\eta)d\eta + \epsilon_1\beta_{12} S_1^0(1 - \nu) \int_y^\xi J_2(\eta)d\eta \\ &\quad + D_1 \int_y^\xi I_1'(\eta)d\eta \\ &\leqslant cJ_1(\xi) - \epsilon_1\beta_{11} S_1^0(1 - \nu) \int_y^\xi \left(J_1(\eta - c\tau) - J_1(\eta)\right) d\eta \\ &\quad - \epsilon_1\beta_{12} S_1^0(1 - \nu) \int_y^\xi \left(J_2(\eta - c\tau) - J_2(\eta)\right) d\eta. \end{aligned} \tag{3.14}$$

Similarly, for $\xi < -2M$, we have

$$\begin{aligned}
&\left(\epsilon_2\beta_{22}S_2^0(1-\nu)-r_2\right)\int_y^\xi J_2(\eta)d\eta+\epsilon_2\beta_{21}S_2^0(1-\nu)\int_y^\xi J_1(\eta)d\eta\\
&\quad+D_2\int_y^\xi I_2'(\eta)d\eta\\
&\quad\leqslant cJ_2(\xi)-\epsilon_2\beta_{22}S_2^0(1-\nu)\int_y^\xi \left(J_2(\eta-c\tau)-J_2(\eta)\right)d\eta\\
&\quad-\epsilon_2\beta_{21}S_2^0(1-\nu)\int_y^\xi \left(J_1(\eta-c\tau)-J_1(\eta)\right)d\eta.
\end{aligned}$$

Next, we show that there exist positive constants a_1, a_2, b_1, b_2 such that

$$a_1\int_{-\infty}^\xi J_1(\eta)d\eta+a_2\int_{-\infty}^\xi J_2(\eta)d\eta\le b_1J_1(\xi)+b_2J_2(\xi),\quad \forall\,\xi<-2M. \tag{3.15}$$

Because $R_0>1$, it follows from Proposition 3.3 that $\epsilon_1\beta_{11}S_1^0+\epsilon_2\beta_{21}S_2^0-r_1>0$ or $\epsilon_2\beta_{22}S_2^0+\epsilon_1\beta_{12}S_1^0-r_2>0$. Therefore, we prove (3.15) by considering the five cases as in Lemma 3.6.

In the following, we only prove **Case 1**. **Cases 2–5** can be treated similarly. Assume $\epsilon_1\beta_{11}S_1^0-r_1>0$. In this case $\nu\in(0,1)$ is taken to satisfy $\epsilon_1\beta_{11}S_1^0(1-\nu)^2-r_1>0$. In view of

$$\begin{aligned}
\int_{-\infty}^\xi \left(J_i(\eta-c\tau)-J_i(\eta)\right)dyd\eta&=\lim_{z\to-\infty}\int_z^\xi \left(J_i(\eta-c\tau)-J_i(\eta)\right)d\eta\\
&=\lim_{z\to-\infty}-\int_z^\xi\int_0^1 I_i(\eta-\theta c\tau)d\theta d\eta=-c\tau\int_0^1 J_i(\xi-\theta c\tau)d\theta
\end{aligned}$$

for $i=1,2$, letting $\xi\to-\infty$ in (3.14) yields

$$\begin{aligned}
&\left(\epsilon_1\beta_{11}S_1^0(1-\nu)-r_1\right)\int_{-\infty}^\xi J_1(\eta)d\eta+\epsilon_1\beta_{12}S_1^0(1-\nu)\int_{-\infty}^\xi J_2(\eta)d\eta+D_1I_1(\xi)\\
&\quad<cJ_1(\xi)+\epsilon_1\beta_{11}S_1^0c\tau\int_0^1 J_1(\xi-\theta c\tau)d\theta+\epsilon_1\beta_{12}S_1^0c\tau\int_0^1 J_2(\xi-\theta c\tau)d\theta.
\end{aligned}\tag{3.16}$$

Since $J_i(\xi-\theta c\tau)$ is non-increasing on $\theta\in[0,1]$, the above inequality (3.16) reduces to

$$\begin{aligned}
&\left(\epsilon_1\beta_{11}S_1^0(1-\nu)-r_1\right)\int_{-\infty}^\xi J_1(\eta)d\eta+\epsilon_1\beta_{12}S_1^0(1-\nu)\int_{-\infty}^\xi J_2(\eta)d\eta+D_1I_1(\xi)\\
&\quad<cJ_1(\xi)+c\tau\epsilon_1\beta_{11}S_1^0J_1(\xi)+c\tau\epsilon_1\beta_{12}S_1^0J_2(\xi),\ \forall\xi<-2M,
\end{aligned}$$

which implies that (3.15) holds.

