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Existence, uniqueness and asymptotic stability of time periodic traveling waves for a periodic Lotka–Volterra competition system with diffusion

Guangyu Zhao, Shigui Ruan *

Department of Mathematics, University of Miami, Coral Gables, FL 33124-4250, USA

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Abstract

We study the existence, uniqueness, and asymptotic stability of time periodic traveling wave solutions to a periodic diffusive Lotka–Volterra competition system. Under certain conditions, we prove that there exists a maximal wave speed c^* such that for each wave speed $c \leq c^*$, there is a time periodic traveling wave connecting two semi-trivial periodic solutions of the corresponding kinetic system. It is shown that such a traveling wave is unique modulo translation and is monotone with respect to its co-moving frame coordinate. We also show that the traveling wave solutions with wave speed $c < c^*$ are asymptotically stable in certain sense. In addition, we establish the nonexistence of time periodic traveling waves for nonzero speed $c > c^*$.

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Résumé

On étudie l'existence, l'unicité, et la stabilité asymptotique des ondes progressives périodiques pour un système compétitif de Lotka–Volterra avec diffusion. Sous certaines conditions, on démontre qu'il existe une vitesse maximale c^* telle que pour chaque vitesse $c < c^*$, il existe une onde périodique progressive en temps connectant deux solutions semi-triviales correspondant à la cinétique du système. On démontre que cette onde (modulo les translations) est unique et est monotone dans le repère lié à l'onde. On montre que les ondes avec une vitesse $c < c^*$ sont asymptotiquement stables (en un certain sens). Enfin, on établit la non existence des ondes périodiques progressives pour des vitesses $c > c^*$.

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1. Introduction

In this paper, we are concerned with time periodic traveling wave solutions to the diffusive Lotka–Volterra competition system:

* Corresponding author.

E-mail addresses: gzhao@math.miami.edu (G. Zhao), ruan@math.miami.edu (S. Ruan).

$$\begin{cases} u_t = u_{xx} + u(r_1(t) - a_1(t)u - b_1(t)v), \\ v_t = dv_{xx} + v(r_2(t) - a_2(t)u - b_2(t)v), \end{cases} \quad (1.1)$$

where $u = u(t, x)$ and $v = v(t, x)$ denote the densities of two competing species in location $x \in \mathbb{R}$ and at time $t \in \mathbb{R}^+$, $d > 0$, and r_i, a_i, b_i ($i = 1, 2$) are T -periodic continuous functions of t , a_i, b_i are positive in $[0, T]$, while r_i may change sign. Nonlinear periodic diffusion systems like (1.1) describe the evolution of two competing species u and v naturally stemming from population dynamics, where the data depend periodically on time. Time periodic traveling waves to system (1.1) are solutions of the form,

$$\begin{pmatrix} u(t, x) \\ v(t, x) \end{pmatrix} = \begin{pmatrix} X(t, x - ct) \\ Y(t, x - ct) \end{pmatrix}, \quad (1.2)$$

satisfying

$$\begin{pmatrix} X(t + T, z) \\ Y(t + T, z) \end{pmatrix} = \begin{pmatrix} X(t, z) \\ Y(t, z) \end{pmatrix}, \quad \begin{pmatrix} X(t, \pm\infty) \\ Y(t, \pm\infty) \end{pmatrix} := \lim_{z \rightarrow \pm\infty} \begin{pmatrix} X(t, z) \\ Y(t, z) \end{pmatrix} = \begin{pmatrix} u^\pm(t) \\ v^\pm(t) \end{pmatrix},$$

where c is an *a priori* unknown constant, referred to as the wave speed, $z = x - ct$ is the co-moving frame coordinate, and $\begin{pmatrix} u^+(t) \\ v^+(t) \end{pmatrix}$ and $\begin{pmatrix} u^-(t) \\ v^-(t) \end{pmatrix}$ are periodic solutions of the corresponding kinetic system:

$$\begin{cases} \frac{du}{dt} = u(r_1(t) - a_1(t)u - b_1(t)v), \\ \frac{dv}{dt} = v(r_2(t) - a_2(t)u - b_2(t)v). \end{cases} \quad (1.3)$$

There have been many interesting studies on traveling wave solutions to diffusive Lotka–Volterra competition systems for which the corresponding kinetic systems are autonomous (see [7,9,11,15,16,24,25,27,28,38,39] and references therein). Recently, an interest in periodic traveling waves of the form (1.2) has been developed, which was stimulated by the observation of periodic traveling waves in a large number of mathematical models arising in various disciplines. Alikakos, Bates and Chen [2] established the existence, uniqueness and stability of time periodic traveling wave solutions for a single reaction diffusion equation with periodic bistable nonlinearities. The time periodic traveling wave solutions were also employed to study the development of interfaces for related higher dimensional equations in bounded domains. Nolen and Xin [35] proved the existence of periodic traveling waves in mean zero space–time periodic shear flows for the KPP nonlinearities. They also utilized a variational principle to characterize the minimal front speed. Liang and Zhao [30] extended the theory of spreading speeds and traveling waves for monotone autonomous semiflows to periodic semiflows in the monostable case (see also [29]). These abstract results were applied to certain periodic diffusive equations. Most recently, Hamel [18] and Hamel and Roques [19] presented a systematic analysis on the qualitative behavior, uniqueness and stability of monostable pulsating fronts for reaction–diffusion equations in periodic media with KPP nonlinearities. The established results provide a complete classification of all KPP pulsating fronts (see [4–6,40] for other related results). Although the study of traveling wave solutions, mostly for the autonomous case, has a longstanding history, there are still very few studies devoted to time periodic traveling wave solutions for diffusive systems with time-periodic reaction terms. Unlike the autonomous case, the presence of time dependent nonlinearities poses significant difficulties and requires new approaches.

In the present work, we consider (1.1) focusing on the case that

$$\frac{\int_0^T r_1(t) dt}{T} > \max_{0 \leq t \leq T} \left(\frac{b_1}{b_2} \right) \frac{\int_0^T r_2(t) dt}{T} > 0, \quad 0 < \frac{\int_0^T r_2(t) dt}{T} \leq \min_{0 \leq t \leq T} \left(\frac{a_2}{a_1} \right) \frac{\int_0^T r_1(t) dt}{T}, \quad (1.4)$$

which implies that (1.3) has only three nonnegative T -periodic solutions $(0, 0)$, $(p(t), 0)$, and $(0, q(t))$, where $p(t)$ and $q(t)$ are explicitly given by:

$$\begin{cases} p(t) = \frac{p_0 e^{\int_0^t r_1(s) ds}}{1 + p_0 \int_0^t e^{\int_0^s r_1(\tau) d\tau} a_1(s) ds}, & p_0 = \frac{e^{\int_0^T r_1(s) ds} - 1}{\int_0^T e^{\int_0^s r_1(\tau) d\tau} a_1(s) ds}, \\ q(t) = \frac{q_0 e^{\int_0^t r_2(s) ds}}{1 + q_0 \int_0^t e^{\int_0^s r_2(\tau) d\tau} b_2(s) ds}, & q_0 = \frac{e^{\int_0^T r_2(s) ds} - 1}{\int_0^T e^{\int_0^s r_2(\tau) d\tau} b_2(s) ds}. \end{cases} \quad (1.5)$$

Note that $(p(t), 0)$ is globally stable in the interior of the positive quadrant $\mathbb{R}_+^2 := \{(u, v) \mid u \geq 0, v \geq 0\}$ (see [10] or [22]). Assume that the inequalities in (1.4) hold, we are primarily interested in periodic traveling waves of (1.1) with $(u^+(t), v^+(t)) = (p(t), 0)$ and $(u^-(t), v^-(t)) = (0, q(t))$. Now let $(U(t, z), W(t, z))$ be defined as follows:

$$U(t, z) = \frac{X(t, z)}{p(t)}, \quad W(t, z) = \frac{q(t) - Y(t, z)}{q(t)}. \tag{1.6}$$

Substituting (1.6) into (1.1) yields that

$$\begin{cases} U_t = U_{zz} + cU_z + U[a_1(t)p(t)(1 - U) - b_1(t)q(t)(1 - W)], \\ W_t = dW_{zz} + cW_z + (1 - W)[a_2(t)p(t)U - b_2(t)q(t)W]; \\ (U(t, z), W(t, z)) = (U(t + T, z), W(t + T, z)), \\ \lim_{z \rightarrow -\infty} (U, W) = (0, 0), \quad \lim_{z \rightarrow \infty} (U, W) = (1, 1). \end{cases} \tag{1.7}$$

The main focus of this paper is on the existence and uniqueness of solutions to (1.7) and their various qualitative properties. The paper is organized as follows. In Section 2, under certain conditions, we establish the existence of $c^* < 0$ such that there exists, for any $c \leq c^*$, a solution to (1.7) which is monotone in z . In Section 3, we study the uniqueness of solutions of (1.7) for $c \leq c^*$ that are constrained to $[0, 1] \times [0, 1]$. To this end, we consider a generalized reaction–diffusion system that retains the most essential features of (1.7). We adopt a dynamical approach to obtain the exact exponential decay rate of a solution as it approaches its unstable limiting state. With this asymptotic property, we employ the sliding method to establish the uniqueness of the aforementioned solution. In addition, we show that the components of such a solution are monotone with respect to the variable z . We also show that the wave speed c^* obtained in Section 1 is the maximal speed such that (1.7) has no solutions with nonzero wave speed $c > c^*$ that are nondecreasing with respect to z . In Section 4, under the same conditions presented in Section 3, we utilize the methods similar to those given in [19] to study the asymptotic stability of time periodic traveling wave solutions of

$$\begin{cases} u_t = u_{xx} + u[a_1(t)p(t)(1 - u) - b_1(t)q(t)(1 - v)], \\ v_t = dv_{xx} + (1 - v)[a_2(t)p(t)u - b_2(t)q(t)v]. \end{cases} \tag{1.8}$$

We first consider the solutions of (1.8) with initial data decaying exponentially as $x \rightarrow -\infty$. We then establish the convergence of such solutions to the periodic traveling waves of (1.8) with speed $c < c^*$ at large time, which indicates that these solutions propagate with constant speed at a long time.

For future reference, we denote a vector by printing a letter in boldface $\mathbf{u} = (u_1, \dots, u_i, \dots, u_n)$, where u_i stands for the i th component of \mathbf{u} . The following notation shall be adopted. Let $I, \Gamma \subseteq \mathbb{R}$ be two (possibly unbounded) intervals and $M \subseteq \mathbb{R}^n$. Denote by $C(I \times \Gamma, M)$ the space of continuous functions $\mathbf{u} : I \times \Gamma \rightarrow M$, $C_b(I \times \Gamma, M)$ is the space of functions $\mathbf{u} \in C(I \times \Gamma, M)$ with $\|\mathbf{u}\|_\infty < \infty$, $C^{k,l}(I \times \Gamma, M)$ is the space of functions $\mathbf{u} \in C(I \times \Gamma, M)$ such that $\mathbf{u}(\cdot, x)$ is k -time continuously differentiable and $\mathbf{u}(t, \cdot)$ is l -time continuously differentiable, $C_b^{k,l}(I \times \Gamma, M)$ is the space of functions $\mathbf{u} \in C^{k,l}(I \times \Gamma, M)$ such that all partial derivatives of \mathbf{u} are uniformly bounded. In particular, given $\alpha \in]0, 1[$, we set:

$$[\mathbf{u}]_\alpha = \sup \left\{ \frac{|\mathbf{u}(t, x) - \mathbf{u}(\tau, y)|}{|t - \tau|^{\alpha/2} + |x - y|^\alpha}, t, \tau \in I, x, y \in \Gamma, (t, x) \neq (\tau, y) \right\}.$$

Denote by $C_b^{\alpha/2, \alpha}(I \times \Gamma, M)$ the space of functions $\mathbf{u} \in C_b(I \times \Gamma, M)$ such that $[\mathbf{u}]_\alpha < \infty$ and $C_b^{1+\alpha/2, 2+\alpha}(I \times \Gamma, M)$ the space of functions $\mathbf{u} \in C_b^{1,2}(I \times \Gamma, M)$ such that $[\mathbf{u}_t]_\alpha < \infty$, $[\mathbf{u}_x]_\alpha < \infty$, and $[\mathbf{u}_{xx}]_\alpha < \infty$. In case that $M = \mathbb{R}^n$, and no confusion occurs, we shall set $C_b(I \times \Gamma) := C_b(I \times \Gamma, \mathbb{R}^n)$, $C_b^{\alpha/2, \alpha}(I \times \Gamma) := C_b^{\alpha/2, \alpha}(I \times \Gamma, \mathbb{R}^n)$, etc. We also set $[a, b]^2 := [a, b] \times [a, b]$, where $-\infty \leq a < b < \infty$. We use the notation,

$$\bar{h} = \frac{1}{T} \int_0^T h(t) dt,$$

for the average of a function h that is integrable in $[0, T]$.

2. Existence of periodic traveling wave solutions

This section is devoted to the existence of time periodic traveling wave solutions to (1.1) connecting the semi-trivial periodic solutions $(0, q(t))$ and $(p(t), 0)$ of (1.3). Here $p(t)$ and $q(t)$ are given by (1.5). Throughout this section, we always assume that

- (A1) $r_i, a_i, b_i \in C^\theta(\mathbb{R}, \mathbb{R})$ for some θ with $0 < \theta < 1$, $r_i(t + T) = r_i(t)$, $a_i(t + T) = a_i(t)$, $b_i(t + T) = b_i(t)$, $i = 1, 2$.
- (A2) $\bar{r}_i > 0$, and $a_i(t) > 0$, $b_i(t) > 0$ for all t , $i = 1, 2$. Moreover, $\bar{r}_1 > \max_t(\frac{b_1}{b_2})\bar{r}_2$ and $\min_t(\frac{a_2}{a_1})\bar{r}_1 \geq \bar{r}_2$.

We thereafter consider

$$\begin{cases} U_t = U_{zz} + cU_z + U[a_1(t)p(t)(1 - U) - b_1(t)q(t)(1 - W)], \\ W_t = dW_{zz} + cW_z + (1 - W)[a_2(t)p(t)U - b_2(t)q(t)W]; \\ (U(t, z), W(t, z)) = (U(t + T, z), W(t + T, z)), \\ \lim_{z \rightarrow -\infty} (U, W) = (0, 0), \quad \lim_{z \rightarrow \infty} (U, W) = (1, 1). \end{cases} \tag{2.1}$$

Definition 2.1. ([12]) If $\mathbf{u} \in \mathbb{R}^n$ and $\mathbf{v} \in \mathbb{R}^n$, the relation $\mathbf{u} < \mathbf{v}$ ($\mathbf{u} \leq \mathbf{v}$ respectively) is to be understood componentwise: $u_i < v_i$ ($u_i \leq v_i$) for each i . The other relations, such as “max”, “min”, “sup”, and “inf”, are similarly to be understood componentwise.

Definition 2.2. ([12]) A vector valued function $\mathbf{w} = (w_1, \dots, w_n) \in C^{1,2}(I \times \Gamma, \mathbb{R}^n)$ is called a *regular super-solution* of

$$\frac{\partial u_i}{\partial t} = d_i(z) \frac{\partial^2 u_i}{\partial z^2} + c_i(z) \frac{\partial u_i}{\partial z} + f_i(t, u_1, \dots, u_n), \quad i = 1, \dots, n, \tag{2.2}$$

provided that

$$d_i(z) \frac{\partial^2 w_i}{\partial z^2} + c_i(z) \frac{\partial w_i}{\partial z} + f_i(t, w_1, \dots, w_n) - \frac{\partial w_i}{\partial t} \leq 0 \quad \text{for } (t, z) \in I \times \Gamma.$$

It is called a *regular sub-solution* of (2.2) if the above inequalities are reversed. Here $d_i, c_i \in C^\theta(\Gamma, \mathbb{R})$, and $f_i \in C^{\theta,1}(I \times \mathbb{R}^n, \mathbb{R})$ for some θ with $0 < \theta < 1$. In particular, there exists $\omega > 0$ such that $d_i(z) \geq \omega$ for all i and $z \in \Gamma$.

Definition 2.3. ([12]) A vector valued function $\mathbf{v} \in C(I \times \Gamma, \mathbb{R}^n)$ is said to be an *irregular super-solution* of (2.2) if there exist regular super-solutions $\mathbf{w}^1, \dots, \mathbf{w}^k$ of (2.2) such that $\mathbf{v} = \min\{\mathbf{w}^1, \dots, \mathbf{w}^k\}$. It is called an *irregular sub-solution* of (2.2) if there exist regular sub-solutions $\mathbf{v}^1, \dots, \mathbf{v}^k$ of (2.2) such that $\mathbf{v} = \max\{\mathbf{v}^1, \dots, \mathbf{v}^k\}$.

Lemma 2.4. Suppose that there exist $\mathbf{u} \in C_b^{\gamma/2, \gamma}([0, T + \epsilon) \times (-\infty, z^0])$ and $\bar{\mathbf{u}} \in C_b^{\gamma/2, \gamma}([0, T + \epsilon) \times \mathbb{R})$ such that \mathbf{u} and $\bar{\mathbf{u}}$ are the irregular super- and sub-solutions of

$$\frac{\partial u_i}{\partial t} = d_i(z) \frac{\partial^2 u_i}{\partial z^2} + c_i(z) \frac{\partial u_i}{\partial z} + f_i(t, u_1, \dots, u_n), \quad i = 1, \dots, n, \tag{2.3}$$

respectively, and $\mathbf{u} \leq \bar{\mathbf{u}}$ for all $(t, z) \in [0, T + \epsilon) \times (-\infty, z^0]$. Here $0 < \gamma < 1$, $\epsilon > 0$, $z^0 \in \mathbb{R}$; d_i, c_i and f_i satisfy the assumptions given in Definition 2.2 with $I = \mathbb{R}$ and $\Gamma = \mathbb{R}$, $\bar{\mathbf{u}} = \min\{\bar{\mathbf{w}}^1, \dots, \bar{\mathbf{w}}^k\}$, and $\mathbf{u} = \max\{\underline{\mathbf{w}}^1, \dots, \underline{\mathbf{w}}^k\}$; $\bar{\mathbf{w}}^l$ and $\underline{\mathbf{w}}^l$ are respectively the regular super- and sub-solutions of (2.3) ($l = 1, \dots, k$). Moreover, assume that $f_i(t, \mathbf{0}) = 0$, $f_i(\cdot + T, \mathbf{u}) = f_i(\cdot, \mathbf{u})$, and $\frac{\partial f_i}{\partial u_j} \geq 0$ in $\prod_{i=1}^n [\underline{\omega}_i, \bar{\omega}_i]$, $i \neq j$, for all $t \in \mathbb{R}$, where $\underline{\omega}_i = \inf u_i$, $\bar{\omega}_i = \sup \bar{u}_i$. In addition, for each $l \in \{1, \dots, k\}$, $\{(t, z) \mid \underline{\omega}_i \leq \bar{w}_i^l \leq \bar{\omega}_i\} = \{(t, z) \mid \underline{\omega}_j \leq \bar{w}_j^l \leq \bar{\omega}_j\}$ and $\{(t, z) \mid \underline{\omega}_i \leq w_i^l \leq \bar{\omega}_i\} = \{(t, z) \mid \underline{\omega}_j \leq w_j^l \leq \bar{\omega}_j\}$, $i \neq j$. Assume further that $\bar{\mathbf{u}} \geq \mathbf{0}$, $\bar{\mathbf{u}}(0, z) \geq \bar{\mathbf{u}}(T, z)$, $\mathbf{u}(T, z) \geq \mathbf{u}(0, z)$, and $\mathbf{u}(t, z^0) \leq \mathbf{0}$ for all $t \in [0, T + \epsilon)$. Then there exists a positive solution $\mathbf{u}^* \in C_b^{1,2}(\mathbb{R} \times \mathbb{R})$ to (2.3) such that $\mathbf{u}^*(\cdot + T, z) = \mathbf{u}^*(\cdot, z)$, $\mathbf{u}^* \leq \bar{\mathbf{u}}$ for all $(t, z) \in [0, T] \times \mathbb{R}$, and $\mathbf{u}^* \geq \mathbf{u}$ for all $(t, z) \in [0, T] \times (-\infty, z^0]$. In addition, if $\bar{\mathbf{u}}$ is nondecreasing with respect to z , and d_i and c_i are constants, then either $(u_i^*)_z > 0$ or $(u_i^*)_z \equiv 0$.