Now we are in the position to prove the main result of the lemma. Let $J(\xi) = J_1(\xi) + J_2(\xi)$. Then inequality (3.15) implies that there exist constants $a > 0$ and $b > 0$ such that

$$a \int_{-\infty}^{\xi} J(\eta) d\eta \leq b J(\xi), \quad \forall \, \xi < -2M.$$

Consequently, we have

$$a \int_{-\infty}^{0} J(\xi + \eta) d\eta \leq b J(\xi), \quad \forall \, \xi < -2M.$$

Since $J(\cdot)$ is increasing, we have $a\eta J(\xi - \eta) \leq bJ(\xi)$ for any $\xi < -2M$ and any $\eta > 0$. Therefore, there exist $\eta_0 > 0$ large enough and $\omega_0 \in (0, 1)$ such that

$$J(\xi - \eta_0) \leqslant \omega_0 J(\xi), \quad \forall \, \xi < -2M.$$

Let $w(\xi) = J(\xi)e^{-\mu_0 \xi}$ with $\mu_0 = \frac{1}{\eta_0} \ln \frac{1}{\omega_0} > 0$. Then we have

$$w(\xi - \eta_0) = J(\xi - \eta_0)e^{-\mu_0(\xi - \eta_0)} \leq \omega_0 J(\xi) e^{-\mu_0(\xi-\eta_0)} = w(\xi), \qquad \xi < -2M.$$

Since $w(\xi) \to 0$ as $\xi \to +\infty$, then there exists a constant $\kappa_0 > 0$ satisfying $w(\xi) \leq \kappa_0$ for $\xi \in \mathbb{R}$, which implies that $J(\xi) \leq \kappa_0 e^{\mu_0 \xi}$ for any $\xi \in \mathbb{R}$. Consequently, there exists $q_0 > 0$ satisfying $\int_{-\infty}^{\xi} J_i(\eta) d\eta \leq q_0 e^{\mu_0 \xi}$ for any $\xi < 0$, $i = 1, 2$. According to (3.16), we get that there exists $p_0 > 0$ such that

$$I_1(\xi) \leq p_0 e^{\mu_0 \xi}, \quad \forall \, \xi \in \mathbb{R}.$$

By a similar way, we have that $I_2(\xi) \leq p_0 e^{\mu_0 \xi}$, $\forall \, \xi \in \mathbb{R}$. Finally, using (3.12) and (3.13), we can obtain

$$\sup_{\xi \in \mathbb{R}} \{I_1(\xi) e^{-\mu_0 \xi}\} < +\infty, \qquad \sup_{\xi \in \mathbb{R}} \{|I_1'(\xi)| e^{-\mu_0 \xi}\} < +\infty, \qquad \sup_{\xi \in \mathbb{R}} \{|I_1''(\xi)| e^{-\mu_0 \xi}\} < +\infty.$$

Similarly, we have

$$\sup_{\xi \in \mathbb{R}} \{I_2(\xi) e^{-\mu_0 \xi}\} < +\infty, \qquad \sup_{\xi \in \mathbb{R}} \{|I_2'(\xi)| e^{-\mu_0 \xi}\} < +\infty, \qquad \sup_{\xi \in \mathbb{R}} \{|I_2''(\xi)| e^{-\mu_0 \xi}\} < +\infty.$$

This completes the proof. □

In the following, we prove the main result of this subsection.

Theorem 3.8 *Assume that* $R_0 > 1$. *For* $c \in (0, c^*)$, *there exists no positive traveling wave solution* $(S_1(x + ct), S_2(x + ct), I_1(x + ct), I_2(x + ct))$ *satisfying* (2.3) *and* (2.4).

Proof We prove the theorem by contradiction. Fix $c \in (0, c^*)$. Suppose on the contrary that there exists a positive solution $(S_1(x+ct), S_2(x+ct), I_1(x+ct), I_2(x+ct))$ of (2.3) satisfying (2.4). By Lemma 3.7, there exists $\mu_0 > 0$ such that

$$\sup_{\xi \in \mathbb{R}}\{I_i(\xi)e^{-\mu_0\xi}\} < +\infty, \quad \sup_{\xi \in \mathbb{R}}\{|I_i'(\xi)|e^{-\mu_0\xi}\} < +\infty,$$
$$\sup_{\xi \in \mathbb{R}}\{|I_i''(\xi)|e^{-\mu_0\xi}\} < +\infty, \quad i = 1, 2.$$

Consider $\mathcal{R}_1(\xi) := S_1^0 - S_1(\xi)$. Then $\mathcal{R}_1(\xi)$ satisfies

$$c\mathcal{R}_1'(\xi) = d_1\mathcal{R}_1''(\xi) - \delta_1\mathcal{R}_1(\xi) + \beta_{11}S_1(\xi)I_1(\xi) + \beta_{12}S_1(\xi)I_2(\xi).$$

Using the inequality

$$\|\mathcal{R}_1'\|_{C((-\infty,0])} \leqslant 2\sqrt{\|\mathcal{R}_1\|_{C((-\infty,0])}\|\mathcal{R}_1''\|_{C((-\infty,0])}}$$

and the fact that

$$\lim_{\xi \to -\infty} \mathcal{R}_1(\xi) = 0,$$

we obtain that

$$\lim_{\xi \to -\infty} \mathcal{R}_1'(\xi) = 0. \tag{3.17}$$

In addition, since $\mathcal{R}_1'(\xi)$ is bounded by the expression of $S_1(\xi)$ for $\xi \in \mathbb{R}$ and (3.17), integrating the above inequality from $-\infty$ to $x (x < 0)$, it follows that there exists a constant $\mathcal{G} > 0$ such that

$$\begin{aligned}\delta_1\int_{-\infty}^{\xi}\mathcal{R}_1(\eta)d\eta &= -c\mathcal{R}_1(\xi) + d_1\mathcal{R}_1'(\xi) + \int_{-\infty}^{\xi}S_1(\eta)\Big[\beta_{11}I_1(\eta) + \beta_{12}I_2(\eta)\Big]d\eta\\ &\leqslant \mathcal{G}, \quad \xi \leqslant 0.\end{aligned}$$

Let

$$E_1(\xi) = \beta_{11}\int_{-\infty}^{\xi}S_1(\eta)I_1(\eta)d\eta + \beta_{12}\int_{-\infty}^{\xi}S_1(\eta)I_2(\eta)d\eta$$

and

$$B_1(\xi) = \delta_1\int_{-\infty}^{\xi}\mathcal{R}_1(\eta)d\eta$$

for any $\xi < 0$. We can see that $E_1(\xi) \le C_M e^{\mu_0\xi}$ for any $\xi \in \mathbb{R}$, where $C_M > 0$ is a constant. By the definition of $\mathcal{R}_1(\xi)$, we have

$$d_1\mathcal{R}_1'(\xi) - c\mathcal{R}_1(\xi) = B_1(\xi) - E_1(\xi), \quad \xi < 0.$$

Solving the last equation yields

$$\begin{aligned}\mathcal{R}_1(\xi) =&\hat{C}_M e^{\frac{c}{d_1}\xi} + \frac{1}{d_1} e^{\frac{c}{d_1}\xi} \int_\xi^0 e^{-\frac{c}{d_1}\eta} \Big[E_1(\eta) - B_1(\xi)\Big] d\eta \\ \leqslant&\hat{C}_M e^{\frac{c}{d_1}\xi} + \frac{1}{d_1} e^{\frac{c}{d_1}\xi} \int_\xi^0 e^{-\frac{c}{d_1}\eta} E_1(\eta) d\eta, \ \ \xi < 0,\end{aligned}$$

where $\hat{C}_M = R_1(0)$. According to $E_1(\xi) = O(e^{\mu_0\xi})$ as $\xi \to -\infty$, it is obvious that $\mathcal{R}_1(\xi) = O(e^{\mu_0'\xi})$ as $\xi \to -\infty$, where $\mu_0' = \min\{\mu_0, \frac{c}{d_1}, \frac{c}{d_2}\}$. In view of $0 \le R_1(\xi) \le S_1^0$, one has

$$\sup_{\xi\in\mathbb{R}}\{R_1(\xi)e^{-\mu_0'\xi}\} < +\infty.$$

Let $\mathcal{R}_2(\xi) := S_2^0 - S_2(\xi)$, $\xi \in \mathbb{R}$. Similarly, we have

$$\sup_{\xi\in\mathbb{R}}\{\mathcal{R}_2(\xi)e^{-\mu_0'\xi}\} < +\infty.$$

In view of $\sup_{\xi\in\mathbb{R}}\{I_i(\xi)e^{-\mu_0\xi}\} < +\infty$, we define the one-sided Laplace transform of I_i by

$$\mathrm{L}_i(\bar\lambda) = \int_{-\infty}^0 e^{-\bar\lambda\xi} I_i(\xi) d\xi, \quad i = 1, 2.$$

Next, we only consider $\bar\lambda \in \mathbb{R}_+$. Since $I_i(\xi) > 0$ for any $\xi \in \mathbb{R}$ and $\mathrm{L}_i(\cdot)$ is increasing on $\mathbb{R}^+$, for each $i = 1, 2$, either there exists a positive constant $\alpha_i > \mu_0$ such that $\mathrm{L}_i(\bar\lambda) < +\infty$ for any $0 \le \bar\lambda < \alpha_i$ and $\lim_{\bar\lambda\to\alpha_i-0} \mathrm{L}_i(\bar\lambda) = +\infty$, or $\mathrm{L}_i(\bar\lambda) < +\infty$ for any $\bar\lambda \ge 0$. Now we further define the two-sided Laplace transform of I_i by

$$\mathcal{L}_i(\bar\lambda) = \int_{-\infty}^{+\infty} e^{-\bar\lambda\xi} I_i(\xi) d\xi, \quad i = 1, 2.$$

We also only consider $\bar\lambda \in \mathbb{R}_+$. Since $I_i(\xi)$ is bounded in $\mathbb{R}$, we have $\int_0^{+\infty} e^{-\bar\lambda\xi} I_i(\xi) d\xi < +\infty$ for any $\bar\lambda > 0$. Thus, $\mathcal{L}_i(\bar\lambda)$ shares the same property with $\mathrm{L}_i(\bar\lambda)$ in $\bar\lambda > 0$, that is, for each $i = 1, 2$, either there exists a positive constant $\alpha_i > \mu_0$ such that $\mathcal{L}_i(\bar\lambda) < +\infty$ for any $0 < \bar\lambda < \alpha_i$ and $\lim_{\bar\lambda\to\alpha_i-0} \mathcal{L}_i(\bar\lambda) = +\infty$, or $\mathcal{L}_i(\bar\lambda) < +\infty$ for any $\bar\lambda > 0$.