Proof. The proof is based on the monotone iterations for parabolic systems. Set $\bar{\mathbf{u}} = \mathbf{u}^0$. Inductively, we define \mathbf{u}^m by:

$$\begin{cases} \frac{\partial u_j^{m+1}}{\partial t} = d_j(z) \frac{\partial^2 u_j^{m+1}}{\partial z^2} + c_j(z) \frac{\partial u_j^{m+1}}{\partial z} - K u_j^{m+1} + f_j(t, \mathbf{u}^m) + K u_j^m; \\ u_j^{m+1}(0, x) = u_j^m(T, x). \end{cases} \quad (2.4)$$

Here K is a positive constant with $K \geq \max_{(t, \mathbf{u}) \in [0, T] \times \Sigma} |\frac{\partial f_j}{\partial u_k}|$ for any j , $\Sigma := \prod_{i=1}^n [\underline{\omega}_i, \bar{\omega}_i]$, and \mathbf{u}^m is understood as a mild solution of (2.4) whose components are given by:

$$u_j^{m+1}(t, z) = G_j(t)u_j^m(T, z) + \int_0^t G_j(t-s)[K u_j^m + f_j(t, \mathbf{u}^m)] ds,$$

where $G_j(t)$ is the analytic semigroup generated by the linear differential operator $A_j : D(A_j) \rightarrow C_b(\mathbb{R})$ defined by

$$D(A_j) = \left\{ C_b(\mathbb{R}) \cap \bigcap_{1 \leq p < \infty} W_{\text{loc}}^{2,p}(\mathbb{R}), A_j u = d_j(z)u_{zz} + c_j(z)u_z - K u \in C_b(\mathbb{R}) \right\}.$$

Thanks to Theorems 5.1.3 and 5.1.4 of [32], $\mathbf{u}^1 \in C_b^{\alpha/2, \alpha}([0, T] \times \mathbb{R}) \cap C_b^{1+\alpha/2, 2+\alpha}([\varepsilon, T] \times \mathbb{R})$ with some $\alpha \in]0, 1[$ for every $\varepsilon \in]0, T[$, and \mathbf{u}^1 satisfies the first equation of (2.4) in $(0, T] \times \mathbb{R}$, whence, for each $m \geq 2$, $\mathbf{u}^m \in C_b^{1+\alpha/2, 2+\alpha}([0, T] \times \mathbb{R})$ satisfies (2.4) in $[0, T] \times \mathbb{R}$.

We now show that $\mathbf{u}^1 \leq \bar{\mathbf{u}}$ for all $(t, z) \in [0, T] \times \mathbb{R}$. Let

$$L = \|\bar{\mathbf{u}} - \mathbf{u}^1\|_{C([0, T] \times \mathbb{R}, \mathbb{R}^n)}, \quad \mathbf{v}^r = \bar{\mathbf{u}} - \mathbf{u}^1 + \frac{L}{\varpi + r^2}(z^2 + \varpi + Nt)e^{\mu t}, \quad r > 0,$$

where $\varpi = (\max_{1 \leq i \leq n} |c_i|_\infty + 1)^2$, $\mu > 1$, and $N > 2 \max_{1 \leq i \leq n} |d_i|_\infty$ are fixed constants such that $2(\max_{1 \leq i \leq n} |d_i|_\infty + \max_{1 \leq i \leq n} |c_i|_\infty |z|) - \mu(z^2 + \varpi) - N < 0$ for all $z \in \mathbb{R}$. Clearly, $\mathbf{v}^r(0, z) > \mathbf{0}$ for all z satisfying $|z| \leq r$. Furthermore, $\mathbf{v}^r(t, \pm r) > \mathbf{0}$ for any $t \in [0, T]$. In fact, it holds that $\mathbf{v}^r(t, z) > \mathbf{0}$ for all $(t, z) \in [0, T] \times [-r, r]$. Assume to the contrary that this is not true, then there exists a point $(t^*, z^*) \in (0, T] \times]-r, r[$ and at least a component v_i^r such that

$$v_i^r(t^*, z^*) = 0 \quad \text{and} \quad \mathbf{v}^r(t, z) \geq \mathbf{0} \quad \text{for all } (t, z) \in [0, t^*] \times [-r, r].$$

Since $\bar{\mathbf{u}}$ is an irregular super-solution of (2.3) and $\bar{\mathbf{u}} = \min\{\bar{\mathbf{w}}^1, \dots, \bar{\mathbf{w}}^k\}$, we may assume without loss of generality that $\bar{u}_i(t^*, z^*) = \bar{w}_i^1(t^*, z^*)$. Now let:

$$\hat{\mathbf{w}} = \bar{\mathbf{w}}^1 - \mathbf{u}^1 + \frac{L}{\varpi + r^2}(z^2 + \varpi + Nt)e^{\mu t}.$$

Obviously, $\hat{w}_i(t^*, z^*) = 0$ and $\hat{\mathbf{w}} \geq \mathbf{0}$ for all $(t, z) \in [0, t^*] \times [-r, r]$. In addition, for any $(t, z) \in (0, t^*) \times]-r, -r[$, it is straightforward to verify that

$$\begin{aligned} & d_j(z) \frac{\partial^2 \hat{w}_j}{\partial z^2} + c_j(z) \frac{\partial \hat{w}_j}{\partial z} - K \hat{w}_j - \frac{\partial \hat{w}_j}{\partial t} \\ & \leq d_j(z) \frac{\partial^2 (\bar{w}_j^1 - u_j^1)}{\partial z^2} + c_j(z) \frac{\partial (\bar{w}_j^1 - u_j^1)}{\partial z} - K (\bar{w}_j^1 - u_j^1) - \frac{\partial (\bar{w}_j^1 - u_j^1)}{\partial t} \\ & \quad + \frac{L}{\varpi + r^2} e^{\mu t} \left[2 \max_{1 \leq j \leq n} |d_j|_\infty + 2 \max_{1 \leq j \leq n} |c_j|_\infty |z| - (K + \mu)(z^2 + \varpi + Nt) - N \right] \\ & < K (\bar{u}_j - \bar{w}_j^1) + \int_0^1 \partial_{\mathbf{u}} f_j(t, s \bar{\mathbf{w}}^1 + (1-s)\bar{\mathbf{u}}) ds (\bar{\mathbf{u}} - \bar{\mathbf{w}}^1), \quad j = 1, \dots, n. \end{aligned}$$

Since $\bar{u}_i(t^*, z^*) = \bar{w}_i^1(t^*, z^*)$ and $\underline{\omega}_i \leq \bar{u}_i(t^*, z^*) \leq \bar{\omega}_i$, it follows from the assumption that $\underline{\omega}_j \leq \bar{w}_j^1(t^*, z^*) \leq \bar{\omega}_j$ with $j \neq i$. This implies that $s \bar{\mathbf{w}}^1 + (1-s)\bar{\mathbf{u}}|_{(t^*, z^*)} \in \prod_{k=1}^n [\underline{\omega}_k, \bar{\omega}_k]$ for any $s \in [0, 1]$. Consequently,

$$\left[d_i(z) \frac{\partial^2 \hat{w}_i}{\partial z^2} + c_i(z) \frac{\partial \hat{w}_i}{\partial z} - K \hat{w}_i - \frac{\partial \hat{w}_i}{\partial t} \right] \Big|_{(t^*, z^*)} < 0.$$

On the other hand, since \hat{w}_i attains its local minimum at (t^*, z^*) , we have:

$$\frac{\partial^2 \hat{w}_i(t^*, z^*)}{\partial z^2} \geq 0, \quad \frac{\partial \hat{w}_i(t^*, z^*)}{\partial z} = 0 \quad \text{and} \quad \frac{\partial \hat{w}_i(t^*, z^*)}{\partial t} \leq 0.$$

Hence it follows that $0 \geq d_i(z) \frac{\partial^2 \hat{w}_i(t^*, z^*)}{\partial z^2} - \frac{\partial \hat{w}_i(t^*, z^*)}{\partial t} < 0$, which is a contradiction. This contradiction confirms that $\mathbf{v}^r(t, z) > \mathbf{0}$ for all $(t, z) \in [0, T] \times [-r, r]$. Since $\mathbf{v}^r \rightarrow \bar{\mathbf{u}} - \mathbf{u}^1$ uniformly in $[0, T] \times [-\sqrt{r}, \sqrt{r}]$ as $r \rightarrow \infty$, we infer that $\bar{\mathbf{u}} - \mathbf{u}^1 \geq \mathbf{0}$. Meanwhile, it follows from the comparison principle that $\mathbf{u}^1 \geq \mathbf{0}$, which along with the assumption shows that $\mathbf{u}^1(t, z^0) \geq \underline{\mathbf{u}}(t, z^0)$ for all $t \in [0, T]$. As $\mathbf{u}^1(0, z) \geq \underline{\mathbf{u}}(0, z)$, repeating the same argument in $[-r, z^0]$ with $r > |z^0|$ yields that $\underline{\mathbf{u}} \leq \mathbf{u}^1$ for all $(t, x) \in [0, T] \times (-\infty, z^0]$. In a similar fashion, it can be shown that $\mathbf{0} \leq \mathbf{u}^{m+1} \leq \mathbf{u}^m \leq \bar{\mathbf{u}}$ for all $(t, x) \in [0, T] \times \mathbb{R}$, and $\underline{\mathbf{u}} \leq \mathbf{u}^m$ for all $(t, x) \in [0, T] \times (-\infty, z^0]$ ($m \geq 1$). That is, the sequence $\{\mathbf{u}^m\}$ is uniformly bounded.

Consequently, for any $m \geq 2$, Theorems 5.1.2 and 5.13 together with Theorem 5.1.4 in [32] imply that

$$\|\mathbf{u}^m\|_{C^{1+\alpha/2, 2+\alpha}([0, T] \times \mathbb{R}, \mathbb{R}^n)} \leq C \left(M + \max_{(t, \mathbf{u}) \in [0, T] \times \Sigma} |\partial_{\mathbf{u}} f(t, \mathbf{u})| \right)$$

for certain positive constants C, M , and $\alpha \in]0, 1[$ depending only upon d_i, c_i , and $\|\bar{\mathbf{u}}\|_{C^{1/2, \gamma}}$. Therefore, there exists a subsequence of $\{\mathbf{u}^m\}$, still labeled by $\{\mathbf{u}^m\}$, such that it converges in $C_{\text{loc}}^{1,2}([0, T] \times \mathbb{R}, \mathbb{R}^n)$ to a function denoted by \mathbf{u}^* . Clearly \mathbf{u}^* satisfies (2.3) in $[0, T] \times \mathbb{R}$. Since $\mathbf{u}^{m+1}(0, z) = \mathbf{u}^m(T, z)$, we find that $\mathbf{u}^*(0, z) = \mathbf{u}^*(T, z)$. Moreover, observe that

$$\begin{aligned} \frac{\partial u_j^{m+1}}{\partial t} \Big|_{t=0} &= \left[d_j(z) \frac{\partial^2 u_j^{m+1}}{\partial z^2} + c_j(z) \frac{\partial u_j^{m+1}}{\partial z} - K u_j^{m+1} + f_j(t, \mathbf{u}^m) + K u_j^m \right] \Big|_{t=0} \\ &= \left[d_j(z) \frac{\partial^2 u_j^m}{\partial z^2} + c_j(z) \frac{\partial u_j^m}{\partial z} - K u_j^m + f_j(t, \mathbf{u}^{m-1}) + K u_j^{m-1} \right] \Big|_{t=T} \\ &= \frac{\partial u_j^m}{\partial t} \Big|_{t=T} \quad (m \geq 1). \end{aligned}$$

Therefore, by taking the limits in these equations, we obtain that $\mathbf{u}_t^*(0, z) = \mathbf{u}_t^*(T, z)$. Namely, \mathbf{u}^* satisfies the periodic boundary conditions. Thus, \mathbf{u}^* can be continued to a smooth T -periodic solution to (2.3).

Now it remains to prove the second part of this lemma. Since d_i and c_i are independent of z , we see that for any $s \in \mathbb{R}$, $u_j^{m+1}(t, z + s)$ satisfies (2.3) with $u_j^{m+1}(0, z + s) = u_j^m(T, z + s)$. As $\mathbf{u}^0(t, z + s) \geq \mathbf{u}^0(t, z)$ for any $s > 0$. The comparison principle yields that $u_j^{m+1}(t, z + s) \geq u_j^{m+1}(t, z)$ as long as $s \geq 0$. Invoking Helly's theorem, upon taking a subsequence of $\{\mathbf{u}^m\}$, still labeled by $\{\mathbf{u}^m\}$, we can actually show that

$$\lim_{m \rightarrow \infty} \|\mathbf{u}^m - \mathbf{u}^*\|_{C([0, T] \times \mathbb{R}, \mathbb{R}^n)} = 0$$

(see Theorem 3.2 in [41]). Thus, $\mathbf{u}^*(t, \cdot)$ is nondecreasing. Since $\frac{\partial f_i}{\partial u_j}(t, \mathbf{u}^*) \geq 0$ with $i \neq j$, the conclusion follows from the (strong) maximum principle immediately. \square

Theorem 2.5. *Suppose that (A1) and (A2) are satisfied. Assume that $0 < d \leq 1$ and $a_1(t)p(t) - b_1(t)q(t) \geq a_2(t)p(t) - b_2(t)q(t) \geq 0$ for any $t \in [0, T]$. Then, for each $c < -2\sqrt{(a_1 p - b_1 q)}$, there exists $(U^c, W^c) \in C_b^{1,2}(\mathbb{R} \times \mathbb{R}, \mathbb{R}^2)$ such that (U^c, W^c) and c solve (2.1). Moreover, $(U_z^c, W_z^c) > (0, 0)$ for all $(t, z) \in \mathbb{R} \times \mathbb{R}$.*

Proof. We utilize Lemma 2.4 to establish the existence of a periodic traveling wave solution of (2.1). In order to apply Lemma 2.4, a pair of ordered (irregular) super- and sub-solutions is needed. Set:

$$\kappa_0 := \frac{1}{T} \int_0^T [a_1(t)p(t) - b_1(t)q(t)] dt, \quad \varphi(t) = \exp\left(\int_0^t [a_1(s)p(s) - b_1(s)q(s)] ds - t\kappa_0 \right).$$

It is clear that

$$\kappa_0\varphi(t) = [a_1(t)p(t) - b_1(t)q(t)]\varphi(t) - \varphi'(t).$$

Let $w(t, z) = m\varphi(t)e^{\lambda_c z}$ and $(\bar{U}, \bar{W}) = \min\{(w, w), (1, 1)\}$, where $\lambda_c = \frac{-c - \sqrt{c^2 - 4\kappa_0}}{2}$, $c < -2\sqrt{\kappa_0}$, and $m > 0$ is an arbitrary constant.

We first show that (\bar{U}, \bar{W}) is an irregular super-solution of

$$\begin{cases} U_t = U_{zz} + cU_z + U[a_1(t)p(t)(1 - U) - b_1(t)q(t)(1 - W)], \\ W_t = dW_{zz} + cW_z + (1 - W)[a_2(t)p(t)U - b_2(t)q(t)W]. \end{cases} \quad (2.5)$$

Since $(1, 1)$ is obviously a solution of (2.5), it suffices to show that $(\bar{u}, \bar{v}) = (w, w)$ is a super-solution of (2.5). In fact, a simple calculation yields that

$$\begin{aligned} w_{zz} + cw_z - w_t + w(a_1p(1 - w) - b_1q(1 - w)) &\leq w_{zz} + cw_z - w_t + w(a_1p - b_1q) \\ &= m\varphi e^{\lambda_c z} (\lambda_c^2 + c\lambda_c + \kappa_0) = 0, \\ dw_{zz} + cw_z - w_t + (1 - w)(a_2pw - b_2qw) &\leq w_{zz} + cw_z - w_t + w(a_2p - b_2q) \\ &\leq m\varphi e^{\lambda_c z} (\lambda_c^2 + c\lambda_c + \kappa_0) = 0. \end{aligned}$$

Moreover, we observe that $\{(t, z) \mid 0 \leq \bar{u} \leq 1\} = \{(t, z) \mid 0 \leq \bar{v} \leq 1\}$.

Next we construct a sub-solution. Let $\psi_d(t)$ be the periodic solution of

$$a_2(t)p(t)\varphi(t) - (b_2(t)q(t) + \kappa_0 + (1 - d)\lambda_c^2)v - \frac{dv}{dt} = 0, \quad d \in (0, 1].$$

Notice that ψ_d exists and is unique since $\overline{b_2(t)q(t) + \kappa_0 + (1 - d)\lambda_c^2} > 0$. In particular, ψ_d is strictly positive since $a_2p\varphi > 0$ for all $t \in \mathbb{R}$. Next we fix ϵ such that $\epsilon \in]0, \min\{\lambda_c, \frac{\sqrt{c^2 - 4\kappa_0}}{2}\}[$ and let $\vartheta = -[(\lambda_c + \epsilon)^2 + c(\lambda_c + \epsilon) + \kappa_0]$. Clearly $\vartheta > 0$. Fix n_1 and n_2 such that $n_1 \geq 1$ and $n_2 = \max\{n_1, n_1 \max_t \frac{\psi_d}{\psi_1}\}$. Set:

$$\begin{aligned} \Lambda_\vartheta &= \frac{\vartheta \min\{n_1 \min_t \varphi, n_2 \min_t \psi_1\}}{(1 + n_2 \max_t \{\frac{\psi_1}{\psi_d}\}) \max_t (\varphi^2 + \psi_d^2) \max_t (a_1p + a_2p + b_1q)}, \\ (\underline{U}(t, z), \underline{W}(t, z)) &= \delta \left(e^{\lambda_c z} \varphi(t) (1 - n_1 e^{\epsilon z}), e^{\lambda_c z} \psi_d(t) \left(1 - n_2 \frac{\psi_1(t)}{\psi_d(t)} e^{\epsilon z} \right) \right), \end{aligned}$$

where $\delta \in (0, \Lambda_\vartheta]$. Notice that $(\underline{U}, \underline{W}) \leq (0, 0)$ for all $z \geq z^0 = -\frac{\ln n_1}{\epsilon}$. Moreover, when $(t, z) \in \mathbb{R} \times (-\infty, z^0]$, we have:

$$\begin{aligned} &\underline{U}[a_1p(1 - \underline{U}) - b_1q(1 - \underline{W})] + \underline{U}_{zz} + c\underline{U}_z - \underline{U}_t \\ &= \delta e^{\lambda_c z} \left\{ (a_1p - b_1q)\varphi(1 - n_1 e^{\epsilon z}) - a_1p\delta e^{\lambda_c z} [\varphi(1 - n_1 e^{\epsilon z})]^2 - \varphi'(1 - n_1 e^{\epsilon z}) \right. \\ &\quad \left. + (\lambda_c^2 + c\lambda_c)\varphi - n_1\varphi e^{\epsilon z} [(\lambda_c + \epsilon)^2 + c(\lambda_c + \epsilon)] \right\} + b_1q(\underline{U}\underline{W}) \\ &\geq \delta e^{\lambda_c z} \left\{ (a_1p - b_1q)\varphi - \kappa_0\varphi - \varphi' - n_1 e^{\epsilon z} [(a_1p - b_1q)\varphi + (\lambda_c + \epsilon)^2 + c(\lambda_c + \epsilon)\varphi - \varphi'] \right. \\ &\quad \left. - \delta\varphi e^{\lambda_c z} (1 - n_1 e^{\epsilon z}) \left[a_1p\varphi(1 - n_1 e^{\epsilon z}) + b_1q\psi_d \left| 1 - n_2 \frac{\psi_1}{\psi_d} e^{\epsilon z} \right| \right] \right\} \\ &= \delta e^{\lambda_c z} \left\{ n_1\vartheta\varphi e^{\epsilon z} - \delta\varphi e^{\lambda_c z} (1 - n_1 e^{\epsilon z}) \left[a_1p\varphi(1 - n_1 e^{\epsilon z}) + b_1q\psi_d \left| 1 - n_2 \frac{\psi_1}{\psi_d} e^{\epsilon z} \right| \right] \right\} \geq 0, \end{aligned}$$

and

$$\begin{aligned} &(1 - \underline{W})[a_2p\underline{U} - b_2q\underline{W}] + d\underline{W}_{zz} + c\underline{W}_z - \underline{W}_t \\ &= \delta e^{\lambda_c z} \left\{ a_2p\varphi(1 - n_1 e^{\epsilon z}) - b_2q(\psi_d - n_2 e^{\epsilon z} \psi_1) - \delta a_2p e^{\lambda_c z} \left[\varphi(1 - n_1 e^{\epsilon z}) \psi_d \left(1 - n_2 \frac{\psi_1}{\psi_d} e^{\epsilon z} \right) \right] \right\} \end{aligned}$$

$$\begin{aligned}
 & + (d\lambda_c^2 + c\lambda_c)\psi_d - n_2\psi_1 e^{\epsilon z} [d(\lambda_c + \epsilon)^2 + c(\lambda_c + \epsilon)] - (\psi'_d - n_2\psi'_1 e^{\epsilon z}) \Big\} + b_2q(\underline{W})^2 \\
 \geq & \delta e^{\lambda_c z} \left\{ a_2 p \varphi - (b_2 q + \kappa_0 + (1-d)\lambda_c^2)\psi_d - \psi'_d - \delta a_2 p e^{\lambda_c z} \left[\varphi(1 - n_1 e^{\epsilon z})\psi_d \left(1 - n_2 \frac{\psi_1}{\psi_d} e^{\epsilon z} \right) \right] \right\} \\
 & - n_2 \delta e^{(\lambda_c + \epsilon)z} \{ a_2 p \varphi - [b_2 q - (\lambda_c + \epsilon)^2 - c(\lambda_c + \epsilon)]\psi_1 - \psi'_1 + (d-1)(\lambda_c + \epsilon)^2 \psi_1 \} \\
 \geq & \delta e^{\lambda_c z} \left\{ n_2 \vartheta \psi_1 e^{\epsilon z} - \delta a_2 p e^{\lambda_c z} \left[\varphi(1 - n_1 e^{\epsilon z})\psi_d \left(1 - n_2 \frac{\psi_1}{\psi_d} e^{\epsilon z} \right) \right] \right\} \geq 0.
 \end{aligned}$$

Thus, $(\underline{U}, \underline{W})$ is a (regular) sub-solution of (2.5) in $\mathbb{R} \times (-\infty, z^0]$.