We firstly show that indeed there are $\alpha_1 = +\infty$ and $\alpha_2 = +\infty$, that is, for both $i = 1, 2$, $\mathcal{L}_i(\bar\lambda) < +\infty$ for any $\bar\lambda > 0$. We prove this claim by a contradiction argument. Without loss of generality, we suppose $0 < \alpha_1 < +\infty$ and $\alpha_1 \le \alpha_2 \le +\infty$ on the contrary. We consider two cases: 1) $0 < \alpha_1 < \alpha_2 \le +\infty$; 2) $0 < \alpha_1 = \alpha_2 < +\infty$. For the first case, assume that $0 < \alpha_1 < \alpha_2 \le +\infty$. In view of

$$\begin{aligned}&D_1 I_1''(\xi) - cI_1'(\xi) - r_1 I_1(\xi) + \epsilon_1\beta_{11} S_1^0 I_1(\xi - c\tau) + \epsilon_1\beta_{12} S_1^0 I_2(\xi - c\tau) \\ &\quad = \epsilon_1\beta_{11}\left(S_1^0 - S_1(\xi - c\tau)\right) I_1(\xi - c\tau) + \epsilon_1\beta_{12}\left(S_1^0 - S_1(\xi - c\tau)\right) I_2(\xi - c\tau),\end{aligned}$$

one has

$$\mathcal{L}_1(\bar{\lambda})\left(D_1\bar{\lambda}^2 - c\bar{\lambda} - r_1 + \epsilon_1\beta_{11}S_1^0 e^{-\lambda c\tau}\right) + \mathcal{L}_2(\bar{\lambda})\epsilon_1\beta_{12}S_1^0 e^{-\lambda c\tau}$$
$$= \epsilon_1\int_{-\infty}^{+\infty} e^{-\bar{\lambda}\xi}\beta_{11}\left(S_1^0 - S_1(\xi - c\tau)\right) I_1(\xi - c\tau)d\xi$$
$$+ \epsilon_1\int_{-\infty}^{+\infty} e^{-\bar{\lambda}\xi}\beta_{12}\left(S_1^0 - S_1(\xi - c\tau)\right) I_2(\xi - c\tau)d\xi. \tag{3.18}$$

Similarly, we have

$$\mathcal{L}_1(\bar{\lambda})\epsilon_2\beta_{21}S_2^0 e^{-\lambda c\tau} + \mathcal{L}_2(\bar{\lambda})\left(D_2\bar{\lambda}^2 - c\bar{\lambda} - r_2 + \epsilon_2\beta_{22}S_2^0 e^{-\lambda c\tau}\right)$$
$$= \epsilon_2\int_{-\infty}^{+\infty} e^{-\bar{\lambda}\xi}\beta_{21}\left(S_2^0 - S_2(\xi - c\tau)\right) I_1(\xi - c\tau)d\xi$$
$$+ \epsilon_2\int_{-\infty}^{+\infty} e^{-\bar{\lambda}\xi}\beta_{22}\left(S_2^0 - S_2(\xi - c\tau)\right) I_2(\xi - c\tau)d\xi. \tag{3.19}$$

Since $0 < S_i^0 - S_i(\xi) \le S_i^0$ for any $\xi \in \mathbb{R}$ and $\sup_{x\in\mathbb{R}}\left\{\left(S_i^0 - S_i(\xi)\right)e^{-\mu_0' x}\right\} < +\infty$, we obtain that

$$\int_{-\infty}^{+\infty} e^{-\bar{\lambda}\xi}\beta_{i1}\left(S_i^0 - S_i(\xi - c\tau)\right) I_1(\xi - c\tau)d\xi < +\infty, \quad \forall\, \bar{\lambda} \in \left(0, \alpha_1 + \mu_0'\right)$$

and

$$\int_{-\infty}^{+\infty} e^{-\lambda\xi}\beta_{i2}\left(S_i^0 - S_i(\xi - c\tau)\right) I_2(\xi - c\tau)d\xi < +\infty, \quad \forall\, \lambda \in \left(0, \alpha_2 + \mu_0'\right).$$

In view of $\alpha_1 < \alpha_2$, letting $\bar{\lambda} \to \alpha_1 - 0$ in (3.19) yields a contradiction because the first term tends to infinity and the other terms have bounded limits as $\bar{\lambda} \to \alpha_1 - 0$. It follows that the assumption $0 < \alpha_1 < \alpha_2 \le +\infty$ is impossible.