Note that both $(\overline{U}, \overline{W})$ and $(\underline{U}, \underline{W})$ are periodic in t , and $(\overline{U}, \overline{W})$ is nondecreasing with respect to z . Moreover, as m is arbitrary, we have that $(\overline{U}, \overline{W}) \geq (\underline{U}, \underline{W})$ for all $(t, z) \in \mathbb{R} \times \mathbb{R}$ as long as m is sufficiently large. Therefore, Lemma 2.4 implies that for each $c < -2\sqrt{\kappa_0}$, there exists $(U^c, W^c) \in C_b^{1,2}(\mathbb{R} \times \mathbb{R}, \mathbb{R}^2)$ such that (U^c, W^c) and c solve (2.5) and $(U^c(\cdot + T, z), W^c(\cdot + T, z)) = (U^c(\cdot, z), W^c(\cdot, z))$. In addition, $(U^c_z, W^c_z) > (0, 0)$ for all $(t, z) \in \mathbb{R} \times \mathbb{R}$. It remains to show that

$$\lim_{z \rightarrow -\infty} (U^c(t, z), W^c(t, z)) = (0, 0), \quad \lim_{z \rightarrow \infty} (U^c(t, z), W^c(t, z)) = (1, 1).$$

Notice that for each $t \in \mathbb{R}$, $\lim_{z \rightarrow \pm\infty} (U^c(t, z), W^c(t, z))$ exist since $(U^c_z, W^c_z) > (0, 0)$. Let

$$(U^c(t, \pm\infty), W^c(t, \pm\infty)) = \lim_{z \rightarrow \pm\infty} (U^c(t, z), W^c(t, z)),$$

respectively. It is easy to see that

$$(U^c(\cdot + T, \infty), W^c(\cdot + T, \infty)) = (U^c(\cdot, \infty), W^c(\cdot, \infty)).$$

Thanks to the regularity of (U^c, W^c) with respect to t and the compactness of $[0, T]$, we find that $(U^c(t, z), W^c(t, z)) \rightarrow (U^c(t, \pm\infty), W^c(t, \pm\infty))$ uniformly in $t \in \mathbb{R}$ as $z \rightarrow \pm\infty$. Since

$$\lim_{z \rightarrow -\infty} (\underline{U}, \underline{W}) = \lim_{z \rightarrow -\infty} (\overline{U}, \overline{W}) = (0, 0),$$

it follows that $(U^c(t, -\infty), W^c(t, -\infty)) = (0, 0)$. On the other hand, since there exists $z_* \leq z^0$ such that $(0, 0) < (\underline{U}, \underline{W}) \leq (U^c, W^c) \leq (\overline{U}, \overline{W})$ for all $z \in (-\infty, z_*]$, and $(0, 0) \leq (U^c, W^c) \leq (1, 1)$ for all $(t, z) \in \mathbb{R} \times \mathbb{R}$, we have $(0, 0) < (U^c(t, \infty), W^c(t, \infty)) \leq (1, 1)$ for all $t \in \mathbb{R}$. Moreover, Barbălat's lemma shows that $\lim_{z \rightarrow \infty} (U^c_{zz}, W^c_{zz}) = \lim_{z \rightarrow \infty} (U^c_z, W^c_z) = (0, 0)$. Thus, $(U^c(t, \infty), W^c(t, \infty))$ is a positive periodic solution of

$$\begin{cases} \frac{du}{dt} = u[a_1(t)p(t)(1-u) - b_1(t)q(t)(1-v)], \\ \frac{dv}{dt} = (1-v)[a_2(t)p(t)u - b_2(t)q(t)v]. \end{cases} \tag{2.6}$$

Due to (A1) and (A2), (1.3) has three and only three nonnegative periodic solutions $(p(t), 0)$, $(q(t), 0)$, and $(0, 0)$, where $p(t)$ and $q(t)$ are given by (1.5). Under the transformations in (1.6), these periodic states are converted to $(1, 1)$, $(0, 0)$ and $(0, 1)$, respectively. They constitute all the periodic solutions of (2.6) confined within $[0, 1] \times [0, 1]$. Consequently, $(U^c(t, \infty), W^c(t, \infty)) = (1, 1)$. The proof is completed. \square

Theorem 2.6. *Suppose that all the assumptions in Theorem 2.5 are satisfied. Assume that $c = c^* = -2\sqrt{(a_1 p - b_1 q)}$. Then there exists $(U^{c^*}, W^{c^*}) \in C_b^{1,2}(\mathbb{R} \times \mathbb{R}, \mathbb{R}^2)$ such that (U^{c^*}, W^{c^*}) and c^* solve (2.1). Moreover, $(U^{c^*}_z, W^{c^*}_z) > (0, 0)$ for all $(t, z) \in \mathbb{R} \times \mathbb{R}$.*

Proof. Let (U^c, W^c) be the time periodic wave solution of (2.1) with $c < c^*$. Since $|U^c|$ and $|W^c|$ are uniformly bounded, it follows from parabolic estimates that

$$\|U^c\|_{C^{1+\frac{\alpha}{2}, 2+\alpha}(\mathbb{R} \times \mathbb{R}, \mathbb{R})} + \|W^c\|_{C^{1+\frac{\alpha}{2}, 2+\alpha}(\mathbb{R} \times \mathbb{R}, \mathbb{R})} < \infty \quad \text{uniformly for } c \in [c^* - 1, c^*),$$

for some $\alpha \in]0, 1[$. Let $\{c_n\}$ be any sequence with $c_n \in [c^* - 1, c^*)$ such that $c_n \rightarrow c^*$ as $n \rightarrow \infty$. By taking a subsequence of $\{(U^{c_n}, W^{c_n})\}$ if necessary (which will be denoted by $\{(U^{c_n}, W^{c_n})\}$ for convenience), we infer that $\{(U^{c_n}, W^{c_n})\}$ converges in $C_{loc}^{1,2}(\mathbb{R}^2, \mathbb{R}) \times C_{loc}^{1,2}(\mathbb{R}^2, \mathbb{R})$ to a function denoted by (U^{c^*}, W^{c^*}) . By taking the limits in (2.5), we see that (U^{c^*}, W^{c^*}) satisfies (2.5). In addition, by Helly's theorem, we can conclude that

$$\lim_{n \rightarrow \infty} [\|U^{c_n} - U^{c^*}\|_{C([0,T] \times \mathbb{R}, \mathbb{R})} + \|W^{c_n} - W^{c^*}\|_{C([0,T] \times \mathbb{R}, \mathbb{R})}] = 0. \tag{2.7}$$

This further implies that $\lim_{n \rightarrow \infty} [\|U^{c_n} - U^{c^*}\|_{C(\mathbb{R} \times \mathbb{R}, \mathbb{R})} + \|W^{c_n} - W^{c^*}\|_{C(\mathbb{R} \times \mathbb{R}, \mathbb{R})}] = 0$ since (U^{c_n}, W^{c_n}) is periodic in t . Clearly, $(U^{c^*}(t + T, \cdot), W^{c^*}(t + T, \cdot)) = (U^{c^*}(t, \cdot), W^{c^*}(t, \cdot))$ and $(U_z^{c^*}, W_z^{c^*}) \geq (0, 0)$. Since $(U^{c^*}(t, z + s), W^{c^*}(t, z + s))$ is a solution as well, where $s > 0$ is arbitrary, the (strong) maximum principle implies that either $(U_z^{c^*}, W_z^{c^*}) > (0, 0)$ or $(U_z^{c^*}, W_z^{c^*}) \equiv (0, 0)$. In light of (2.7) and the fact that $\lim_{z \rightarrow -\infty} (U^{c_n}, W^{c_n}) = (0, 0)$ and $\lim_{z \rightarrow \infty} (U^{c_n}, W^{c_n}) = (1, 1)$ for each n , there exists $M > 0$ such that $(U^{c^*}, W^{c^*}) \leq (\frac{1}{4}, \frac{1}{4})$ for all $(t, z) \in \mathbb{R} \times (-\infty, -M]$ while $(U^{c^*}, W^{c^*}) \geq (\frac{1}{2}, \frac{1}{2})$ for all $(t, z) \in \mathbb{R} \times [M, \infty)$. Hence we must have $(U_z^{c^*}, W_z^{c^*}) > (0, 0)$. Moreover, it is easy to see that

$$\lim_{z \rightarrow -\infty} (U^{c^*}, W^{c^*}) = (0, 0), \quad \lim_{z \rightarrow \infty} (U^{c^*}, W^{c^*}) = (1, 1).$$

The proof is completed. \square

Corollary 2.7. *Suppose that (A1) is satisfied. Assume that r_1 and r_2 are strictly positive in $[0, T]$ such that $a_1(t) \min_t \frac{r_1}{a_1} \geq b_2(t) \max_t \frac{r_2}{b_2}$ for all $t \in [0, T]$. Moreover, assume that $0 < d \leq 1$, $\frac{a_2(t)}{a_1(t)} \geq 1 > \frac{b_1(t)}{b_2(t)}$, and $[b_2(t) - b_1(t)] \min_t \frac{r_2}{b_2} \geq [a_2(t) - a_1(t)] \max_t \frac{r_1}{a_1}$ for all $t \in [0, T]$. Then, for each $c \leq c^* = -2\sqrt{(a_1 p - b_1 q)}$, there exists $(U^c, W^c) \in C_b^{1,2}(\mathbb{R} \times \mathbb{R})$ such that (U^c, W^c) and c solve (2.1). In addition, $(U_z^c, W_z^c) > (0, 0)$ for all $(t, z) \in \mathbb{R} \times \mathbb{R}$.*

Proof. Let $p(t)$ and $q(t)$ be given by (1.5). Since $p(t)$ and $q(t)$ are periodic functions that satisfy:

$$p'(t) = p(t)(r_1(t) - a_1(t)p(t)), \quad q'(t) = q(t)(r_2(t) - b_2(t)q(t)),$$

respectively, the comparison principle implies that

$$\min_{0 \leq t \leq T} \frac{r_1(t)}{a_1(t)} \leq p(t) \leq \max_{0 \leq t \leq T} \frac{r_1(t)}{a_1(t)}, \quad \min_{0 \leq t \leq T} \frac{r_2(t)}{b_2(t)} \leq q(t) \leq \max_{0 \leq t \leq T} \frac{r_2(t)}{b_2(t)}. \tag{2.8}$$

This together with the assumption implies that

$$a_1(t)p(t) \geq a_1(t) \min_{0 \leq t \leq T} \frac{r_1(t)}{a_1(t)} \geq b_2(t) \max_{0 \leq t \leq T} \frac{r_2(t)}{b_2(t)} \geq b_2(t)q(t).$$

As $\bar{r}_1 = \overline{a_1 p}$ and $\bar{r}_2 = \overline{b_2 q}$, by the assumption that $\frac{a_2(t)}{a_1(t)} \geq 1 > \frac{b_1(t)}{b_2(t)}$ for all $t \in \mathbb{R}$, we readily verify that (A2) holds. It follows from the assumption and (2.8) that

$$[b_2(t) - b_1(t)]q(t) \geq [b_2(t) - b_1(t)] \min_{0 \leq t \leq T} \frac{r_2(t)}{b_2(t)} \geq [a_2(t) - a_1(t)] \max_{0 \leq t \leq T} \frac{r_1(t)}{a_1(t)} \geq [a_2(t) - a_1(t)]p(t).$$

Namely, $a_1(t)p(t) - b_1(t)q(t) \geq a_2(t)p(t) - b_2(t)q(t)$ for all $t \in [0, T]$. In addition, we have $a_2(t)p(t) \geq a_1(t)p(t) \geq b_2(t)q(t)$. Therefore, the conclusion follows from Theorems 2.5 and 2.6. \square

3. Uniqueness and monotonicity of periodic traveling wave solutions

In this section we study the uniqueness and monotonicity of periodic traveling waves of (2.1). We consider the following general system:

$$\begin{cases} u_t = u_{zz} + cu_z + g(t, u, v), \\ v_t = dv_{zz} + cv_z + h(t, u, v), \\ (0, 0) \leq (u, v) \leq (1, 1); \\ (u(t, z), v(t, z)) = (u(t + T, z), v(t + T, z)), \\ \lim_{z \rightarrow -\infty} (u, v) = (0, 0), \quad \lim_{z \rightarrow \infty} (u, v) = (1, 1), \end{cases} \tag{3.1}$$

where $0 < d \leq 1$, $g \in C^{\theta,2}(\mathbb{R} \times \mathbb{R}^2, \mathbb{R})$, $h \in C^{\theta,2}(\mathbb{R} \times \mathbb{R}^2, \mathbb{R})$ for some $\theta \in]0, 1[$, and $g(t + T, u, v) = g(t, u, v)$, $h(t + T, u, v) = h(t, u, v)$ for any $(t, u, v) \in \mathbb{R} \times \mathbb{R}^2$.

Throughout this section, we assume that

- (H1) $\overline{g_u(t, 1, 1)} = h(t, 0, 0) = h(t, 1, 1) = 0$ for any $t \in \mathbb{R}$. Moreover, $g(t, 0, v) \equiv 0$ for all $(t, v) \in \mathbb{R} \times \mathbb{R}^+$, and $\overline{g_u(t, 0, 0)} > 0$.
- (H2) $g_v(t, u, v) \geq 0$ for all $(t, u, v) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}$, and $h_u(t, u, v) \geq 0$ for all $(t, u, v) \in \mathbb{R} \times \mathbb{R} \times [0, 1]$.

In what follows, we set:

$$\kappa = \overline{g_u(t, 0, 0)}, \quad \phi(t) = e^{\int_0^t g_u(s,0,0) ds - \kappa t}, \quad c^* = -2\sqrt{\kappa}, \quad \lambda_c = \frac{-c - \sqrt{c^2 - 4\kappa}}{2} \quad \text{if } c \leq c^*. \quad (3.2)$$

- (H3) $\overline{h_v(t, 0, 0)} < 0$ and $h_u(t, 0, 0) \not\equiv 0$.

(H4) Let:

$$\bar{\mathbf{w}} = \begin{cases} (me^{\lambda_c z} \phi(t), me^{\lambda_c z} \phi(t)), & (t, z) \in \mathbb{R} \times \mathbb{R}, \quad \text{if } c < c^*, \\ ((m - nz)e^{\lambda_c z} \phi(t), (m - nz)e^{\lambda_c z} \phi(t)), & (t, z) \in \mathbb{R} \times (-\infty, \frac{m}{n} - \frac{2}{\sqrt{\kappa}}], \quad \text{if } c = c^*, \end{cases}$$

where m and n are arbitrary positive constants. Assume that $\bar{\mathbf{w}}$ is a (regular) super-solution of

$$\begin{cases} u_t = u_{zz} + cu_z + g(t, u, v), \\ v_t = dv_{zz} + cv_z + h(t, u, v). \end{cases} \quad (3.3)$$

- (H5) Let ν be a characteristic exponent of

$$\frac{d\mathbf{w}}{dt} - A(t)\mathbf{w} = 0,$$

where $A(t) = \begin{pmatrix} g_u(t,1,1) & g_v(t,1,1) \\ h_u(t,1,1) & h_v(t,1,1) \end{pmatrix}$. Let $\begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix}$ be the eigensolution associated with ν . Assume that $\nu < 0$, and both φ_1 and φ_2 are strictly positive in $[0, T]$.

Lemma 3.1. *Suppose that $(\underline{u}, \underline{v}) \in C^{1,2}(\mathbb{R} \times \mathbb{R}, [0, 1]^2)$ is a regular sub-solution of (3.3) and is T -periodic in t . Assume that $(\bar{u}, \bar{v}) \in C_b^{\theta/2, \theta}(\mathbb{R} \times \mathbb{R})$ is T -periodic in t , and $(\bar{u}(t, \cdot), \bar{v}(t, \cdot))$ is nondecreasing, where $0 < \theta < 1$. In addition, there exists $\bar{\sigma} \in \mathbb{R}$ such that $(\bar{u}(t, z), \bar{v}(t, z)) = (1, 1)$ for any $(t, z) \in \mathbb{R} \times [\bar{\sigma}, +\infty)$. Moreover, $(\bar{u}(t, z), \bar{v}(t, z))$ satisfies:*

$$\begin{cases} \bar{u}_t \geq \bar{u}_{zz} + c\bar{u}_z + g(t, \bar{u}, \bar{v}), \\ \bar{v}_t \geq d\bar{v}_{zz} + c\bar{v}_z + h(t, \bar{u}, \bar{v}), \end{cases}$$

whenever $(\bar{u}, \bar{v}) \leq (1, 1)$, and $\{(t, z) \in \mathbb{R}^2 \mid \bar{u}(t, z) < 1\} = \{(t, z) \in \mathbb{R}^2 \mid \bar{v}(t, z) < 1\}$. If there exists $\sigma < \bar{\sigma}$ such that $(\underline{u}(t, \sigma), \underline{v}(t, \sigma)) < (\bar{u}(t, \sigma), \bar{v}(t, \sigma))$ for all $t \in \mathbb{R}$. Then $(\underline{u}, \underline{v}) < (\bar{u}, \bar{v})$ for any $(t, z) \in \mathbb{R} \times [\sigma, +\infty)$.

Proof. The proof is similar to that of Lemma 3.1(1) of [18]. Define:

$$\begin{aligned} \theta^* &= \inf\{\theta \in [0, \infty) \mid \underline{u}(t, z - \theta) \leq \bar{u}(t, z) \text{ for all } (t, z) \in \mathbb{R} \times [\sigma + \theta, +\infty)\}, \\ \theta_* &= \inf\{\theta \in [0, \infty) \mid \underline{v}(t, z - \theta) \leq \bar{v}(t, z) \text{ for all } (t, z) \in \mathbb{R} \times [\sigma + \theta, +\infty)\}. \end{aligned}$$

From the assumptions it follows that $(\underline{u}(t, z - \theta), \underline{v}(t, z - \theta)) < (\bar{u}(t, z), \bar{v}(t, z))$ for all $(t, z, \theta) \in \mathbb{R} \times [\sigma + \theta, +\infty) \times [\bar{\sigma} - \sigma, +\infty)$. Thus, $\theta^*, \theta_* \in [0, \bar{\sigma} - \sigma)$. Assume without loss of generality that $\theta^* = \max\{\theta^*, \theta_*\}$. We claim that $\theta^* = 0$.

Assume to the contrary that $\theta^* \neq 0$. Then there exist two sequences $\{\theta_n\}_{n \in \mathbb{N}}$ and $\{(t_n, z_n)\}_{n \in \mathbb{N}}$ such that $\theta_n \rightarrow \theta^*$ as $n \rightarrow \infty$, $0 < \theta_n < \theta^*$, $z_n \geq \sigma + \theta_n$, and

$$\underline{u}(t_n, z_n - \theta_n) > \bar{u}(t_n, z_n), \quad \lim_{n \rightarrow \infty} [\underline{u}(t_n, z_n - \theta_n) - \bar{u}(t_n, z_n)] = 0.$$

Notice that $\{z_n\}$ is bounded and hence there exists a subsequence of $\{z_n\}$, still labeled by $\{z_n\}$, such that $z_n \rightarrow z^* \in [\sigma + \theta^*, +\infty)$ as $n \rightarrow \infty$. Since (\bar{u}, \bar{v}) and $(\underline{u}, \underline{v})$ are T -periodic in t , we may assume without loss of generality that $t_n \in [0, T]$ for each n , and that $t_n \rightarrow t^*$ as $n \rightarrow \infty$. Thus, $\underline{u}(t^*, z^* - \theta^*) = \bar{u}(t^*, z^*)$ and $\underline{v}(t^*, z^* - \theta^*) \leq \bar{v}(t^*, z^*)$. In view of the assumptions, we have:

$$(\underline{u}(t^*, \sigma), \underline{v}(t^*, \sigma)) < (\bar{u}(t^*, \sigma), \bar{v}(t^*, \sigma)) \leq (\bar{u}(t^*, \sigma + \theta^*), \bar{v}(t^*, \sigma + \theta^*)). \tag{3.4}$$

Therefore, $z^* > \sigma + \theta^*$. In particular, since $\underline{u}(t^*, z^* - \theta^*) = \bar{u}(t^*, z^*) < 1$ and $(\bar{u}(t, \cdot), \bar{v}(t, \cdot))$ is monotone and (\bar{u}, \bar{v}) is Hölder continuous, $[t^* - \varepsilon, t^* + \varepsilon] \times [\sigma + \theta^*, z^* + \varepsilon] \subset \{(t, z) \mid \bar{u} < 1\}$ for some $\varepsilon > 0$ which is sufficiently small. Write,

$$(\underline{u}^{-\theta^*}(t, z), \underline{v}^{-\theta^*}(t, z)) = (\underline{u}(t, z - \theta^*), \underline{v}(t, z - \theta^*)), \quad (\tilde{u}, \tilde{v}) = (\underline{u}^{-\theta^*} - \bar{u}, \underline{v}^{-\theta^*} - \bar{v}).$$

Note that $(\tilde{u}(t, z), \tilde{v}(t, z)) \leq (0, 0)$ for any $(t, z) \in \mathbb{R} \times [\sigma + \theta^*, +\infty)$ and $\tilde{u}(t^*, z^*) = 0$. Since $g_v \geq 0$, we have:

$$\left[\int_0^1 g_u(t, \tau \underline{u}^{-\theta^*} + (1 - \tau)\bar{u}, \tau \underline{v}^{-\theta^*} + (1 - \tau)\bar{v}) d\tau \right] \tilde{u} + \tilde{u}_{zz} + c\tilde{u}_z - \tilde{u}_t \geq 0 \quad \text{in } \{\bar{u} < 1\} \cap \mathbb{R} \times [\sigma + \theta^*, +\infty),$$

it then follows from the (strong) maximum principle that $\underline{u}(t, z - \theta^*) = \bar{u}(t, z)$ for all $(t, z) \in (-\infty, t^*] \times [\sigma + \theta^*, z^*] \cap \{(t, z) \mid \bar{u} < 1\}$. Therefore, we have $\underline{u}(t^*, \sigma) = \bar{u}(t^*, \sigma + \theta^*)$, which however contradicts (3.4). Hence $\theta^* = 0$. Moreover, thanks to the maximum principle, it is easy to see that $(\underline{u}, \underline{v}) < (\bar{u}, \bar{v})$ for all $(t, z) \in \mathbb{R} \times [\sigma, +\infty)$. The proof is completed. \square

Lemma 3.2. ([13]) *Let the differential operators $L_k := \sum_{i,j=1}^n a_{i,j}^k(t, \mathbf{x}) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i^k \frac{\partial}{\partial x_i} - \frac{\partial}{\partial t}$ ($k = 1, 2, \dots, l$) be uniformly parabolic in an open domain $] \tau, M[\times \Omega$ of $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^n$, that is, there is $\alpha_0 > 0$ such that $a_{i,j}^k(t, \mathbf{x}) \xi_i \xi_j \geq \alpha_0 \sum_{i=1}^n \xi_i^2$ for any n -tuples of real numbers $(\xi_1, \xi_2, \dots, \xi_n)$, where $-\infty < \tau < M \leq \infty$ and Ω is open and bounded. Suppose that $a_{i,j}^k, b_i^k \in C(] \tau, M[\times \Omega, \mathbb{R})$ and $\max_{(t, \mathbf{x}) \in] \tau, M[\times \Omega} |b_i^k(t, \mathbf{x})| + |a_{i,j}^k(t, \mathbf{x})| \leq \beta_0$ for some $\beta_0 > 0$. Assume that $\mathbf{w} = (w_1, w_2, \dots, w_l) \in C(] \tau, M[\times \bar{\Omega}, \mathbb{R}^l) \cap C^{1,2}(] \tau, M[\times \Omega, \mathbb{R}^l)$ satisfies:*

$$\sum_{s=1}^l c^{k,s}(t, \mathbf{x}) w_s + L_k w_k \leq 0, \quad (t, x) \in] \tau, M[\times \Omega, \quad k = 1, 2, \dots, l, \tag{3.5}$$

where $c^{k,s} \in C(] \tau, M[\times \Omega, \mathbb{R})$ and $c^{k,s} \geq 0$ if $k \neq s$, and $\max_{(t, \mathbf{x}) \in] \tau, M[\times \Omega} |c^{k,s}(t, \mathbf{x})| \leq \gamma_0$ ($k, s = 1, 2, \dots, l$) for some $\gamma_0 > 0$. Let D and U be domains in Ω such that $D \subset \subset U$, $\text{dist}(\bar{D}, \partial U) > \varrho$, and $|D| > \varepsilon$ for certain positive constants ϱ and ε . Let θ be a positive constant with $\tau + 4\theta < M$. Then there exist positive constants p, ω_1 and ω_2 , determined only by $\alpha_0, \beta_0, \gamma_0, \varrho, \varepsilon, n, \text{diam}\Omega$ and θ , such that

$$\inf_{] \tau + 3\theta, \tau + 4\theta[\times D} w_k \geq \omega_1 \| (w_k)^+ \|_{L^p(] \tau + \theta, \tau + 2\theta[\times D)} - \omega_2 \max_{j=1, \dots, k} \sup_{\partial_P(] \tau, \tau + 4\theta[\times U)} (w_j)^-.$$

Here $(w_k)^+ = \max\{w_k, 0\}$, $(w_k)^- = \max\{-w_k, 0\}$, and $\partial_P(] \tau, \tau + 4\theta[\times U) = \{\tau\} \times U \cup [\tau, \tau + 4\theta] \times \partial U$. Moreover, if all inequalities in (3.5) are replaced by equalities, then the conclusion holds with $p = \infty$, and with ω_1, ω_2 independent of ε .

Proof. See Lemma 3.6 of [13] for a proof. \square

Lemma 3.3. *Suppose that (H1)–(H4) are satisfied. Assume that $(u, v) \in C_b^{1,2}(\mathbb{R} \times \mathbb{R})$ and c solve (3.1). Then there exist positive constants K_i ($i = 1, 2, 3$) and $M > 0$ such that for all $(t, z) \in \mathbb{R} \times (-\infty, -M]$,*

$$K_1 e^{\lambda_c z} \leq u(t, z) \leq K_2 e^{\lambda_c z}, \quad 0 < v(t, z) \leq K_3 e^{\lambda_c z}, \quad \text{if } c < c^*, \tag{3.6}$$

and

$$K_1 |z| e^{\lambda_c z} \leq u(t, z) \leq K_2 |z| e^{\lambda_c z}, \quad 0 < v(t, z) \leq K_3 |z| e^{\lambda_c z}, \quad \text{if } c = c^*. \tag{3.7}$$

Proof. Thanks to (H1), (3.1) can be written as

$$\begin{cases} u_t = u_{zz} + cu_z + \int_0^1 (g_u(t, \tau u, \tau v) d\tau)u + \int_0^1 (g_v(t, \tau u, \tau v) d\tau)v, \\ v_t = dv_{zz} + cv_z + \int_0^1 (h_u(t, \tau u, \tau v) d\tau)u + \int_0^1 (h_v(t, \tau u, \tau v) d\tau)v. \end{cases} \quad (3.8)$$

Let $D =]z - \frac{1}{4}, z + \frac{1}{4}[$, $U =]z - \frac{1}{2}, z + \frac{1}{2}[$, $\Omega =]z - 1, z + 1[$ with $z \in \mathbb{R}$, $\tau = 0$, and $\theta = T$. Since $u(\cdot, z)$ and $v(\cdot, z)$ are periodic and u and v are both positive and bounded by 1, applying Lemma 3.2 to (3.8) yields,

$$(u(t, z), v(t, z)) \leq N_1(u(t', z), v(t', z)) \quad \text{for all } z \in \mathbb{R} \text{ and all } t, t' \in \mathbb{R}, \quad (3.9)$$

where N_1 is a positive constant independent of u and v . Now let

$$\alpha = \overline{h_v(t, 0, 0)}, \quad \tilde{\phi}(t) = \exp\left(\int_0^t h_v(s, 0, 0) ds - t\alpha\right), \quad \hat{u} = \int_0^T \frac{u}{\phi} dt, \quad \hat{v} = \int_0^T \frac{v}{\tilde{\phi}} dt. \quad (3.10)$$

Then a straightforward calculation yields,

$$\begin{cases} \hat{u}_{zz} + c\hat{u}_z + \kappa\hat{u} + \int_0^T \frac{g(t, u, v) - g_u(t, 0, 0)u}{\phi} dt = 0, \\ d\hat{v}_{zz} + c\hat{v}_z + \alpha\hat{v} + \int_0^T \frac{h_u(t, 0, 0)u}{\tilde{\phi}} dt + \int_0^T \frac{h(t, u, v) - h_u(t, 0, 0)u - h_v(t, 0, 0)v}{\tilde{\phi}} dt = 0. \end{cases} \quad (3.11)$$

Note that $\alpha < 0$ because of (H3). Moreover, thanks to (H1), it is not difficult to see that

$$\begin{aligned} |[g(t, u, v) - g_u(t, 0, 0)u]\phi^{-1}| &\leq C(|u|^2 + |u||v|), \\ |[h(t, u, v) - h_u(t, 0, 0)u - h_v(t, 0, 0)v]\tilde{\phi}^{-1}| &\leq C(|u|^2 + |uv| + |v|^2) \end{aligned}$$

for some constant $C > 0$, which is independent of u, v, t and z . In view of (3.9) and the positivity of u and v , it follows that

$$\begin{aligned} \int_0^T |[g(t, u, v) - g_u(t, 0, 0)u]\phi^{-1}| dt &\leq C(|\hat{u}|^2 + |\hat{u}||\hat{v}|), \\ \int_0^T |[h(t, u, v) - h_u(t, 0, 0)u - h_v(t, 0, 0)v]\tilde{\phi}^{-1}| dt &\leq C(|\hat{u}|^2 + |\hat{u}||\hat{v}| + |\hat{v}|^2), \end{aligned}$$

for some positive constant C independent of t and z . Hence, we have:

$$\int_0^T |[g(t, u, v) - g_u(t, 0, 0)u]\phi^{-1}| dt = o(|\hat{u}|) \quad \text{as } z \rightarrow -\infty. \quad (3.12)$$

Furthermore, choose $\varepsilon > 0$ such that $\varepsilon \leq -\frac{\alpha}{2}$. Then there exists $M_\varepsilon > 0$ such that

$$\int_0^T |h_u(t, 0, 0)u\tilde{\phi}^{-1}| dt + \int_0^T |[h(t, u, v) - h_u(t, 0, 0)u - h_v(t, 0, 0)v]\tilde{\phi}^{-1}| dt \leq C_\varepsilon\hat{u} + \varepsilon\hat{v}, \quad z \leq -M_\varepsilon,$$

where $C_\varepsilon = \max_{0 \leq t \leq T} [h_u(t, 0, 0) + \varepsilon] \tilde{\phi}^{-1}$. Consequently,

$$d\hat{v}_{zz} + c\hat{v}_z + \frac{\alpha}{2}\hat{v} + C_\varepsilon\hat{u} \geq 0, \quad z \leq -M_\varepsilon. \tag{3.13}$$

In order to prove (3.6) and (3.7), we need to distinguish between two cases.

Case I. $c < c^* = -2\sqrt{\kappa}$. We begin by showing that

$$\hat{u}(z) = m_1 e^{\lambda_c z} + O(e^{(\lambda_c + \epsilon)z}) \quad \text{as } z \rightarrow -\infty \tag{3.14}$$

for certain positive constants m_1 and ϵ . In view of (3.12), from the standard differential equation theory (see Theorem 2.4 of [17]), it follows that $\hat{u} = O(e^{(\lambda_c - \delta)z})$ as $z \rightarrow -\infty$, where $\delta > 0$ is sufficiently small such that $\delta < \frac{\lambda_c}{2}$. Since $\hat{u} = O(e^{(\lambda_c - \delta)z})$ as $z \rightarrow -\infty$, there exist positive constants C_δ and M_δ such that $\hat{u} \leq C_\delta e^{(\lambda_c - \delta)z}$ whenever $z \leq -M_\delta$. Assume without loss of generality that $M_\varepsilon \geq M_\delta$, then it is easy to verify that for any $m \geq \frac{2C_\varepsilon C_\delta}{\alpha}$, $w = m e^{(\lambda_c - \delta)z}$ satisfies:

$$dw_{zz} + cw_z + \frac{\alpha}{2}w + C_\varepsilon\hat{u} \leq 0, \quad z \leq -M_\varepsilon. \tag{3.15}$$

Since \hat{v} is bounded, we can select an $m_\delta > 0$ such that $m_\delta e^{-(\lambda_c - \delta)M_\varepsilon} \geq \hat{v}(-M_\varepsilon)$.

We next show that $w_\delta = m_\delta e^{(\lambda_c - \delta)z} \geq \hat{v}(z)$ for all $z \leq -M_\varepsilon$, namely, $\inf_{z \leq -M_\varepsilon} \{w_\delta - \hat{v}\} \geq 0$. Suppose that this is not true, then, since $w_\delta - \hat{v} \rightarrow 0$ as $z \rightarrow -\infty$, there exists a finite point $z^* < -M_\varepsilon$ such that $w_\delta(z^*) - \hat{v}(z^*) = \inf_{z \leq -M_\varepsilon} \{w_\delta - \hat{v}\} < 0$. Notice that

$$d(w - \hat{v})_{zz} + c(w - \hat{v})_z + \frac{\alpha}{2}(w - \hat{v}) \leq 0, \quad z \leq -M_\varepsilon.$$

As $w_\delta - \hat{v}$ attains its global minimum in $(-\infty, -M_\varepsilon)$ at $z = z^*$, we have that $(w - \hat{v})_{zz}(z^*) \geq 0$ and $(w - \hat{v})_z(z^*) = 0$, which forces that $\frac{\alpha}{2}(w_\delta - \hat{v})(z^*) \leq 0$. This is clearly impossible since $\frac{\alpha}{2} < 0$. Therefore, $\hat{v}(z) \leq m_\delta e^{(\lambda_c - \delta)z}$ when $z \leq -M_\varepsilon$. In other words, $\hat{v} = O(e^{(\lambda_c - \delta)z})$ as $z \rightarrow -\infty$. This implies that

$$\int_0^T |[g(t, u, v) - g_u(t, 0, 0)u] \phi^{-1}| dt = O(e^{2(\lambda_c - \delta)z}) \quad \text{as } z \rightarrow -\infty.$$

Hence, (3.14) follows from Proposition 6.1 of [33] (see also [34]). Clearly, $m_1 \geq 0$ since $\hat{u} > 0$. If $m_1 > 0$, then for sufficient large m , it is easy to see $m e^{\lambda_c z}$ satisfies (3.15) for $z \leq 0$ with $|z|$ sufficiently large, and by arguing in the same way, we can infer that $\hat{v}(z) = O(e^{\lambda_c z})$ as $z \rightarrow -\infty$. Furthermore, (3.14) implies that there exist positive constants C and K such that

$$\inf_{t \in [0, T]} u(t, z) \leq \frac{2C}{T} \sup_{t \in [0, T]} \phi(t) e^{\lambda_c z}, \quad \sup_{t \in [0, T]} u(t, z) \geq \frac{C}{2T} \inf_{t \in [0, T]} \phi(t) e^{\lambda_c z}$$

whenever $z \leq -K$. Thus the first compound inequality of (3.6) follows from (3.9) immediately provided that $m_1 \neq 0$. Likewise, we can show that $0 < v(t, z) \leq K_3 e^{\lambda_c z}$ for some constant $K_3 > 0$ for $z \leq 0$ with $|z|$ sufficiently large.

Therefore, to obtain (3.6), it suffices to show that $m_1 \neq 0$. To this end, we adopt a technique developed in [18] to reach a contradiction by assuming $m_1 = 0$ (see Propositions 3.2 and 3.3 of [18]). Assume by contradiction that $m_1 = 0$. Then, (3.14) together with (3.9) yields that there exist positive constants C and K such that $\sup_{t \in \mathbb{R}} u(t, z) \leq C e^{(\lambda_c + \epsilon)z}$ whenever $z \leq -K$, where $\epsilon > 0$ is sufficiently small. Moreover, with the same reasoning, we can show that $\hat{v} = O(e^{(\lambda_c + \epsilon)z})$ as $z \rightarrow -\infty$, which along with (3.9) implies that $\sup_{t \in \mathbb{R}} v(t, z) = O(e^{(\lambda_c + \epsilon)z})$ as $z \rightarrow -\infty$. Therefore, there exists a sequence $\{(t_n, z_n)\}_{n \in \mathbb{N}} \in \mathbb{R} \times \mathbb{R}^-$ such that

$$z_n \rightarrow -\infty, \quad \varepsilon_n = u(t_n, z_n) e^{-\lambda_c z_n} \rightarrow 0, \quad \epsilon_n = v(t_n, z_n) e^{-\lambda_c z_n} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Since both ε_n and ϵ_n are positive, it follows from (3.9) that

$$(u(t, z_n), v(t, z_n)) \leq N_1 (\varepsilon_n e^{\lambda_c z_n}, \epsilon_n e^{\lambda_c z_n}) < \frac{2N_1 (\varepsilon_n + \epsilon_n) \phi(t)}{\inf_{t \in [0, T]} \phi(t)} (e^{\lambda_c z_n}, e^{\lambda_c z_n})$$

for all $n \in \mathbb{N}$ and all $t \in \mathbb{R}$. Next we let:

$$(\bar{u}_n(t, z), \bar{v}_n(t, z)) = \min \left\{ \frac{2N_1(\varepsilon_n + \epsilon_n)\phi(t)}{\inf_{t \in [0, T]} \phi(t)} (e^{\lambda_c z}, e^{\lambda_c z}), (1, 1) \right\}.$$

Clearly, $(u(t, z_n), v(t, z_n)) < (\bar{u}_n(t, z_n), \bar{v}_n(t, z_n))$ for all $t \in \mathbb{R}$, $(\bar{u}_n, \bar{v}_n) \in C(\mathbb{R} \times \mathbb{R})$ is T -periodic in t , and $(\bar{u}_n(t, \cdot), \bar{v}_n(t, \cdot))$ is nondecreasing. In addition, there exists $\bar{\sigma}_n$ for which $(\bar{u}_n(t, z), \bar{v}_n(t, z)) = (1, 1)$ for all $(t, z) \in \mathbb{R} \times [\bar{\sigma}_n, +\infty)$. Notice that $\{(t, z) \mid \bar{u}_n < 1\} = \{(t, z) \mid \bar{v}_n < 1\}$ since $\bar{u}_n = \bar{v}_n$ for all $(t, z) \in \mathbb{R} \times \mathbb{R}$. Moreover, due to (H4), (\bar{u}_n, \bar{v}_n) is a regular super-solution of (3.3) whenever $(\bar{u}_n, \bar{v}_n) < (1, 1)$. Thus, $(\bar{u}_n(t, \cdot), \bar{v}_n(t, \cdot))$ has all the properties specified by Lemma 3.1. Hence Lemma 3.1 implies that

$$0 < (u(t, z), v(t, z)) < (\bar{u}_n(t, z), \bar{v}_n(t, z)) = \min \left\{ \frac{2N_1(\varepsilon_n + \epsilon_n)\phi(t)}{\inf_{t \in [0, T]} \phi(t)} (e^{\lambda_c z}, e^{\lambda_c z}), (1, 1) \right\},$$

for all $(t, z) \in \mathbb{R} \times [z_n, +\infty)$. Since $z_n \rightarrow -\infty$ and $\varepsilon_n + \epsilon_n \rightarrow 0$ as $n \rightarrow \infty$, it follows that $(u(t, z), v(t, z)) = (0, 0)$ for any $(t, z) \in \mathbb{R} \times \mathbb{R}^-$, which is impossible. Thus, $m_1 \neq 0$.