Consider the second case, that is, assume that $0 < \alpha_1 = \alpha_2 =: \alpha_0 < +\infty$. If one of inequalities $D_1\alpha_0^2 - c\alpha_0 - r_1 + \epsilon_1\beta_{11}S_1^0 e^{-\alpha_0 c\tau} \ge 0$ and $D_2\alpha_0^2 - c\alpha_0 - r_2 + \epsilon_2\beta_{22}S_2^0 e^{-\alpha_0 c\tau} \ge 0$ holds, then letting $\bar{\lambda} \to \alpha_1 - 0$ in (3.18) or (3.19) yields a contradiction. If both inequalities

$$D_1\alpha_0^2 - c\alpha_0 - r_1 + \epsilon_1\beta_{11}S_1^0 e^{-\alpha_0 c\tau} < 0 \quad \text{and} \quad D_2\alpha_0^2 - c\alpha_0 - r_2 + \epsilon_2\beta_{22}S_2^0 e^{-\alpha_0 c\tau} < 0 \tag{3.20}$$

hold, then we rewrite (3.18) and (3.19) as

$$M(\bar{\lambda}, c)\begin{pmatrix}\mathcal{L}_1(\bar{\lambda})\\ \mathcal{L}_2(\bar{\lambda})\end{pmatrix} - \begin{pmatrix}\mathcal{L}_1(\bar{\lambda})\\ \mathcal{L}_2(\bar{\lambda})\end{pmatrix} = \begin{pmatrix}\frac{h_1(\bar{\lambda})}{m_1(\bar{\lambda},c)}\\ \frac{h_2(\bar{\lambda})}{m_2(\bar{\lambda},c)}\end{pmatrix}, \quad \bar{\lambda} \in (0, \alpha_0),$$

where $h_i(\bar{\lambda}) := \epsilon_i \sum_{j=1,2} \int_{-\infty}^{+\infty} e^{-\bar{\lambda}\xi} \beta_{ij} \left(S_i^0 - S_i(\xi - c\tau)\right) I_j(\xi - c\tau) d\xi$. It is obvious that $\alpha_0 < \mu_c$ due to (3.20). See Sect. 2 for the definitions of $M(\lambda, c)$ and $\rho(\lambda, c)$. Since $c \in (0, c^*)$ and $R_0 > 1$, it follows from Proposition 2.3 that $\inf_{\bar{\lambda}\in[0,\alpha_0]} \rho(\bar{\lambda}, c) > 1$. Since the matrix $M(\bar{\lambda}, c)$ is positive, then we can show that either

$$\frac{\epsilon_1\beta_{11}S_1^0 e^{-\bar{\lambda}c\tau}}{m_1(\bar{\lambda}, c)}\mathcal{L}_1(\bar{\lambda}) + \frac{\epsilon_1\beta_{12}S_1^0 e^{-\bar{\lambda}c\tau}}{m_1(\bar{\lambda}, c)}\mathcal{L}_2(\bar{\lambda}) \geq \rho(\bar{\lambda}, c)\mathcal{L}_1(\bar{\lambda}), \quad \bar{\lambda} \in (0, \alpha_0)$$

holds or

$$\frac{\epsilon_2\beta_{21}S_2^0 J_2(\bar{\lambda}, c)}{m_2(\bar{\lambda}, c)}\mathcal{L}_1(\bar{\lambda}) + \frac{\epsilon_2\beta_{22}S_2^0 J_2(\bar{\lambda}, c)}{m_2(\bar{\lambda}, c)}\mathcal{L}_2(\bar{\lambda}) \geq \rho(\bar{\lambda}, c)\mathcal{L}_2(\bar{\lambda}), \quad \bar{\lambda} \in (0, \alpha_0)$$

holds. Hence, for any $\bar{\lambda} \in (0, \alpha_0)$, there holds either

$$\left(\rho(\bar{\lambda}, c) - 1\right)\mathcal{L}_1(\bar{\lambda}) \leq \frac{h_1(\bar{\lambda})}{m_1(\bar{\lambda}, c)} \tag{3.21}$$

or

$$\left(\rho(\bar{\lambda}, c) - 1\right)\mathcal{L}_2(\bar{\lambda}) \leq \frac{h_2(\bar{\lambda})}{m_2(\bar{\lambda}, c)}. \tag{3.22}$$

Since $\inf_{\bar{\lambda}\in[0,\alpha_0]} m_i(\bar{\lambda}, c) > 0$ and $h_i(\bar{\lambda})$ is well defined in $[0, \alpha_0 + \mu_0')$, letting $\bar{\lambda} \to \alpha_0 - 0$ in (3.21) and (3.22) yields a contradiction due to $\lim_{\bar{\lambda}\to\alpha_0-0} \mathcal{L}_i(\bar{\lambda}) = +\infty$. Thus, we have proved that the assumption $0 < \alpha_1 = \alpha_2 =: \alpha_0 < +\infty$ is also impossible.

Now we complete the proof of the theorem. Note that we have proved that for each $i = 1, 2$, $\mathcal{L}_i(\bar{\lambda}) < +\infty$ for any $\bar{\lambda} > 0$. Using (2.28) we get that for each $\xi \in \mathbb{R}$, $I_i(\xi + y)e^{-\bar{\Lambda}y}$ is decreasing in $y \in \mathbb{R}$ and $I_i(\xi + y)e^{\bar{\Lambda}y}$ is increasing in $y \in \mathbb{R}$. Consequently, we obtain that