Case II. $c = c^* = -2\sqrt{\kappa}$. The proof is similar to case I. Since $c = -2\sqrt{\kappa}$, $\lambda^2 + c\lambda + \kappa = 0$ has a repeated root $\lambda_c = \sqrt{\kappa}$. Similar as deriving (3.14), we have:

$$\hat{u}(z) = m_1 |z| e^{\lambda_c z} + O(e^{\lambda_c z}) \quad \text{as } z \rightarrow -\infty$$

for some positive constant m_1 . Notice that $m|z|e^{\lambda_c z}$ satisfies (3.15) provided that m is sufficiently large and z is negative. Thus, it follows from the same reasoning that $\hat{v} = O(|z|e^{\lambda_c z})$ as $z \rightarrow -\infty$ if $m_1 \neq 0$. Furthermore, thanks to (3.9), (3.7) follows provided that $m_1 \neq 0$. Thus, to show (3.7), we need to prove that $m_1 \neq 0$.

Assume by contradiction again that $m_1 = 0$. Then, we have that $\hat{u} = O(e^{\lambda_c z})$, $\hat{v} = O(e^{\lambda_c z})$ as $z \rightarrow -\infty$, which with (3.9) implies that $\sup_{t \in \mathbb{R}} u(t, z) = O(e^{\lambda_c z})$, $\sup_{t \in \mathbb{R}} v(t, z) = O(e^{\lambda_c z})$ as $z \rightarrow -\infty$. Hence, there exists a sequence $(t_n, z_n)_{n \in \mathbb{N}} \in \mathbb{R} \times \mathbb{R}^-$ such that

$$z_n \rightarrow -\infty, \quad \varepsilon_n = u(t_n, z_n) |z_n|^{-1} e^{-\lambda_c z_n} \rightarrow 0, \quad \epsilon_n = v(t_n, z_n) |z_n|^{-1} e^{-\lambda_c z_n} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Using (3.9) again, we find that

$$(u(t, z_n), v(t, z_n)) \leq N_1(\varepsilon_n |z_n| e^{\lambda_c z_n}, \epsilon_n |z_n| e^{\lambda_c z_n}) < \frac{2N_1(\varepsilon_n + \epsilon_n) |z_n| \phi(t)}{\inf_{t \in [0, T]} \phi(t)} (e^{\lambda_c z_n}, e^{\lambda_c z_n}) \tag{3.16}$$

for all $n \in \mathbb{N}$ and all $t \in \mathbb{R}$.

Now let θ_n be the least positive number such that

$$\frac{N_1(\varepsilon_n + \epsilon_n) |z_n| e^{\lambda_c \frac{\theta_n}{2}} \theta_n}{(\theta_n - z_n)} = 1. \tag{3.17}$$

It is easy to see that the sequence $\{\theta_n\}_{n \in \mathbb{N}}$ converges to $+\infty$ as $n \rightarrow +\infty$. Next we set:

$$u_n^*(t, z) = v_n^*(t, z) = \frac{2N_1(\varepsilon_n + \epsilon_n) |z_n|}{\inf_{t \in [0, T]} \phi(t) (\theta_n - z_n)} (\theta_n - z) e^{\lambda_c z} \phi(t).$$

Notice that

$$u_n^*(t, 0) = v_n^*(t, 0) = \frac{2N_1(\varepsilon_n + \epsilon_n) |z_n| \theta_n}{\inf_{t \in [0, T]} \phi(t) (\theta_n - z_n)} \phi(t) = \frac{2\phi(t)}{\inf_{t \in [0, T]} \phi(t)} e^{-\lambda_c \frac{\theta_n}{2}}$$

and

$$u_n^*\left(t, \frac{\theta_n}{2}\right) = v_n^*\left(t, \frac{\theta_n}{2}\right) = \frac{N_1(\varepsilon_n + \epsilon_n) |z_n| \theta_n}{\inf_{t \in [0, T]} \phi(t) (\theta_n - z_n)} e^{\lambda_c \frac{\theta_n}{2}} \phi(t) = \frac{\phi(t)}{\inf_{t \in [0, T]} \phi(t)} \geq 1.$$

In addition,

$$(u_n^*)_z = (v_n^*)_z = e^{\lambda_c z} \phi(t) \frac{2N_1(\varepsilon_n + \epsilon_n) |z_n|}{\inf_{t \in [0, T]} \phi(t) (\theta_n - z_n)} [\lambda_c (\theta_n - z) - 1] \geq 0,$$

for all $(t, z) \in \mathbb{R} \times (-\infty, \frac{\theta_n}{2}]$ provided that n is sufficiently large. Apparently, $(u_n^*, v_n^*) \geq (0, 0)$ for all $(t, z) \in \mathbb{R} \times (-\infty, \frac{\theta_n}{2}]$. Now we let:

$$(\bar{u}_n, \bar{v}_n) = \min\{(u_n^*(t, z), v_n^*(t, z)), (1, 1)\} \quad \text{if } z \leq \frac{\theta_n}{2} \quad \text{and} \quad (\bar{u}_n, \bar{v}_n) = (1, 1) \quad \text{if } z \geq \frac{\theta_n}{2}.$$

It is clear that $(\bar{u}_n, \bar{v}_n) \in C(\mathbb{R} \times \mathbb{R}, [0, 1]^2)$ is T -periodic in t and is nondecreasing in z for n sufficiently large, and $(\bar{u}_n, \bar{v}_n) = (u_n^*, v_n^*)$ as long as $(\bar{u}_n, \bar{v}_n) < (1, 1)$. Furthermore, $\{(t, z) \mid \bar{u}_n < 1\} = \{(t, z) \mid \bar{v}_n < 1\} \subseteq \mathbb{R} \times (-\infty, \frac{\theta_n}{2}]$, and by virtue of (H4), $(u_n^*(t, z), v_n^*(t, z))$ is a super-solution of (3.3) whenever $(t, z) \in \mathbb{R} \times (-\infty, \frac{\theta_n}{2}]$. Thus, (\bar{u}_n, \bar{v}_n) enjoys all the properties required by Lemma 3.1 when n is sufficiently large. In particular, (3.16) shows that $(u(t, z_n), v(t, z_n)) < (\bar{u}_n(t, z_n), \bar{v}_n(t, z_n))$. Therefore, if n is sufficiently large, applying Lemma 3.1 with $\sigma = z_n$ yields that $(u(t, z), v(t, z)) < (\bar{u}_n(t, z), \bar{v}_n(t, z))$ for any $(t, z) \in \mathbb{R} \times [z_n, +\infty)$. Since $(u_n^*(t, 0), v_n^*(t, 0)) < (1, 1)$, we have $(\bar{u}_n(t, 0), \bar{v}_n(t, 0)) = (u_n^*(t, 0), v_n^*(t, 0))$. Hence

$$(u(t, 0), v(t, 0)) < (u_n^*(t, 0), v_n^*(t, 0)) = \frac{4\phi(t)}{\inf_{t \in [0, T]} \phi(t)} e^{-\lambda_c \frac{\theta_n}{2}} (1, 1).$$

As $\lim_{n \rightarrow \infty} e^{-\lambda_c \frac{\theta_n}{2}} = 0$, it follows that $(u(t, 0), v(t, 0)) = (0, 0)$, so we reached a contradiction, which implies that $m_1 \neq 0$. Therefore, (3.7) follows. This completes the proof. \square

Proposition 3.4. *Suppose that all assumptions of Lemma 3.3 are satisfied. Let $(u, v) \in C_b^{1,2}(\mathbb{R} \times \mathbb{R})$ and c solve (3.1). Then there exist positive constants M_1 and M such that for all $(t, z) \in \mathbb{R} \times (-\infty, -M]$,*

$$|u_z(t, z)| \leq M_1 e^{\lambda_c z}, \quad |v_z(t, z)| \leq M_1 e^{\lambda_c z}, \quad \text{if } c < c^*, \tag{3.18}$$

and

$$|u_z(t, z)| \leq M_1 |z| e^{\lambda_c z}, \quad |v_z(t, z)| \leq M_1 |z| e^{\lambda_c z}, \quad \text{if } c = c^*. \tag{3.19}$$

Proof. Thanks to (3.8) and the interior parabolic L^p -estimates (see Theorem 7.22 of [31]), for any $z \in \mathbb{R}$ and any $p \in]3, \infty[$, we have:

$$\left[\int_T^{2T} \int_{z-\frac{1}{2}}^{z+\frac{1}{2}} (|u_{ss}(\tau, s)|^p + |u_s(\tau, s)|^p + |u_\tau(\tau, s)|^p) ds d\tau \right]^{\frac{1}{p}} \leq C \left[\int_0^{2T} \int_{z-1}^{z+1} (|u|^p + |v|^p) ds d\tau \right]^{\frac{1}{p}},$$

$$\left[\int_T^{2T} \int_{z-\frac{1}{2}}^{z+\frac{1}{2}} (|v_{ss}(\tau, s)|^p + |v_s(\tau, s)|^p + |v_\tau(\tau, s)|^p) ds d\tau \right]^{\frac{1}{p}} \leq C \left[\int_0^{2T} \int_{z-1}^{z+1} (|u|^p + |v|^p) ds d\tau \right]^{\frac{1}{p}}$$

for some positive constant C independent of u, v, t and z . In view of (3.6) and (3.7), there exists $M > 0$ such that

$$\left[\int_0^{2T} \int_{z-1}^{z+1} (|u|^p + |v|^p) ds d\tau \right]^{\frac{1}{p}} \leq C' \sup_{(t,z) \in [0, 2T] \times [z-1, z+1]} (|u| + |v|) \leq M' |z|^\iota e^{\lambda_c z}, \quad z \leq -M$$

for certain positive constants C' and M' , which are independent of z , where $\iota = 0$ if $c < c^*$ and $\iota = 1$ if $c = c^*$. As a result, we have:

$$\left[\int_T^{2T} \int_{-\frac{1}{2}}^{\frac{1}{2}} (|u_{\eta\eta}(\tau, \eta + z)|^p + |u_\eta(\tau, \eta + z)|^p + |u_\tau(\tau, \eta + z)|^p) d\eta d\tau \right]^{\frac{1}{p}} \leq M' |z|^\iota e^{\lambda_c z}, \quad z \leq -M,$$

$$\left[\int_T^{2T} \int_{-\frac{1}{2}}^{\frac{1}{2}} (|v_{\eta\eta}(\tau, \eta + z)|^p + |v_\eta(\tau, \eta + z)|^p + |v_\tau(\tau, \eta + z)|^p) d\eta d\tau \right]^{\frac{1}{p}} \leq M' |z|^\iota e^{\lambda_c z}, \quad z \leq -M.$$

Consequently, the Sobolev embedding theorem implies that

$$\left\{ |u_s(\tau, s)|_{C([T, 2T] \times [z - \frac{1}{2}, z + \frac{1}{2}])} + |v_s(\tau, s)|_{C([T, 2T] \times [z - \frac{1}{2}, z + \frac{1}{2}])} \right\} \leq M_1 |z|^l e^{\lambda_c z}, \quad z \leq -M,$$

for some positive constants M_1 (see [1] or [21]). Since both u and v are T -periodic in t , (3.18) and (3.19) follow. The proof is complete. \square

Next we proceed to establish the exact exponential decay rate for a solution of (3.1) as $z \rightarrow -\infty$. To achieve this goal, we employ a dynamical system approach by using the variable z as an evolution variable (see [36,37,41]). Let $\mathcal{A} : D(\mathcal{A}) \subset Y \rightarrow Y$ be the linear operator defined by:

$$\mathcal{A} := \begin{pmatrix} 0 & I \\ \partial_t - g_u(t, 0, 0) & -c \end{pmatrix}. \tag{3.20}$$

Here $Y = L_T^2 \times L_T^2$, $L_T^2 := \{j(t + T) = j(t), \int_0^T |j(t + s)|^2 ds < \infty\}$ equipped with the norm $\|j\|_{L_T^2} = (\int_0^T |j(s)|^2 ds)^{\frac{1}{2}}$. It is easy to see that \mathcal{A} is closed and densely defined with the domain $D(\mathcal{A}) = H_T^1 \times L_T^2$, $H_T^1 = \{j \in L_T^2, \sup_t \int_0^T |j'(t + s)|^2 ds < \infty\}$, where $'$ stands for the weak derivative of j . Now let $w = u_z$, then the first equation of (3.1) can be cast as a first order system:

$$\frac{d}{dz} \begin{pmatrix} u \\ w \end{pmatrix} = \mathcal{A} \begin{pmatrix} u \\ w \end{pmatrix} + \begin{pmatrix} 0 \\ g_u(t, 0, 0)u - g(t, u, v) \end{pmatrix}.$$

We will take the Laplace transform of this system to obtain the asymptotic expansion of u . To this end, we first examine the spectrum of \mathcal{A} .

Lemma 3.5. *Let \mathcal{A} be defined by (3.20). Then $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) = \{\lambda \in \mathbb{C} \mid \lambda^2 + c\lambda = -\kappa + \frac{2\pi ni}{T}, n \in \mathbb{N}\}$, where $i = \sqrt{-1}$. Moreover, $\lambda_c \in \sigma_p(\mathcal{A})$ and $\ker(\lambda_c I - \mathcal{A}) = \text{span}\left\{\begin{pmatrix} \phi \\ \lambda_c \phi \end{pmatrix}\right\}$. In particular, $\ker(\lambda_c I - \mathcal{A})^n = \ker(\lambda_c I - \mathcal{A})$ for $n = 2, \dots$, provided $c < c^*$. If $c = c^*$, then $\ker(\lambda_c I - \mathcal{A}) \subset \ker(\lambda_c I - \mathcal{A})^2$, $\ker(\lambda_c I - \mathcal{A})^2 \setminus \ker(\lambda_c I - \mathcal{A}) = \text{span}\left\{\begin{pmatrix} \phi \\ (\lambda_c - 1)\phi \end{pmatrix}\right\}$, and $\ker(\lambda_c I - \mathcal{A})^n = \ker(\lambda_c I - \mathcal{A})^2$ for $n = 3, \dots$*

Proof. Let $L : H_T^1 \rightarrow L_T^2$ be defined by:

$$L := \partial_t - g_u(t, 0, 0).$$

Notice that $\lambda \in \rho(\mathcal{A})$ if and only if $\lambda^2 + c\lambda \in \rho(L)$. Indeed, if $\lambda \in \rho(\mathcal{A})$, then for any $\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \in Y$, there exists $\begin{pmatrix} u_p \\ w_p \end{pmatrix}$ satisfying,

$$\begin{cases} \lambda u - w = p_1, \\ g_u(t, 0, 0)u - \partial_t u + cw + \lambda w = p_2, \end{cases} \tag{3.21}$$

which implies that $g_u(t, 0, 0)u_p - \partial_t u_p + (c\lambda + \lambda^2)u_p = (\lambda + c)p_1 + p_2$. Since $\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ is arbitrary, $\lambda^2 + c\lambda \in \rho(L)$.

On the other hand, if $\lambda^2 + c\lambda \in \rho(L)$, then it is clear that

$$\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} [(\lambda^2 + c\lambda)I - L]^{-1}[(\lambda + c)p_1 + p_2] \\ \lambda[(\lambda^2 + c\lambda)I - L]^{-1}[(\lambda + c)p_1 + p_2] - p_1 \end{pmatrix}$$

solves (3.21). Thus, $\lambda \in \rho(\mathcal{A})$. This also implies that $\lambda \in \sigma(\mathcal{A})$ if and only if $\lambda^2 + c\lambda \in \sigma(L)$. Since L has compact resolvent, for any $\omega \in \mathbb{C}$, $\omega I - L$ is Fredholm of index zero. Thus we have $\sigma(L) = \sigma_p(L)$ and $\sigma_p(L) = \{-\kappa + \frac{2n\pi i}{T}, n \in \mathbb{N}\}$. In addition, if $\lambda^2 + c\lambda \in \sigma_p(L)$, that is, $\lambda^2 + c\lambda = -\kappa + \frac{2n\pi i}{T}$ for some n , then $\phi(t)e^{\frac{2n\pi it}{T}}$ is the eigenfunction associated with $\lambda^2 + c\lambda$. Namely, $\begin{pmatrix} \phi(t)e^{\frac{2n\pi it}{T}} \\ \lambda\phi(t)e^{\frac{2n\pi it}{T}} \end{pmatrix}$ is the eigenfunction corresponding to the eigenvalue λ of \mathcal{A} . Hence the first part this lemma follows. Apparently, $\lambda_c \in \sigma(\mathcal{A})$ since $\lambda_c^2 + c\lambda_c + \kappa = 0$. If $\begin{pmatrix} u_1 \\ w_1 \end{pmatrix} \in \ker(\lambda_c I - \mathcal{A})$. Then we must have that $\lambda_c u_1 = w_1$ and $g_u(t, 0, 0)u_1 - \partial_t u_1 - \kappa u_1 = 0$. Since $\ker(-\kappa I - L) = \text{span}\{\phi\}$, it follows that $\ker(\lambda_c I - \mathcal{A}) = \text{span}\left\{\begin{pmatrix} \phi \\ \lambda_c \phi \end{pmatrix}\right\}$.

Now assume that $c < c^*$, we claim that $\ker(\lambda_c I - \mathcal{A})^n = \ker(\lambda_c I - \mathcal{A})$, $n = 2, \dots$. To this end, it suffices to show that $\ker(\lambda_c I - \mathcal{A})^2 = \ker(\lambda_c I - \mathcal{A})$. Assume to the contrary that this is not true, then there exists $\begin{pmatrix} u_2 \\ w_2 \end{pmatrix} \in \ker(\lambda_c I - \mathcal{A})^2 \setminus \ker(\lambda_c I - \mathcal{A})$. Since $(\lambda_c I - \mathcal{A})\begin{pmatrix} u_2 \\ w_2 \end{pmatrix} \in \ker(\lambda_c I - \mathcal{A})$ and $\ker(\lambda_c I - \mathcal{A}) = \text{span}\left\{\begin{pmatrix} \phi \\ \lambda_c \phi \end{pmatrix}\right\}$, it follows that $\lambda_c u_2 - w_2 = k\phi$ and $g_u(t, 0, 0)u_2 - \partial_t u_2 + cw_2 + \lambda_c w_2 = k\lambda_c \phi$ for some nonzero constant k . Without loss of the generality, we assume that $k = 1$. A direct calculation shows that $-\kappa u_2 + g_u(t, 0, 0)u_2 - \partial_t u_2 = (2\lambda_c + c)\phi$, that is, $(-\kappa I - L)u_2 = (2\lambda_c + c)\phi$. Let $L^* := -\partial_t - g_u(t, 0, 0)$ be the formal adjoint operator of L . Note that $-\kappa$ is also an eigenvalue of L^* and ϕ^{-1} is the corresponding eigenfunction. Therefore, the Fredholm alternative implies that $\int_0^T (2\lambda_c + c)\phi\phi^{-1} dt = (2\lambda_c + c)T = 0$, this is impossible since $2\lambda_c + c = -\sqrt{c^2 - 4\kappa} < 0$. Thus, $\ker(\lambda_c I - \mathcal{A})^2 = \ker(\lambda_c I - \mathcal{A})$. By induction, we can show that $\ker(\lambda_c I - \mathcal{A})^n = \ker(\lambda_c I - \mathcal{A})$ for all $n \in \mathbb{N}^+$.

It remains to prove the assertion for the case of $c = c^*$. We first show that $\ker(\lambda_c I - \mathcal{A}) \subset \ker(\lambda_c I - \mathcal{A})^2$. Again assume that $\begin{pmatrix} u_2 \\ w_2 \end{pmatrix} \in \ker(\lambda_c I - \mathcal{A})^2$. Then the same arguments as the above yield that $\lambda_c u_2 - w_2 = k\phi$ and $-\kappa u_2 + g_u(t, 0, 0)u_2 - \partial_t u_2 = (2\lambda_c + c)k\phi$ for some nonzero constant k . Since $2\lambda_c + c = 0$, we have $-\kappa u_2 + g_u(t, 0, 0)u_2 - \partial_t u_2 = 0$, this implies that $u_2 = m\phi$ for some constant m and hence $w_2 = (m\lambda_c - k)\phi$. Therefore, every element belonging to $\ker(\lambda_c I - \mathcal{A})^2$ must be in the form $\begin{pmatrix} m\phi \\ (m\lambda_c - k)\phi \end{pmatrix}$. Note that

$$\begin{pmatrix} m\phi \\ (m\lambda_c - k)\phi \end{pmatrix} = k \begin{pmatrix} \phi \\ (\lambda_c - 1)\phi \end{pmatrix} + (m - k) \begin{pmatrix} \phi \\ \lambda_c \phi \end{pmatrix}.$$

It then follows that

$$\ker(\lambda_c I - \mathcal{A})^2 = \text{span} \left\{ \begin{pmatrix} \phi \\ (\lambda_c - 1)\phi \end{pmatrix}, \begin{pmatrix} \phi \\ \lambda_c \phi \end{pmatrix} \right\}.$$

Next, we prove that $\ker(\lambda_c I - \mathcal{A})^n = \ker(\lambda_c I - \mathcal{A})^2$ for all $n \in \mathbb{N}^+$ with $n \geq 2$. With the same reasoning, it is sufficient to show that $\ker(\lambda_c I - \mathcal{A})^3 = \ker(\lambda_c I - \mathcal{A})^2$. Again assume by contradiction that this is not true, then there exists $\begin{pmatrix} u_3 \\ w_3 \end{pmatrix} \in \ker(\lambda_c I - \mathcal{A})^3 \setminus \ker(\lambda_c I - \mathcal{A})^2$. Thus,

$$\lambda_c u_3 - w_3 = m\phi, \quad g_u(t, 0, 0)u_3 - \partial_t u_3 + cw_3 + \lambda_c w_3 = (m\lambda_c - k)\phi,$$

for some constants m and k . Note that $k \neq 0$, otherwise, it follows from the above arguments that $\begin{pmatrix} u_3 \\ w_3 \end{pmatrix}$ is a linear combination of $\begin{pmatrix} \phi \\ (\lambda_c - 1)\phi \end{pmatrix}$ and $\begin{pmatrix} \phi \\ \lambda_c \phi \end{pmatrix}$. A straightforward computation shows that $-\kappa u_3 - \partial_t u_3 + g_u(t, 0, 0)u_3 = -k\phi$. Here we used the fact that $2\lambda_c + c = 0$. Thus applying the Fredholm alternative again yields that $-k = 0$, which is a contradiction. This contradiction confirms that $\ker(\lambda_c I - \mathcal{A})^3 = \ker(\lambda_c I - \mathcal{A})^2$. The proof is completed. \square

Proposition 3.6. Let $\Theta_\epsilon = \{\lambda \in \mathbb{C} \mid \lambda_c - 2\epsilon \leq \text{Re } \lambda \leq \lambda_c + 2\epsilon, \epsilon \in \mathbb{R}^+\}$ be the vertical strip containing the vertical line $\text{Re } \lambda = \lambda_c$. Then there exists $\epsilon' \in]0, \frac{\lambda_c}{2}[$ sufficiently small such that $\Theta_{\epsilon'} \cap \sigma(\mathcal{A}) = \{\lambda_c\}$. Furthermore, if $c < c^*$, then the Laurent series for $(\lambda I - \mathcal{A})^{-1}$ at $\lambda = \lambda_c$ is given by:

$$(\lambda I - \mathcal{A})^{-1} = \sum_{n=0}^{\infty} (\lambda - \lambda_c)^n S^{n+1} - (\lambda - \lambda_c)^{-1} P. \tag{3.22}$$

In case that $c = c^*$, the Laurent series for $(\lambda I - \mathcal{A})^{-1}$ at $\lambda = \lambda_c$ is given by:

$$(\lambda I - \mathcal{A})^{-1} = \sum_{n=0}^{\infty} (\lambda - \lambda_c)^n S^{n+1} - (\lambda - \lambda_c)^{-1} P - (\lambda - \lambda_c)^{-2} D. \tag{3.23}$$

Here

$$S = \frac{1}{2\pi i} \int_{\Gamma} \frac{(\lambda I - \mathcal{A})^{-1}}{(\lambda - \lambda_c)} d\lambda, \quad P = -\frac{1}{2\pi i} \int_{\Gamma} (\lambda I - \mathcal{A})^{-1} d\lambda, \quad D = (\mathcal{A} - \lambda_c)P,$$

and $\Gamma: |\lambda - \lambda_c| = \gamma < 2\epsilon'$.

Proof. In terms of Lemma 3.5, $\lambda \in \sigma(\mathcal{A})$ if and only if $\lambda^2 + c\lambda = -\kappa + \frac{2n\pi i}{T}$. Let $\lambda = \lambda_r + i\lambda_m$, where $\lambda_r, \lambda_m \in \mathbb{R}$. Thus, if $\lambda \in \sigma(\mathcal{A})$, then λ_r and λ_m have to satisfy:

$$\lambda_r^2 - \lambda_m^2 + c\lambda_r + \kappa = 0, \quad (2\lambda_r + c)\lambda_m = \frac{2n\pi}{T}. \tag{3.24}$$

Let $\lambda_r = \lambda_c + \varepsilon$ with $|\varepsilon| > 0$. Then $|\lambda_r^2 + c\lambda_r + \kappa| = |\varepsilon - \sqrt{c^2 - 4\kappa}||\varepsilon|$ and $|\frac{2n\pi}{T(2\lambda_r+c)}| = |\frac{2n\pi}{T(2\varepsilon - \sqrt{c^2-4\kappa})}|$. Hence, (3.24) has no solution as long as ε is sufficiently small. This shows that there exists $\varepsilon' > 0$ such that $\Theta_{\varepsilon'} \cap \sigma(\mathcal{A}) = \{\lambda_c\}$. Thus, $\lambda = \lambda_c$ is the only singular point of $\lambda I - \mathcal{A}$ in $\Theta_{\varepsilon'}$. By [26], the Laurent series for $(\lambda I - \mathcal{A})^{-1}$ at $\lambda = \lambda_c$ is given by:

$$(\lambda I - \mathcal{A})^{-1} = \sum_{n=0}^{\infty} (\lambda - \lambda_c)^n S^{n+1} - \frac{P}{(\lambda - \lambda_c)} - \sum_{n=1}^{\infty} \frac{D^n}{(\lambda - \lambda_c)^{(n+1)}}.$$

Here P is the spectral projection. Denote the range of P by $\mathcal{R}(P)$. As λ_c is an isolated eigenvalue of \mathcal{A} of finite algebraic multiplicity, λ_c is a pole of $(\lambda - \mathcal{A})^{-1}$ (see Proposition 1.8 of [23]). In particular, by Lemma 3.5, λ_c is a simple pole of $(\lambda - \mathcal{A})^{-1}$ and $\mathcal{R}(P) = \ker(\lambda_c I - \mathcal{A})$ provided $c < c^*$. Thus, if $c < c^*$, then $D^n = 0$ for all $n \in \mathbb{N}^+$, and (3.22) follows. In case that $c = c^*$, λ_c is a pole of $(\lambda - \mathcal{A})^{-1}$ of order 2 and $\mathcal{R}(P) = \ker(\lambda_c I - \mathcal{A})^2$. Then $D^n = 0$ for all $n \geq 2$, which yields (3.23). The proof is completed. \square

Remark 3.7. Let $\lambda \in \rho(\mathcal{A})$ and $\lambda = \mu + i\eta$ with $\mu, \eta \in \mathbb{R}$. Denote by \mathcal{S} the subspace of elements in Y of the form $\begin{pmatrix} 0 \\ j \end{pmatrix}$ and let $(\lambda I - \mathcal{A})_{\mathcal{S}}^{-1}$ be the restriction of $(\lambda I - \mathcal{A})^{-1}$ to \mathcal{S} . Then we notice that for certain positive constants C and Σ ,

$$\|(\lambda I - \mathcal{A})_{\mathcal{S}}^{-1}\| \leq \frac{C}{|\eta|} \quad \text{if } |\eta| \geq \Sigma,$$

provided that $\mu \in [\lambda_c - 2\varepsilon', \lambda_c + 2\varepsilon']$. Indeed, if $\begin{pmatrix} u \\ w \end{pmatrix} = (\lambda I - \mathcal{A})_{\mathcal{S}}^{-1} \begin{pmatrix} 0 \\ j \end{pmatrix}$, then $w = \lambda u$ and $u = (\lambda^2 + c\lambda - L)^{-1} j$. Define:

$$\langle \iota, J \rangle := \int_0^T \frac{\iota(t) \overline{J(t)}}{\phi^2(t)} dt, \quad \iota, J \in L_T^2 \mathbb{C},$$

where $L_T^2 \mathbb{C}$ is the complexification of L_T^2 and \overline{j} stands for the complex conjugate of j . Then

$$\begin{aligned} \operatorname{Re} \langle (L + \kappa I)J, J \rangle &= \operatorname{Re} \int_0^T \left[\frac{J'}{\phi} - \frac{g_u(t, 0, 0)J - \kappa J}{\phi} \right] \frac{\overline{J}}{\phi} dt \\ &= \operatorname{Re} \int_0^T \left[\left(\frac{J}{\phi} \right)' - J \left(\frac{1}{\phi} \right)' - \frac{g_u(t, 0, 0)J - \kappa J}{\phi} \right] \frac{\overline{J}}{\phi} dt \\ &= \operatorname{Re} \int_0^T \left(\frac{J}{\phi} \right)' \frac{\overline{J}}{\phi} dt = 0. \end{aligned}$$

Similarly, we can show that $\operatorname{Re} \langle -(L + \kappa I)J, J \rangle = 0$. In view of Proposition C.7.2 of [20], both $-(L + \kappa I)$ and $L + \kappa I$ are m -accretive and $\{\operatorname{Re} \lambda \neq 0\} \subseteq \rho(L + \kappa I)$. Hence we have that $\|(\lambda^2 + c\lambda - L)^{-1}\| \leq \frac{M}{|\operatorname{Re}(\lambda^2 + c\lambda + \kappa)|}$ for some constant $M > 0$ as long as $\operatorname{Re}(\lambda^2 + c\lambda + \kappa) \neq 0$. Using the fact that $\lambda_c^2 + c\lambda_c = -\kappa$, we find that

$$\|(\lambda I - \mathcal{A})_{\mathcal{S}}^{-1}\| \leq \frac{M(|\mu + i\eta| + 1)}{|\mu^2 - \lambda_c^2 - c(\lambda_c - \mu) - \eta^2|}.$$

Let $\Delta(\mu) = \mu^2 - \lambda_c^2 - c(\lambda_c - \mu)$. Then Δ is bounded for $\mu \in [\lambda_c - 2\varepsilon', \lambda_c + 2\varepsilon']$. Consequently, $\lim_{|\eta| \rightarrow \infty} \frac{|\Delta(\mu) - \eta^2|}{(|\mu + i\eta| + 1)|\eta|} = 1$ uniformly for $\mu \in [\lambda_c - 2\varepsilon', \lambda_c + 2\varepsilon']$. Therefore, there exist positive constants Σ and C such that

$$\|(\lambda I - \mathcal{A})_S^{-1}\| \leq \frac{C}{|\eta|} \quad \text{if } |\eta| \geq \Sigma,$$

whenever $\mu \in [\lambda_c - 2\epsilon', \lambda_c + 2\epsilon']$.

Theorem 3.8. *Suppose that (H1)–(H4) are satisfied. Let $c \leq c^* = -2\sqrt{\kappa}$. Assume that $(u, v) \in C_b^{1,2}(\mathbb{R} \times \mathbb{R})$ and c solve (3.1). Then*

$$\lim_{z \rightarrow -\infty} \frac{u(t, z)}{k_1 e^{\lambda_c z} \phi(t)} = 1, \quad \lim_{z \rightarrow -\infty} \frac{v(t, z)}{k_1 e^{\lambda_c z} \phi_d(t)} = 1, \quad \text{uniformly in } t \in \mathbb{R}, \text{ if } c < c^*, \quad (3.25)$$

$$\lim_{z \rightarrow -\infty} \frac{u_z(t, z)}{k_1 e^{\lambda_c z} \phi(t)} = \lambda_c, \quad \lim_{z \rightarrow -\infty} \frac{v_z(t, z)}{k_1 e^{\lambda_c z} \phi_d(t)} = \lambda_c, \quad \text{uniformly in } t \in \mathbb{R}, \text{ if } c < c^*, \quad (3.26)$$

and

$$\lim_{z \rightarrow -\infty} \frac{u(t, z)}{k_1 |z| e^{\lambda_c z} \phi(t)} = 1, \quad \lim_{z \rightarrow -\infty} \frac{v(t, z)}{k_1 |z| e^{\lambda_c z} \phi_d(t)} = 1, \quad \text{uniformly in } t \in \mathbb{R}, \text{ if } c = c^*, \quad (3.27)$$

$$\lim_{z \rightarrow -\infty} \frac{u_z(t, z)}{k_1 |z| e^{\lambda_c z} \phi(t)} = \lambda_c, \quad \lim_{z \rightarrow -\infty} \frac{v_z(t, z)}{k_1 |z| e^{\lambda_c z} \phi_d(t)} = \lambda_c, \quad \text{uniformly in } t \in \mathbb{R}, \text{ if } c = c^*, \quad (3.28)$$

for some positive constant k_1 . Here

$$\begin{cases} \phi_d(t) = \phi_d(0) e^{\int_0^t (h_v(s,0,0) - \varrho) ds} + \int_0^t e^{\int_s^t (h_v(\tau,0,0) - \varrho) d\tau} h_u(s, 0, 0) \phi(s) ds, \\ \phi_d(0) = (1 - e^{\int_0^T (h_v(t,0,0) - \varrho) dt})^{-1} \int_0^T e^{\int_s^T (h_v(\tau,0,0) - \varrho) d\tau} h_u(s, 0, 0) \phi(s) ds, \end{cases} \quad (3.29)$$

and $\varrho = \kappa + (1 - d)\lambda_c^2$.

Proof. The proof will be divided into two steps.

Step 1. We first show that there exists a constant $k_1 > 0$ such that

$$\lim_{z \rightarrow -\infty} \frac{u(t, z)}{k_1 e^{\lambda_c z} \phi(t)} = 1 \quad \text{if } c < c^* \quad \text{and} \quad \lim_{z \rightarrow -\infty} \frac{u(t, z)}{k_1 |z| e^{\lambda_c z} \phi(t)} = 1 \quad \text{if } c = c^*, \quad \text{uniformly in } t.$$

Due to (H1), we see that

$$\left[\int_0^1 g_u(t, \tau u, v) d\tau \right] u + u_{zz} + cu_z - u_t = 0.$$

By virtue of the (interior) parabolic L^p estimates and (3.9), it is easy to see that

$$\sup_{(t,z) \in \mathbb{R} \times \mathbb{R}} \frac{|u_{zz}|}{|u|} + \frac{|u_z|}{|u|} \leq C_T \quad (3.30)$$

for some constant C_T . Let $\chi(z) \in C_b^3(\mathbb{R}, \mathbb{R})$ such that $\chi \equiv 1$ if $z \leq 0$ and $\chi \equiv 0$ if $z > 1$, and $|\chi'| + |\chi''| + |\chi'''| < \infty$ for all z . Now set:

$$w = u_z, \quad u^\diamond = \chi u, \quad w^\diamond = (\chi u)_z.$$

A direct computation shows that

$$w_z^\diamond + cw^\diamond - u_t^\diamond = -g_u(t, 0, 0)u^\diamond + \chi[g_u(t, 0, 0)u - g(t, u, v)] + \chi''u + 2\chi'u_z + c\chi'u. \quad (3.31)$$

Let

$$\tilde{g}(t, z) = \chi[g_u(t, 0, 0)u - g(t, u, v)] + \chi''u + 2\chi'u_z + c\chi'u. \quad (3.32)$$

Then, as shown before, by using z as an evolution variable, we can rewrite (3.31) as a first order system,

$$\frac{d}{dz} \begin{pmatrix} u^\diamond \\ w^\diamond \end{pmatrix} = \mathcal{A} \begin{pmatrix} u^\diamond \\ w^\diamond \end{pmatrix} + \begin{pmatrix} 0 \\ \tilde{g}(t, z) \end{pmatrix},$$

where $(u^\diamond, w^\diamond) : z \rightarrow (u^\diamond(\cdot, z), w^\diamond(\cdot, z)) \in Y$. Once again, we distinguish between two cases.

Case I. $c < c^* = -2\sqrt{\kappa}$. Thanks to (3.6) and (3.30), there exist positive constants C and M such that $\sup_{t \in \mathbb{R}} (|u_z^\diamond| + |w_z^\diamond|) \leq C e^{\lambda_c z}$ as long as $z \leq -M$, which implies that $(e^{-\lambda z} u^\diamond, e^{-\lambda z} w^\diamond) \in W^{1,1}(\mathbb{R}, Y) \cap W^{1,\infty}(\mathbb{R}, Y)$ provided $\lambda \in [\lambda_c - 2\epsilon', \lambda_c)$, where ϵ' is given in Proposition 3.6. In addition, Proposition 3.6 shows that $\lambda \in \rho(\mathcal{A})$ if $\lambda_c - 2\epsilon' \leq \text{Re } \lambda < \lambda_c$. Hence we can take the two-sided Laplace transform of (u^\diamond, w^\diamond) with respect to z and obtain that

$$\begin{pmatrix} \int_{\mathbb{R}} e^{-\lambda s} u^\diamond(\cdot, s) ds \\ \int_{\mathbb{R}} e^{-\lambda s} w^\diamond(\cdot, s) ds \end{pmatrix} = \mathfrak{F}(\lambda) := (\lambda I - \mathcal{A})^{-1} \begin{pmatrix} 0 \\ \int_{\mathbb{R}} e^{-\lambda s} \tilde{g}(\cdot, s) ds \end{pmatrix},$$

where $\lambda_c - 2\epsilon' \leq \text{Re } \lambda < \lambda_c$. It follows from (3.6), (3.18), and (3.32) that $\sup_{(t,z) \in \mathbb{R}^2} (|\tilde{g}| + |\tilde{g}_z|) < \infty$ and $\sup_{t \in \mathbb{R}} |\tilde{g}| + |\tilde{g}_z| = O(e^{2\lambda_c z})$ as $z \rightarrow -\infty$. This implies that both $\int_{\mathbb{R}} e^{-\lambda s} \tilde{g} ds$ and $\int_{\mathbb{R}} e^{-\lambda s} \tilde{g}_z ds$ are analytic for λ with $\text{Re } \lambda \in]0, \lambda_c + 2\epsilon'[$. Let $\lambda = \mu + i\eta$ with $\mu, \eta \in \mathbb{R}$. Note that $\int_{\mathbb{R}} e^{-\lambda s} \tilde{g} ds = \int_{\mathbb{R}} e^{-i\eta s} e^{-\mu s} \tilde{g} ds$ is identical with the Fourier transform of $e^{-\mu s} \tilde{g}$ if μ is regarded as fixed. It is clear that $e^{-\mu s} \tilde{g} \in W^{1,1}(\mathbb{R}, L_T^2) \cap W^{1,\infty}(\mathbb{R}, L_T^2)$ for any $\mu \in [\epsilon', \lambda_c + \frac{3\epsilon'}{2}]$, in particular, $\|e^{-\mu s} \tilde{g}\|_{W^{1,1}(\mathbb{R}, L_T^2)}$ are uniformly bounded for all $\mu \in [\epsilon', \lambda_c + \frac{3\epsilon'}{2}]$, we then have that

$$\left\| \int_{\mathbb{R}} e^{-\lambda s} \tilde{g} ds \right\|_{L_T^2} \leq \frac{C}{|\eta|}, \quad \lambda = \mu + i\eta,$$

for some positive constant C , where $\mu \in [\epsilon', \lambda_c + \frac{3\epsilon'}{2}]$. It then follows from Remark 3.7 that

$$\left\| (\lambda I - \mathcal{A})^{-1} \begin{pmatrix} 0 \\ \int_{\mathbb{R}} e^{-\lambda s} \tilde{g} ds \end{pmatrix} \right\|_Y \leq \frac{C_1}{|\eta|^2}, \quad \lambda = \mu + i\eta, \quad |\eta| \geq \Sigma, \tag{3.33}$$

for some constants C_1 and λ with $\mu \in [\lambda_c - \epsilon', \lambda_c + \frac{3\epsilon'}{2}] \setminus \{\lambda_c\}$. This implies that $\mathfrak{F}(\mu + i\eta) \in L^1(\mathbb{R}, Y) \cap L^\infty(\mathbb{R}, Y)$ for fixed $\mu \in [\lambda_c - \epsilon', \lambda_c + \frac{3\epsilon'}{2}] \setminus \{\lambda_c\}$.