$$\begin{aligned}
&D_i I_i''(\xi) - cI_i'(\xi) - r_i I_i(\xi) + \epsilon_i\beta_{i1}S_i^0 I_1(\xi - c\tau) + \epsilon_i\beta_{i2}S_i^0 I_2(\xi - c\tau)\\
&\quad = \epsilon_i\beta_{i1}\left(S_i^0 - S_i(\xi - c\tau)\right) I_1(\xi - c\tau) + \epsilon_i\beta_{i2}\left(S_i^0 - S_i(\xi - c\tau)\right) I_2(\xi - c\tau)\\
&\quad < \epsilon_i\beta_{i1}S_i^0 I_1(\xi - c\tau) + \epsilon_i\beta_{i2} I_2(\xi - c\tau)\\
&\quad = \epsilon_i\beta_{i1}S_i^0 e^{\bar{\Lambda}c\tau}e^{-\bar{\Lambda}c\tau} I_1(\xi - c\tau) + \epsilon_i\beta_{i2}S_i^0 e^{\bar{\Lambda}c\tau}e^{-\bar{\Lambda}c\tau} I_2(\xi - c\tau)\\
&\quad < \epsilon_i\beta_{i1}S_i^0 e^{\bar{\Lambda}c\tau} I_1(\xi) + \epsilon_i\beta_{i2}S_i^0 e^{\bar{\Lambda}c\tau} I_2(\xi)
\end{aligned} \tag{3.23}$$

for any $\xi \in \mathbb{R}$, $i = 1, 2$. Using (3.23), we have

$$\begin{aligned}
&\int_{-\infty}^{+\infty} e^{-\bar{\lambda}\xi} I_1(\xi)\left(-m_1(\bar{\lambda}, c) + \epsilon_1\beta_{11}S_1^0 e^{-\bar{\lambda}c\tau} - \epsilon_1\beta_{11}S_1^0 e^{\bar{\Lambda}c\tau}\right) d\xi\\
&\quad + \int_{-\infty}^{+\infty} e^{-\bar{\lambda}\xi} I_2(\xi)\left(\epsilon_1\beta_{12}S_1^0 e^{-\bar{\lambda}c\tau} - \epsilon_1\beta_{12}S_1^0 e^{\bar{\Lambda}c\tau}\right) d\xi \leq 0
\end{aligned}$$

and

$$\int_{-\infty}^{+\infty} e^{-\bar{\lambda}\xi} I_1(\xi)\left(\epsilon_2\beta_{21}S_2^0 e^{-\bar{\lambda}c\tau} - \epsilon_2\beta_{21}S_2^0 e^{\bar{\Lambda}c\tau}\right)d\xi$$
$$+\int_{-\infty}^{+\infty} e^{-\bar{\lambda}\xi} I_2(\xi)\left(-m_2(\bar{\lambda},c) + \epsilon_2\beta_{22}S_2^0 e^{-\bar{\lambda}c\tau} - \epsilon_2\beta_{22}S_2^0 e^{\bar{\Lambda}c\tau}\right)d\xi \le 0.$$

Adding the last two inequalities, we obtain that

$$\int_{-\infty}^{+\infty} e^{-\bar{\lambda}\xi} I_1(\xi)\chi_1(\bar{\lambda})d\xi + \int_{-\infty}^{+\infty} e^{-\bar{\lambda}\xi} I_2(\xi)\chi_2(\bar{\lambda})d\xi \le 0, \tag{3.24}$$

where

$$\begin{aligned}\chi_1(\bar{\lambda}) &:= -m_1(\bar{\lambda},c) + \epsilon_1\beta_{11}S_1^0 e^{-\bar{\lambda}c\tau} + \epsilon_2\beta_{21}S_2^0 e^{-\bar{\lambda}c\tau}\\ &\quad - \epsilon_1\beta_{11}S_1^0 e^{\bar{\Lambda}c\tau} - \epsilon_2\beta_{21}S_2^0 e^{\bar{\Lambda}c\tau},\\ \chi_2(\bar{\lambda}) &:= -m_2(\bar{\lambda},c) + \epsilon_1\beta_{12}S_1^0 e^{-\bar{\lambda}c\tau} + \epsilon_2\beta_{22}S_2^0 e^{-\bar{\lambda}c\tau}\\ &\quad - \epsilon_1\beta_{12}S_1^0 e^{\bar{\Lambda}c\tau} - \epsilon_2\beta_{22}S_2^0 e^{\bar{\Lambda}c\tau}\end{aligned}$$

However, letting $\bar{\lambda} \to +\infty$ in (3.24) yields a contradiction because $\lim_{\bar{\lambda}\to+\infty} \chi_i(\bar{\lambda}) = +\infty$. This completes the proof. □

4 Numerical simulations and discussion

In this section we firstly provide some numerical simulations to confirm the existence of traveling wave solutions of system (1.2) connecting the disease–free equilibrium and the endemic equilibrium. For this purpose, we take the parameters of the model as below:

$$\begin{aligned}&d_1 = 0.04,\ d_2 = 0.04,\ D_1 = 0.04,\ D_2 = 0.04,\ \tau = 1,\\ &\epsilon_1 = 0.57,\ \epsilon_2 = 0.67,\ \lambda_1 = 1.6,\ \lambda_2 = 1.5,\ r_1 = 0.55,\ r_2 = 0.4,\\ &\beta_{11} = 1,\ \beta_{12} = 0.8,\ \beta_{21} = 0.3,\ \beta_{22} = 0.2,\ \delta_1 = 0.4,\ \delta_2 = 0.3.\end{aligned}$$