Now select μ such that $\mu \in [\lambda_c - \epsilon', \lambda_c)$. By taking the inverse Laplace transform of \mathfrak{F} , we obtain that

$$\begin{pmatrix} u^\diamond(\cdot, z) \\ w^\diamond(\cdot, z) \end{pmatrix} = \frac{1}{2\pi i} \int_{\mu - i\infty}^{\mu + i\infty} e^{\lambda z} (\lambda I - \mathcal{A})^{-1} G(\lambda) d\lambda,$$

where $G(\lambda) = \begin{pmatrix} 0 \\ \int_{\mathbb{R}} e^{-\lambda s} \tilde{g} ds \end{pmatrix}$. Since $(u, w) \equiv (u^\diamond, w^\diamond)$ for all $z \leq 0$, we have:

$$\begin{pmatrix} u(\cdot, z) \\ w(\cdot, z) \end{pmatrix} = \frac{1}{2\pi i} \int_{\mu - i\infty}^{\mu + i\infty} e^{\lambda z} (\lambda I - \mathcal{A})^{-1} G(\lambda) d\lambda, \quad z \leq 0. \tag{3.34}$$

By virtue of (3.33), we find that

$$\lim_{\eta \rightarrow \pm\infty} \int_{\mu}^{\lambda_c + \epsilon'} \|e^{(\tau + i\eta)z} ((\tau + i\eta)I - \mathcal{A})^{-1} G(\tau + i\eta)\|_Y d\tau = 0, \quad z \leq 0. \tag{3.35}$$

Furthermore, it follows from (3.22) that

$$(\lambda I - \mathcal{A})^{-1} G(\lambda) = \sum_{n=0}^{\infty} (\lambda - \lambda_c)^n S^{n+1} G(\lambda) - \frac{PG(\lambda_c)}{(\lambda - \lambda_c)} + \frac{P[G(\lambda_c) - G(\lambda)]}{(\lambda - \lambda_c)}, \quad |\lambda - \lambda_c| \leq \gamma < 2\epsilon'.$$

As shown above, $G(\lambda)$ is analytic for λ with $\text{Re } \lambda \in]0, \lambda_c + 2\epsilon'[$ and $PG \in \ker(\lambda_c I - \mathcal{A}) = \text{span}\left\{\begin{pmatrix} \phi \\ \lambda_c \phi \end{pmatrix}\right\}$. Thanks to (3.35), by shifting the path of integral in (3.34) to $\text{Re } \lambda = \lambda_c + \epsilon'$ and using the residue theorem, we obtain that

$$\begin{pmatrix} u(t, z) \\ w(t, z) \end{pmatrix} = k_1 e^{\lambda_c z} \begin{pmatrix} \phi(t) \\ \lambda_c \phi(t) \end{pmatrix} + \frac{e^{(\lambda_c + \epsilon')z}}{2\pi} \int_{-\infty}^{+\infty} e^{i\eta z} ((\lambda_c + \epsilon' + i\eta)I - \mathcal{A})^{-1} G(\lambda_c + \epsilon' + i\eta) d\eta,$$

for $z \leq 0$, where $k_1 \in \mathbb{R}$ is a constant. Let $\zeta(t, z) = u(t, z) - k_1 e^{\lambda_c z} \phi(t)$ for all $(t, z) \in \mathbb{R} \times \mathbb{R}^-$. Since $\zeta(t, z)$ is T -periodic in t , (3.33) implies that

$$\left(\int_{z-1}^{z+1} \int_0^{2T} |\zeta(\tau, s)|^2 d\tau ds \right)^{\frac{1}{2}} \leq C e^{(\lambda_c + \epsilon')z}, \quad z \leq 0$$

for some positive constant C , which is independent of z .

In addition, notice that $\zeta(t, z)$ is a bounded periodic solution of

$$[g(t, u, v) - g_u(t, 0, 0)u] + g_u(t, 0, 0)\zeta + \zeta_{zz} + c\zeta_z - \zeta_t = 0, \quad (t, z) \in \mathbb{R} \times \mathbb{R}^-.$$

In light of (3.6), we see that $[g(t, u, v) - g_u(t, 0, 0)u] = O(e^{2\lambda_c z})$ as $z \rightarrow -\infty$. It then follows from the interior parabolic estimates that

$$\left(\int_{z-\frac{1}{2}}^{z+\frac{1}{2}} \int_T^{2T} |\zeta_{ss}(\tau, s)|^2 d\tau ds + |\zeta_s(\tau, s)|^2 d\tau ds + |\zeta_\tau(\tau, s)|^2 d\tau ds \right)^{\frac{1}{2}} \leq C e^{(\lambda_c + \epsilon')z}, \quad z \leq 0.$$

Here C is a positive constant independent of z . Thus, the Sobolev embedding theorem implies that there exists some positive constant C such that $\sup_{t \in [0, T]} |\zeta(t, z)| \leq C e^{(\lambda_c + \epsilon')z}$ for all $z \in \mathbb{R}^-$. This implies that $k_1 \geq 0$ since $u > 0$. By (3.6) and the fact that ζ is also T -periodic in t , we infer that $k_1 > 0$, and

$$\lim_{z \rightarrow -\infty} \frac{u(t, z)}{k_1 e^{\lambda_c z} \phi(t)} = 1 \quad \text{uniformly in } t \in \mathbb{R}.$$

Now let $\tilde{\zeta}(t, z) = u_z(t, z) - k_1 \lambda_c e^{\lambda_c z} \phi(t)$ for all $(t, z) \in \mathbb{R} \times \mathbb{R}^-$. As $w(t, z) = u_z(t, z)$ for all $(t, z) \in \mathbb{R} \times \mathbb{R}^-$, it follows from (3.33) that

$$\left(\int_{z-1}^{z+1} \int_0^{2T} |\tilde{\zeta}(t, z)|^2 dt dz \right)^{\frac{1}{2}} \leq C e^{(\lambda_c + \epsilon')z}, \quad z \leq 0,$$

for some positive constant C . Moreover, $\tilde{\zeta}(t, z)$ satisfies that

$$[g_u(t, u, v)u_z - g_u(t, 0, 0)u_z + g_v(t, u, v)v_z] + g_u(t, 0, 0)\tilde{\zeta} + c\tilde{\zeta}_z - \tilde{\zeta}_t = 0.$$

Due to (3.6) and (3.18), $[g_u(t, u, v)u_z - g_u(t, 0, 0)u_z + g_v(t, u, v)v_z] = O(e^{2\lambda_c z})$ as $z \rightarrow -\infty$. With the same reasoning, we can infer that $\sup_{t \in [0, T]} |\tilde{\zeta}(t, z)| \leq C' e^{(\lambda_c + \epsilon')z}$ for some positive constant C' and $z \leq 0$. Therefore,

$$\lim_{z \rightarrow -\infty} \frac{u_z(t, z)}{k_1 \phi(t) e^{\lambda_c z}} = \lambda_c \quad \text{uniformly in } t \in \mathbb{R}.$$

Case II. $c = c^* = -2\sqrt{\kappa}$. The proof for this case is almost same as in Case I. Notice that

$$\begin{aligned} (\lambda I - \mathcal{A})^{-1} G(\lambda) &= \sum_{n=0}^{\infty} (\lambda - \lambda_c)^n S^{n+1} G(\lambda) - \frac{PG(\lambda_c)}{(\lambda - \lambda_c)} + \frac{P[G(\lambda_c) - G(\lambda)]}{(\lambda - \lambda_c)} \\ &\quad - \frac{DG(\lambda_c)}{(\lambda - \lambda_c)^2} + \frac{D[G(\lambda_c) - G(\lambda)]}{(\lambda - \lambda_c)^2}, \quad |\lambda - \lambda_c| \leq \gamma < 2\epsilon'. \end{aligned}$$

Since $PG \in \ker(\lambda_c I - \mathcal{A})^2 = \text{span}\left\{ \begin{pmatrix} \phi \\ \lambda_c \phi \end{pmatrix}, \begin{pmatrix} \phi \\ (\lambda_c - 1)\phi \end{pmatrix} \right\}$ and $DG = (\mathcal{A} - \lambda_c I)PG \in \ker(\lambda_c I - \mathcal{A}) = \text{span}\left\{ \begin{pmatrix} \phi \\ \lambda_c \phi \end{pmatrix} \right\}$, we find that

$$\begin{aligned} \begin{pmatrix} u(t, z) \\ w(t, z) \end{pmatrix} &= -k_1 z e^{\lambda_c z} \begin{pmatrix} \phi(t) \\ \lambda_c \phi(t) \end{pmatrix} + k_2 e^{\lambda_c z} \begin{pmatrix} \phi(t) \\ (\lambda_c - 1)\phi(t) \end{pmatrix} + k_3 e^{\lambda_c z} \begin{pmatrix} \phi(t) \\ \lambda_c \phi(t) \end{pmatrix} \\ &+ \frac{e^{(\lambda_c + \epsilon')z}}{2\pi} \int_{-\infty}^{+\infty} e^{i\eta z} ((\lambda_c + \epsilon' + i\eta)I - \mathcal{A})^{-1} G(\lambda_c + \epsilon' + i\eta) d\eta, \quad z \leq 0, \end{aligned}$$

for certain constants k_1, k_2 and k_3 . In view of (3.7) and (3.19), by the same reasoning, we readily conclude that $k_1 > 0$, and

$$\lim_{z \rightarrow -\infty} \frac{u(t, z)}{k_1 |z| e^{\lambda_c z} \phi(t)} = 1, \quad \lim_{z \rightarrow -\infty} \frac{u_z(t, z)}{k_1 |z| e^{\lambda_c z} \phi(t)} = \lambda_c \quad \text{uniformly in } t \in \mathbb{R}.$$

Step 2. It remains to prove the claimed asymptotic behaviors for v . We start with the case that $c < c^*$. Again let $\alpha = h_v(t, 0, 0)$. By (H3), we have $\varrho = (d - 1)\lambda_c^2 - \kappa + \alpha < 0$. Therefore, the following equation,

$$[h_v(t, 0, 0) - \kappa + (d - 1)\lambda_c^2]w + h_u(t, 0, 0)\phi - w_t = 0$$

has a unique periodic solution $\phi_d(t)$ given by (3.29). Then $k_1 e^{\lambda_c z} \phi_d(t)$ satisfies:

$$h_v(t, 0, 0)w + h_u(t, 0, 0)k_1 e^{\lambda_c z} \phi + dw_{zz} + cw_z - w_t = 0. \tag{3.36}$$

Now let $\xi(t, z) = [v(t, z) - k_1 e^{\lambda_c z} \phi_d(t)]\tilde{\phi}(t)^{-1}$ and $\zeta(t, z) = [u(t, z) - k_1 e^{\lambda_c z} \phi(t)]\tilde{\phi}(t)^{-1}$ for all $(t, z) \in \mathbb{R} \times \mathbb{R}^-$, where $\tilde{\phi}(t) = \exp(\int_0^t h_v(\tau, 0, 0) d\tau - \alpha t)$. Then we find that

$$R(t, z) + \alpha \xi + d\xi_{zz} + c\xi_z - \xi_t = 0, \quad z \leq 0, \tag{3.37}$$

where $R(t, z) = [h(t, u, v) - h_u(t, 0, 0)u - h_v(t, 0, 0)v]\tilde{\phi}^{-1} + h_u(t, 0, 0)\zeta$. Due to Lemma 3.3, it is clear that $\sup_{t \in \mathbb{R}} |\xi(t, z)| = O(e^{\lambda_c z})$ and $\sup_{t \in \mathbb{R}} |\zeta(t, z)| = O(e^{(\lambda_c + \epsilon')z})$ as $z \rightarrow -\infty$. Moreover, $\sup_{t \in \mathbb{R}} [h(t, u, v) - h_u(t, 0, 0)u - h_v(t, 0, 0)v] = O(e^{2\lambda_c z})$ as $z \rightarrow -\infty$. Therefore, there exist two positive constants M and C_M such that $|R(t, z)| \leq C_M e^{(\lambda_c + \epsilon')z}$ for all $t \in \mathbb{R}$ whenever $z \leq -M$.

We next show that $\sup_{t \in \mathbb{R}} |\xi(t, z)| = o(e^{\lambda_c z})$ as $z \rightarrow -\infty$. To this end, we use the arguments similar to those given in the proof of Lemma 3.3. Fix δ with $0 < \delta \leq \epsilon'$ such that $d(\lambda_c + \delta)^2 + c(\lambda_c + \delta) + \alpha < 0$ and write $\Lambda_\delta = d(\lambda_c + \delta)^2 + c(\lambda_c + \delta) + \alpha$. Then it is easy to see that $\pm C e^{(\lambda_c + \delta)z}$ respectively satisfy that

$$R(t, z) + \alpha w + dw_{zz} + cw_z - w_t \leq 0 (\geq 0), \quad z \leq -M, \tag{3.38}$$

where $C \geq \frac{C_M}{|\Lambda_\delta|}$. As $|\xi(t, z)|$ is bounded for all $(t, z) \in \mathbb{R} \times \mathbb{R}^-$, there exists $C_\delta \geq \frac{C_M}{|\Lambda_\delta|}$ such that $C_\delta e^{-(\lambda_c + \delta)M} \geq |\xi(t, -M)|$ for all $t \in \mathbb{R}$. We now claim that

$$-C_\delta e^{(\lambda_c + \delta)z} \leq \xi(t, z) \leq C_\delta e^{(\lambda_c + \delta)z} \quad \text{for all } (t, z) \in \mathbb{R} \times (-\infty, M]. \tag{3.39}$$

Indeed, let $w^\pm(t, z) = \pm C_\delta e^{(\lambda_c + \delta)z} - \xi(t, z)$. Then we obviously have:

$$\alpha w^- + dw_{zz}^- + cw_z^- - w_t^- \geq 0, \quad \alpha w^+ + dw_{zz}^+ + cw_z^+ - w_t^+ \leq 0 \quad \text{for all } (t, z) \in \mathbb{R} \times (-\infty, M].$$

Since w^\pm are both T -periodic in t , it is sufficient to show that $w^+(t, z) \geq 0 (w^-(t, z) \leq 0)$ for all $(t, z) \in [0, 2T] \times (-\infty, -M]$. We shall only provide a proof for $w^+(t, z) \geq 0$. Assume to the contrary that $\inf_{(t, z) \in [0, 2T] \times (-\infty, -M]} w^+(t, z) < 0$. Then by virtue of the fact that $\lim_{z \rightarrow -\infty} \sup_{t \in [0, 2T]} w^+(t, z) = 0$, there exists a point $(\underline{t}, \underline{z})$ with $0 < \underline{t} < 2T$ and $\underline{z} < -M$ such that $w^+(\underline{t}, \underline{z}) = \inf_{(t, z) \in [0, 2T] \times (-\infty, -M]} w^+(t, z)$. This gives rise to that $[\alpha w^+ + dw_{zz}^+ + cw_z^+ - w_t^+]|_{(\underline{t}, \underline{z})} > 0$, which is a contradiction. Hence, $w^+(t, z) \geq 0$ for all $(t, z) \in [0, 2T] \times (-\infty, -M]$. Similarly, we have $w^-(t, z) \leq 0$ for all $(t, z) \in [0, 2T] \times (-\infty, -M]$. Thus, (3.39) holds. Since $\xi(t, z) = [v(t, z) - k_1 e^{\lambda_c z} \phi_d(t)]\tilde{\phi}(t)^{-1}$, it follows that

$$\lim_{z \rightarrow -\infty} \frac{v(t, z)}{k_1 e^{\lambda_c z} \phi_d(t)} = 1, \quad \text{uniformly in } t \in \mathbb{R}, \text{ if } c < c^*.$$

The proof for v_z is similar. Let $\tilde{\xi}(t, z) = [v_z(t, z) - k_1 \phi_d(t) \lambda_c e^{\lambda_c z}] \tilde{\phi}(t)^{-1}$ and $\tilde{\zeta}(t, z) = [u_z(t, z) - k_1 \phi(t) \lambda_c e^{\lambda_c z}] \tilde{\phi}(t)^{-1}$. Then $\tilde{\xi}$ satisfies:

$$\tilde{R}(t, z) + \alpha \tilde{\xi} + d\tilde{\xi}_{zz} + c\tilde{\xi}_z - \tilde{\xi}_t = 0, \quad z \leq 0,$$

where $\tilde{R}(t, z) = \{[h_u(t, u, v) - h_u(t, 0, 0)]u_z + [h_v(t, u, v) - h_v(t, 0, 0)]v_z\}\tilde{\phi}^{-1} + h_u(t, 0, 0)\tilde{\zeta}$. Following the same lines as the above, we can deduce that $\sup_{t \in \mathbb{R}} |\tilde{\xi}(t, z)| = O(e^{(\lambda_c + \delta)z})$ as $z \rightarrow -\infty$. Therefore, we have:

$$\lim_{z \rightarrow -\infty} \frac{v_z(t, z)}{k_1 e^{\lambda_c z} \phi_d(t)} = \lambda_c, \quad \text{uniformly in } t \in \mathbb{R}, \text{ if } c < c^*.$$

We now consider the case that $c = c^*$. Let $\tilde{\psi}_d(t)$ be the periodic solution of

$$h_u(t, 0, 0)\phi + 2(1-d)\phi_d + [h_v(t, 0, 0) - \kappa + (d-1)\lambda_c^2]w - \frac{dw}{dt} = 0.$$

Note that $\tilde{\psi}_d(t)$ exists and is unique since $\overline{h_v(t, 0, 0)} - \kappa + (d-1)\lambda_c^2 < 0$. It is straightforward to check that $k_1 e^{\lambda_c z} (|z|\phi_d(t) + \lambda_c \tilde{\psi}_d(t))$ solves:

$$h_v(t, 0, 0)w + h_u(t, 0, 0)k_1 e^{\lambda_c z} (|z| + \lambda_c)\phi + dw_{zz} + cw_z - w_t = 0, \quad (t, z) \in \mathbb{R} \times \mathbb{R}^-.$$

Accordingly, we set:

$$\xi(t, z) = \frac{v - k_1 e^{\lambda_c z} (|z|\phi_d + \lambda_c \tilde{\psi}_d(t))}{\tilde{\phi}}, \quad \zeta(t, z) = \frac{u - k_1 e^{\lambda_c z} (|z| + \lambda_c)\phi}{\tilde{\phi}}, \quad (t, z) \in \mathbb{R} \times \mathbb{R}^-.$$

Then we obtain (3.37) again. Notice that $\sup_{t \in \mathbb{R}} |\xi(t, z)| = O(|z|e^{\lambda_c z})$, and $\sup_{t \in \mathbb{R}} |\zeta(t, z)| = O(e^{\lambda_c z})$ as $z \rightarrow -\infty$. Therefore, $\sup_{t \in \mathbb{R}} |R(t, z)| = O(e^{\lambda_c z})$ as $z \rightarrow -\infty$. Consequently, $\pm C e^{\lambda_c z}$ satisfy (3.38) respectively provided that C is sufficiently large. With the same reasoning, we can deduce that $\sup_{t \in \mathbb{R}} |\xi(t, z)| = O(e^{\lambda_c z})$ and $\sup_{t \in \mathbb{R}} |\xi_z(t, z)| = O(e^{\lambda_c z})$ as $z \rightarrow -\infty$. Hence, we have:

$$\lim_{z \rightarrow -\infty} \frac{v(t, z)}{k_1 |z| e^{\lambda_c z} \phi_d(t)} = 1, \quad \lim_{z \rightarrow -\infty} \frac{v_z(t, z)}{k_1 |z| e^{\lambda_c z} \phi_d(t)} = \lambda_c, \quad \text{uniformly in } t \in \mathbb{R}, \text{ if } c = c^*.$$

The proof is completed. \square

Proposition 3.9. Suppose that (H1), (H2) and (H5) are satisfied. Let (\bar{u}, \bar{v}) and $(\underline{u}, \underline{v}) \in C_b^{1,2}(\mathbb{R} \times \mathbb{R})$ be respectively the regular super-solution and sub-solution of (3.1). In particular, both (\bar{u}, \bar{v}) and $(\underline{u}, \underline{v})$ are T -periodic in t , and $\liminf_{z \rightarrow \infty} \{\inf_{t \in [0, T]} (\bar{u} - \underline{u})\} \geq 0$, $\liminf_{z \rightarrow \infty} \{\inf_{t \in [0, T]} (\bar{v} - \underline{v})\} \geq 0$. Let,

$$\rho^* := \sup \left\{ \rho \left| g_u(t, \cdot, \cdot) - g_u(t, 1, 1) \right| + \left| g_v(t, \cdot, \cdot) - g_v(t, 1, 1) \right| \leq \frac{n^* |\underline{v}|}{2}, \forall (t, \cdot, \cdot) \in \mathbb{R} \times [1 - \rho, 1 + \rho]^2 \right\},$$

$$\rho_* := \sup \left\{ \rho \left| h_u(t, \cdot, \cdot) - h_u(t, 1, 1) \right| + \left| h_v(t, \cdot, \cdot) - h_v(t, 1, 1) \right| \leq \frac{n^* |\underline{v}|}{2}, \forall (t, \cdot, \cdot) \in \mathbb{R} \times [1 - \rho, 1 + \rho]^2 \right\},$$

where $\rho \in \mathbb{R}^+$ and $n^* = \frac{\min\{\min_{t \in \mathbb{R}} \varphi_1, \min_{t \in \mathbb{R}} \varphi_2\}}{\max_{t \in \mathbb{R}} (\varphi_1 + \varphi_2)}$. If there exists $z' \in \mathbb{R}$ such that

$$(\bar{u}(t, z), \bar{v}(t, z)) \in [1 - \rho^0, 1]^2 \quad \text{and} \quad (\underline{u}(t, z), \underline{v}(t, z)) \in [1 - \rho^0, 1]^2,$$

for all $(t, z) \in \mathbb{R} \times [z', \infty)$, and $(\bar{u}(t, z'), \bar{v}(t, z')) \geq (\underline{u}(t, z'), \underline{v}(t, z'))$ for all $t \in \mathbb{R}$, where $\rho^0 = \min\{\rho^*, \rho_*\}$. Then $(\bar{u}(t, z), \bar{v}(t, z)) \geq (\underline{u}(t, z), \underline{v}(t, z))$ for all $(t, z) \in \mathbb{R} \times [z', +\infty)$.