Using these parameters, we obtain the basic reproduction number $R_0 \approx 4.7 > 1$, the minimal speed $c^* \approx 0.459$, the disease-free equilibrium $(S_1^0, S_2^0, I_1^0, I_2^0) = (4, 5, 0, 0)$ and the endemic equilibrium $(S_1^*, S_2^*, I_1^*, I_2^*) = (0.4817, 1.3595, 1.4586, 1.8291)$. To simulate the traveling wave solutions of system (1.2), we further truncate the spatial domain $\mathbb{R}$ by $[0, 800]$ and the time domain $\mathbb{R}^+$ by $[0, 200]$. For the sake of convenience, we use the following piecewise functions as initial conditions:

$$S_i(t,x) = \begin{cases} S_i^*, & 0 \leqslant x < 400, -\tau \leqslant t \leqslant 0, i = 1, 2,\\ S_i^0, & 400 \leqslant x \leqslant 800, -\tau \leqslant t \leqslant 0, i = 1, 2\end{cases}$$

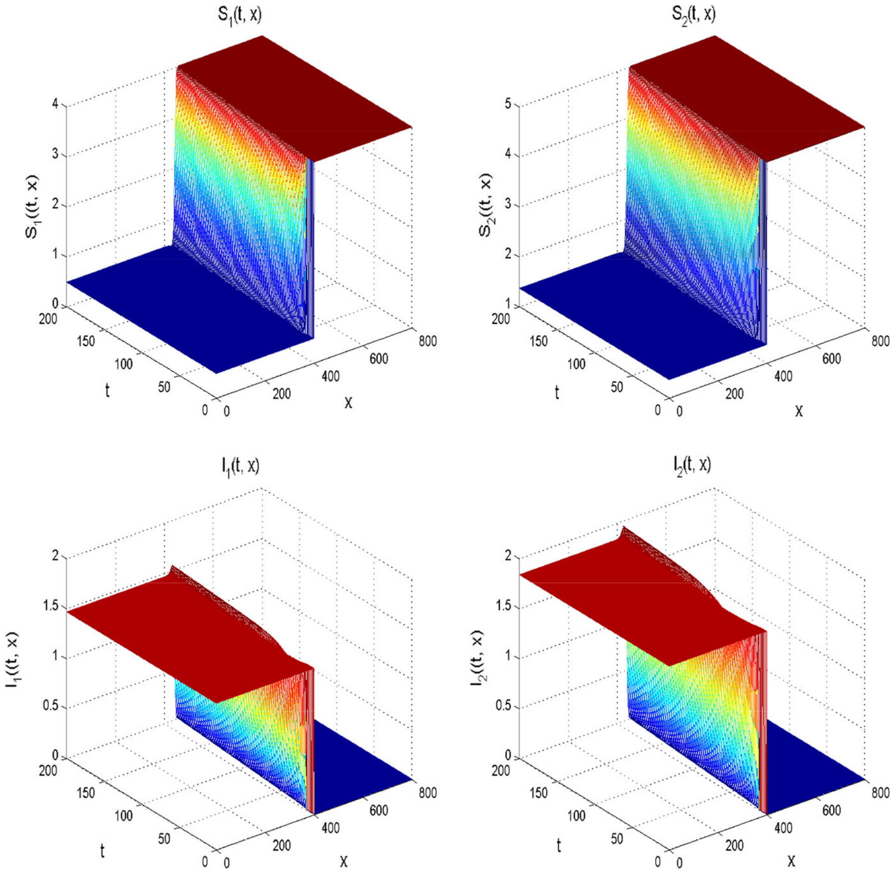

Fig. 1 Numerical simulations of solutions for system (1.2)

and

$$I_i(t,x) = \begin{cases} I_i^*, & 0 \leqslant x < 400,\ -\tau \leqslant t \leqslant 0,\ i = 1, 2, \\ 0, & 400 \leqslant x \leqslant 800,\ -\tau \leqslant t \leqslant 0,\ i = 1, 2. \end{cases}$$

In addition, we take Neumann boundary condition for system (1.2). Figure 1 illustrates the simulation results of (1.2) with the given parameters, which shows that system (1.2) admits a traveling wave solution (S_1, S_2, I_1, I_2) with wave speed $c = c^*$. Note that the traveling wave of system (1.2) is not monotonic, see Fig. 2.

In addition, the dependence of the minimal wave speed c^* on the parameters can be discussed by similar arguments to those in (Zhao et al. 2017, Section 5). Notice that $S_i^0 = \frac{\lambda_i}{\delta_i}$, $r_i = \widetilde{m}_i + \vartheta_i$ and $\epsilon_i = e^{-r_i\tau}$. Then by virtue of (2.8) and Proposition 2.3, it is easy to see that the minimal wave speed c^* depends on the parameters λ_i, δ_i, D_i, β_{ij}, $\widetilde{m}_i$, ϑ_i and τ, where $i, j = 1, 2$. For the sake of convenience, we denote $\rho(\lambda, c)$ by ρ, where $\rho(\lambda, c)$ is defined in (2.8). In addition, we always assume $R_0 > 1$ in the following. Then by direct calculations, we have

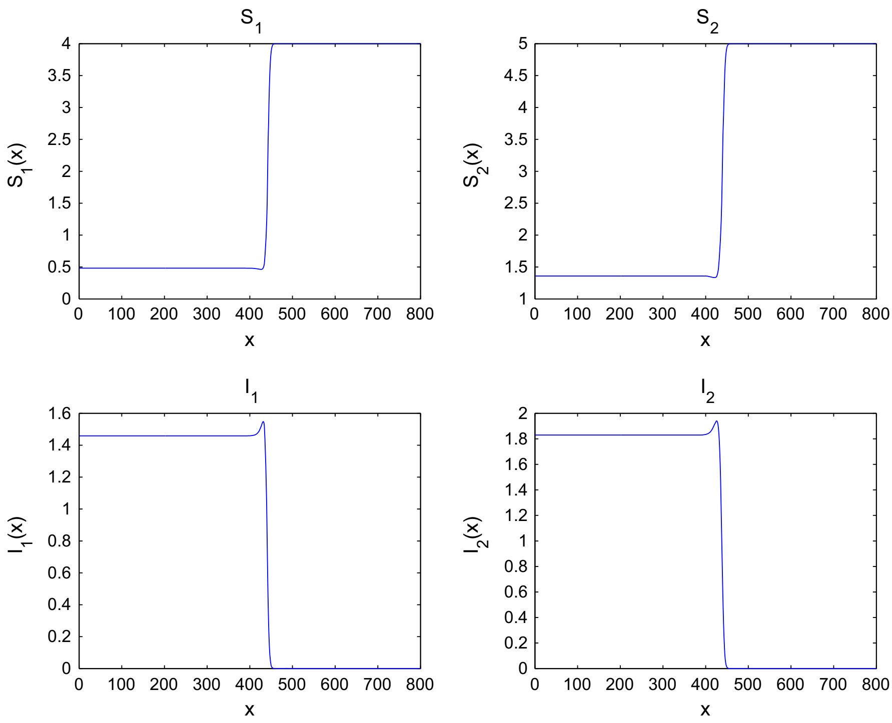

Fig. 2 Cross section curves of solutions of system (1.2) at $t = 200$ in Fig. 1

$$\frac{\partial \rho}{\partial S_i^0} > 0, \quad \frac{\partial \rho}{\partial D_i} > 0, \quad \frac{\partial \rho}{\partial \beta_{ij}} > 0, \quad \frac{\partial \rho}{\partial r_i} < 0, \quad \frac{\partial \rho}{\partial \tau} < 0$$

when $c \geqslant 0$ and $\mu \in (0, \mu_c)$, where $i, j = 1, 2$. Consequently, by Proposition 2.3 we know that c^* is increasing on $\lambda_i > 0$, $D_i > 0$ and $\beta_{ij} > 0$ respectively, and is decreasing on $\delta_i > 0$, $\widetilde{m}_i > 0$, ϑ_i and $\tau > 0$, where $i, j = 1, 2$. This implies that all the recruitment λ_i of the susceptible individuals, the diffusion rates D_i of the infective individuals and the transmission rates β_{ij} can increase the spread speed of the disease, while the death rates δ_i of the susceptible individuals, the recovery rates $\widetilde{m}_i$ and the death rates ϑ_i of the infectious individuals, and the latent period τ can decrease the spread speed of the disease.

Extensive studies have been carried out to study the effects of spatial heterogeneity, host heterogeneity, and latency on the transmission dynamics of infectious diseases. However, there are few results about their combined effects on the spatial spread of infectious diseases. In this paper, we described the spatial heterogeneity by using reaction-diffusion equations and a constant recruitment of the host population, the host heterogeneity by using two host groups, and the latency by using a discrete time delay. More specifically, we considered a two-group diffusive SIR model with time delay and constant recruitment and studied the existence and nonexistence of traveling wave solutions of the model. When the basic reproduction number $R_0 > 1$, we proved that there exists a positive number c^* such that for each wave speed $c \geq c^*$, the model admits a nontrivial traveling wave solution with wave speed c. In particular, we used Lyapunov functional method to establish the convergence of traveling waves as $x \to +\infty$. We also showed the nonexistence of nonnegative traveling wave solutions of this model when $R_0 \leq 1$ or $R_0 > 1$ and $0 < c < c^*$.

Here we would like to provide some comparisons between the results in our earlier paper (Zhao et al. 2017) and those in this paper. Note that (Zhao et al. 2017, (2.11)) includes the mobility of the latent individuals but not the natural death of the individuals, while in this paper (see (1.2)) we considered the recruitment and natural death of the individuals but not the mobility of the latent individuals. Therefore, for system (1.2) of this paper the minimal wave speed depends on the recruitment λ_i and the death rates δ_i of the susceptible individuals but not the mobility of the latent individuals, while for the model (2.11) of Zhao et al. (2017) the minimal wave speed depends on the mobility of the latent individuals but not the natural death rates δ_i of the susceptible individuals.

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