Proof. As both (\bar{u}, \bar{v}) and $(\underline{u}, \underline{v})$ are T -periodic in t , it suffices to prove that

$$\inf_{(t, z) \in [0, 2T] \times [z', +\infty)} \{\bar{u} - \underline{u}\} \geq 0 \quad \text{and} \quad \inf_{(t, z) \in [0, 2T] \times [z', +\infty)} \{\bar{v} - \underline{v}\} \geq 0. \tag{3.40}$$

Let

$$u^\tau(t, z) = \bar{u}(t, z) - \underline{u}(t, z) + \tau \varphi_1(t), \quad v^\tau(t, z) = \bar{v}(t, z) - \underline{v}(t, z) + \tau \varphi_2(t).$$

Since both $\bar{u} - \underline{u}$ and $\bar{v} - \underline{v}$ are bounded, there exists $M > 0$ such that $(u^\tau(t, z), v^\tau(t, z)) \geq (0, 0)$ for all $(t, z) \in [0, 2T] \times [z', +\infty)$ as long as $\tau \geq M$. Now define:

$$\tau^* = \inf \{ \tau \in [0, \infty) \mid (u^\tau(t, z), v^\tau(t, z)) \geq (0, 0) \text{ for all } (t, z) \in [0, 2T] \times [z', +\infty) \}.$$

Notice that τ^* is bounded. To complete the proof, it suffices to show that $\tau^* = 0$.

Assume to the contrary that this is not true. Then it is easy to see that

$$\text{either } \inf_{(t,z) \in [0,2T] \times [z',+\infty)} u^{\tau^*}(t,z) = 0 \quad \text{or} \quad \inf_{(t,z) \in [0,2T] \times [z',+\infty)} v^{\tau^*}(t,z) = 0.$$

Assume without loss of generality that $\inf_{(t,z) \in [0,2T] \times [z',+\infty)} v^{\tau^*} = 0$. Due to the fact that $\liminf_{z \rightarrow \infty} \{ \inf_{t \in [0,2T]} v^{\tau^*} \} \geq \tau^* \min_t \varphi_2 > 0$, there exists $(t^*, z^*) \in (0, 2T) \times (z', \infty)$ such that $v^{\tau^*}(t^*, z^*) = 0$. On the other hand, since

$$\begin{aligned} & \tau^* \left\{ c(\varphi_2)_z + d(\varphi_2)_{zz} - (\varphi_2)_t + \left[\int_0^1 h_u(t, s\bar{u} + (1-s)\underline{u}, s\bar{v} + (1-s)\underline{v}) ds \right] \varphi_1 \right. \\ & \quad \left. + \left[\int_0^1 h_v(t, s\bar{u} + (1-s)\underline{u}, s\bar{v} + (1-s)\underline{v}) ds \right] \varphi_2 \right\} \\ & = \tau^* \left\{ v\varphi_2 + \left[\int_0^1 h_u(t, s\bar{u} + (1-s)\underline{u}, s\bar{v} + (1-s)\underline{v}) - h_u(t, 1, 1) ds \right] \varphi_1 \right. \\ & \quad \left. + \left[\int_0^1 h_v(t, s\bar{u} + (1-s)\underline{u}, s\bar{v} + (1-s)\underline{v}) - h_v(t, 1, 1) ds \right] \varphi_2 \right\} \leq 0 \end{aligned}$$

for all $(t, z) \in \mathbb{R} \times [z', \infty)$, we have,

$$\begin{aligned} & \left[\int_0^1 h_v(t, s\bar{u} + (1-s)\underline{u}, s\bar{v} + (1-s)\underline{v}) ds \right] v^{\tau^*} + c v_z^{\tau^*} + d v_{zz}^{\tau^*} - v_t^{\tau^*} \\ & \leq - \left[\int_0^1 h_u(t, s\bar{u} + (1-s)\underline{u}, s\bar{v} + (1-s)\underline{v}) ds \right] u^{\tau^*} \leq 0 \end{aligned}$$

for all $(t, z) \in \mathbb{R} \times [z', \infty)$. Therefore, the strong maximum principle implies that $v^{\tau^*}(t, z) \equiv 0$ for all $(t, z) \in [0, t^*] \times [z', \infty)$. This is impossible since $v^{\tau^*}(t, z^*) > 0$. Hence we must have $\tau^* = 0$. The proof is completed. \square

Theorem 3.10. Assume that (H1)–(H5) are satisfied. Let $c \leq c^* = -2\sqrt{\kappa}$. Suppose that $(u, v) \in C_b^{1,2}(\mathbb{R} \times \mathbb{R})$ and c solve (3.1). Then $(u_z, v_z) > (0, 0)$ for all $(t, z) \in \mathbb{R} \times \mathbb{R}$.

Proof. In terms of (3.26) and (3.28), there exists $\underline{z} \in \mathbb{R}$ such that $(u_z, v_z) > (0, 0)$ for all $(t, z) \in \mathbb{R} \times (-\infty, \underline{z}]$. On the other hand, since $\lim_{z \rightarrow \infty} (u, v) = (1, 1)$, the continuity and the positivity of (u, v) imply that

$$\min \left\{ \inf_{t \in \mathbb{R}, z \geq \underline{z}} u(t, z), \inf_{t \in \mathbb{R}, z \geq \underline{z}} v(t, z) \right\} > 0.$$

Thus, there exists $z_* \leq \underline{z}$ such that

$$\max \left\{ \sup_{t \in \mathbb{R}, z \leq z_*} u(t, z), \sup_{t \in \mathbb{R}, z \leq z_*} v(t, z) \right\} \leq \min \left\{ \inf_{t \in \mathbb{R}, z \geq \underline{z}} u(t, z), \inf_{t \in \mathbb{R}, z \geq \underline{z}} v(t, z) \right\}.$$

Consequently,

$$(u(t, z), v(t, z)) \leq (u(t, z + s), v(t, z + s)) \quad \text{for all } (t, z, s) \in \mathbb{R} \times (-\infty, z_*) \times \mathbb{R}^+.$$

We next show that there exists $s \geq 0$ for which $(u(t, z), v(t, z)) \leq (u(t, z + s), v(t, z + s))$ for all $(t, z) \in \mathbb{R} \times \mathbb{R}$. Indeed, let $z^* \geq 0$ be sufficiently large such that

$$|u(t, z) - 1| + |v(t, z) - 1| \leq \rho^0 \quad \text{for all } (t, z) \in \mathbb{R} \times [z^*, \infty).$$

Here ρ^0 is specified in Proposition 3.9. Since $(u(t, z), v(t, z))$ is bounded in $\mathbb{R} \times [z_*, z^*]$ (its components are bounded in $\mathbb{R} \times [z_*, z^*]$), and $\lim_{z \rightarrow \infty} (u(t, z), v(t, z)) = (1, 1)$ uniformly in $t \in \mathbb{R}$, there exists $\bar{s} \geq 0$ such that $(u(t, z), v(t, z)) \leq (u(t, z + s), v(t, z + s))$ for all $(t, z, s) \in \mathbb{R} \times [z_*, z^*] \times [\bar{s}, \infty)$. Then, Proposition 3.9 implies that $(u(t, z), v(t, z)) \leq (u(t, z + s), v(t, z + s))$ for all $(t, z, s) \in \mathbb{R} \times [z^*, \infty) \times [\bar{s}, \infty)$. Hence, $(u(t, z), v(t, z)) \leq (u(t, z + s), v(t, z + s))$ for all $(t, z, s) \in \mathbb{R} \times \mathbb{R} \times [\bar{s}, \infty)$.

Now we define

$$s^* = \inf\{s \in \mathbb{R}^+, (u(t, z), v(t, z)) \leq (u(t, z + \eta), v(t, z + \eta)) \text{ for all } (t, z, \eta) \in \mathbb{R} \times \mathbb{R} \times [s, +\infty)\}.$$

We claim that $s^* = 0$. Assume to the contrary that $s^* > 0$. We next show that there exists a finite point (t', z') such that either $u(t', z' + s^*) = u(t', z')$ or $v(t', z' + s^*) = v(t', z')$ provided that $s^* > 0$. Indeed, if $s^* > 0$, let $\gamma \in]0, s^*[$ be fixed, as shown before, $(u_z, v_z) > (0, 0)$ when $(t, z) \in \mathbb{R} \times (-\infty, \underline{z}]$ for some $\underline{z} \in \mathbb{R}$. Hence, there exists $M > 0$ sufficiently large such that $(u(t, z + s^* - \gamma), v(t, z + s^* - \gamma)) > (u(t, z), v(t, z))$ for all $(t, z) \in \mathbb{R} \times (-\infty, -M]$ and $(u(t, \cdot + s^*), v(t, \cdot + s^*))$ is monotone in $(-\infty, -M]$. Now if $(u(t, z + s^*), v(t, z + s^*)) > (u(t, z), v(t, z))$ for all $(t, z) \in \mathbb{R} \times \mathbb{R}$, then there exists $\delta \in]0, s^*[$, which is sufficiently small, such that

$$(u(t, z + s^* - \eta), v(t, z + s^* - \eta)) \geq (u(t, z), v(t, z)) \quad \text{for all } (t, z, \eta) \in \mathbb{R} \times [-2M, 2M] \times [0, \delta].$$

Without loss of generality, assume that $\delta \leq \gamma$. As $(u(t, \cdot + s^*), v(t, \cdot + s^*))$ is monotone in $(-\infty, -M]$, we have:

$$(u(t, z + s^* - \eta), v(t, z + s^* - \eta)) \geq (u(t, z + s^* - \gamma), v(t, z + s^* - \gamma)) > (u(t, z), v(t, z))$$

for all $(t, z, \eta) \in \mathbb{R} \times (-\infty, -M] \times [0, \delta]$, and hence $(u(t, z + s^* - \eta), v(t, z + s^* - \eta)) \geq (u(t, z), v(t, z))$ for all $(t, z, \eta) \in \mathbb{R} \times (-\infty, 2M] \times [0, \delta]$. Since $2M$ is sufficiently large, applying Proposition 3.9 again yields that

$$(u(t, z + s^* - \eta), v(t, z + s^* - \eta)) \geq (u(t, z), v(t, z)) \quad \text{for all } (t, z, \eta) \in \mathbb{R} \times \mathbb{R} \times [0, \delta].$$

In particular, in view of the definition of s^* , we see that

$$(u(t, z + s^* - \delta + \eta), v(t, z + s^* - \delta + \eta)) \geq (u(t, z), v(t, z)) \quad \text{for all } (t, z, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+,$$

which is impossible since it contradicts the definition of s^* .

Therefore, there exists (t', z') such that either $u(t', z' + s^*) = u(t', z')$ or $v(t', z' + s^*) = v(t', z')$ if $s^* \neq 0$. Now write:

$$(u^{s^*}(t, z), v^{s^*}(t, z)) = (u(t, z + s^*), v(t, z + s^*)), \quad (u^\circ, v^\circ) = (u^{s^*} - u, v^{s^*} - v).$$

Clearly, if $s^* \neq 0$, then either $u^\circ(t', z') = 0$ or $v^\circ(t', z') = 0$. Assume without loss of generality that $u^\circ(t', z') = 0$. Notice that

$$\left[\int_0^1 g_u(t, \tau u^{s^*} + (1 - \tau)u, \tau v^{s^*} + (1 - \tau)v) d\tau \right] u^\circ + u_{zz}^\circ + cu_z^\circ - u_t^\circ \leq 0.$$

Here we have used (H2) that $g_v \geq 0$ in $\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+$. It then follows from the maximum principle that $u^\circ(t, z) \equiv 0$, which is impossible since $u^\circ > 0$ for all $z \leq -M$. Therefore, we must have $s^* = 0$, and consequently, $(u(t, z), v(t, z)) \leq (u(t, z + s), v(t, z + s))$ for any $s \geq 0$. In particular, it is clear that $(u(t, z), v(t, z)) < (u(t, z + s), v(t, z + s))$ as long as $s > 0$. This completes the proof. \square

Theorem 3.11. Assume that (H1)–(H5) are satisfied. Let $c \leq c^* = -2\sqrt{\kappa}$. Suppose that $(u, v) \in C_b^{1,2}(\mathbb{R} \times \mathbb{R})$ and c solve (3.1). Then (u, v) is unique up to the translation with respect to z .

Proof. Let (u_1, v_1) and (u_2, v_2) be two solutions of (3.1) for some $c \leq c^*$. Then, in terms of Theorem 3.8, there exist positive constants k_1 and k_2 such that

$$\begin{aligned} \lim_{z \rightarrow -\infty} \frac{u_1(t, z)}{k_1 |z|^l e^{\lambda cz} \phi(t)} &= 1, & \lim_{z \rightarrow -\infty} \frac{v_1(t, z)}{k_1 |z|^l e^{\lambda cz} \phi_d(t)} &= 1, & \text{uniformly in } t, \\ \lim_{z \rightarrow -\infty} \frac{u_2(t, z)}{k_2 |z|^l e^{\lambda cz} \phi(t)} &= 1, & \lim_{z \rightarrow -\infty} \frac{v_2(t, z)}{k_2 |z|^l e^{\lambda cz} \phi_d(t)} &= 1, & \text{uniformly in } t. \end{aligned} \tag{3.41}$$

Here $\iota = 1$ if $c = c^*$, and $\iota = 0$ if $c < c^*$. We now proceed to establish the desired conclusion by means of the sliding method developed in [3]. The proof will be broken up into three steps.

Step 1. We show that there exists $s \in \mathbb{R}$ such that

$$(u_1(t, z + s), v_1(t, z + s)) \geq (u_2(t, z), v_2(t, z))$$

for all $(t, z) \in \mathbb{R} \times \mathbb{R}$. Choose $\underline{s} \in \mathbb{R}$ such that $k_1 e^{\lambda_c \underline{s}} > k_2$. Thanks to (3.41), we have:

$$\lim_{z \rightarrow -\infty} \frac{u_1(t, z + \underline{s})}{k_1 e^{\lambda_c \underline{s}} |z + \underline{s}|^\iota e^{\lambda_c z} \phi(t)} = 1, \quad \lim_{z \rightarrow -\infty} \frac{v_1(t, z + \underline{s})}{k_1 e^{\lambda_c \underline{s}} |z + \underline{s}|^\iota e^{\lambda_c z} \phi_d(t)} = 1.$$

Clearly, there exists $M > 0$ sufficiently large such that

$$(u_1(t, z + \underline{s}), v_1(t, z + \underline{s})) \geq (u_2(t, z), v_2(t, z)) \quad \text{for all } (t, z) \in \mathbb{R} \times (-\infty, -M]$$

and

$$|u_1(t, z + \underline{s}) - 1| + |v_1(t, z + \underline{s}) - 1| + |u_2(t, z) - 1| + |v_2(t, z) - 1| \leq \rho^0, \quad \forall (t, z) \in \mathbb{R} \times [M, \infty),$$

where ρ^0 is specified in Proposition 3.9. Since both u_2 and v_2 are bounded in $\mathbb{R} \times [-2M, 2M]$ and $\lim_{z \rightarrow \infty} (u_1(t, z), v_1(t, z)) = (1, 1)$, there is $\bar{s} \in \mathbb{R}$ such that $(u_1(t, z + \bar{s}), v_1(t, z + \bar{s})) \geq (u_2(t, z), v_2(t, z))$ for all $(t, z) \in \mathbb{R} \times [-2M, 2M]$. Without loss of generality, we may assume that $\underline{s} \leq \bar{s}$. As $(u_1(t, \cdot), v_1(t, \cdot))$ is monotone, we have:

$$(u_1(t, z + \bar{s}), v_1(t, z + \bar{s})) \geq (u_1(t, z + \underline{s}), v_1(t, z + \underline{s})) \geq (u_2(t, z), v_2(t, z)),$$

for all $(t, z) \in \mathbb{R} \times (-\infty, -M]$. Hence, $(u_1(t, z + \bar{s}), v_1(t, z + \bar{s})) \geq (u_2(t, z), v_2(t, z))$ for all $(t, z) \in \mathbb{R} \times (-\infty, 2M]$. In view of the selection of M , it follows from Proposition 3.9 that

$$(u_1(t, z + \bar{s}), v_1(t, z + \bar{s})) \geq (u_2(t, z), v_2(t, z)) \quad \text{for all } (t, z) \in \mathbb{R} \times \mathbb{R}.$$

Step 2. Define $s^* = \inf\{s \in \mathbb{R} \mid (u_1^s(t, z), v_1^s(t, z)) \geq (u_2(t, z), v_2(t, z)), \forall (t, z) \in \mathbb{R} \times \mathbb{R}\}$. Here $(u_1^s(t, z), v_1^s(t, z)) = (u_1(t, z + s), v_1(t, z + s))$, $s \in \mathbb{R}$. Clearly, s^* is bounded. In addition, (3.41) shows that $k_1 e^{\lambda_c s^*} \geq k_2$, otherwise, there is $(\underline{t}, \underline{z})$ such that $u_1^{s^*}(\underline{t}, \underline{z}) < u_2(\underline{t}, \underline{z})$. We next show that $k_1 e^{\lambda_c s^*} = k_2$. Suppose that this is not true, that is, $k_1 e^{\lambda_c s^*} > k_2$. Then $(u_1^{s^*}, v_1^{s^*}) > (u_2, v_2)$ for all $(t, z) \in \mathbb{R} \times \mathbb{R}$. If not, there exists $(\bar{t}, \bar{z}) \in \mathbb{R} \times \mathbb{R}$ such that either $u_1^{s^*}(\bar{t}, \bar{z}) = u_2(\bar{t}, \bar{z})$ or $v_1^{s^*}(\bar{t}, \bar{z}) = v_2(\bar{t}, \bar{z})$. Since

$$\int_0^1 g_u(t, \tau u_1^{s^*} + (1 - \tau)u_2, v_1^{s^*} + (1 - \tau)v_2) d\tau [u_1^{s^*} - u_2] + (u_1^{s^*} - u_2)_{zz} + c(u_1^{s^*} - u_2)_z - (u_1^{s^*} - u_2)_t \leq 0,$$

and

$$\int_0^1 h_v(t, \tau u_1^{s^*} + (1 - \tau)u_2, \tau v_1^{s^*} + (1 - \tau)v_2) d\tau [v_1^{s^*} - v_2] + d(v_1^{s^*} - v_2)_{zz} + c(v_1^{s^*} - v_2)_z - (v_1^{s^*} - v_2)_t \leq 0,$$

it follows from the maximum principle that either $u_1^{s^*} \equiv u_2$ or $v_1^{s^*} \equiv v_2$, which together with (H2) implies $(u_1^{s^*}, v_1^{s^*}) \equiv (u_2, v_2)$. This is impossible since it contradicts the assumption that $k_1 e^{\lambda_c s^*} > k_2$. Thus, $k_1 e^{\lambda_c s^*} > k_2$ implies that $(u_1^{s^*}, v_1^{s^*}) > (u_2, v_2)$ for any $(t, z) \in \mathbb{R} \times \mathbb{R}$. Now if $k_1 e^{\lambda_c s^*} > k_2$, then $k_1 e^{\lambda_c (s^* - l)} > k_2$ as long as $l \in]0, \lambda_c s^* - \ln \frac{k_2}{k_1}[$.

Let $\delta \in]0, \lambda_c s^* - \ln \frac{k_2}{k_1}[$ be fixed. Select θ such that $\theta \in]\frac{k_2}{k_1 e^{\lambda_c (s^* - \delta)}}, 1[$. As $\lim_{z \rightarrow -\infty} \frac{|z + s^* - \delta|}{|z|} = 1$, there exists $K_\theta > 0$ for which $\frac{|z + s^* - \delta|}{|z|} \geq \theta$ when $z \leq -K_\theta$. Choose $\varepsilon > 0$ such that $\varepsilon < \frac{1}{2} \frac{\theta k_1 e^{\lambda_c (s^* - \delta)} - k_2}{k_1 e^{\lambda_c (s^* - \delta)} + k_2}$. By virtue of Theorem 3.8, there exists $K_\varepsilon > 0$ such that

$$\left| \frac{u_1^{s^* - \delta}(t, z)}{|z + s^* - \delta|^\iota k_1 e^{\lambda_c (s^* - \delta)} e^{\lambda_c z} \phi(t)} - 1 \right| + \left| \frac{v_1^{s^* - \delta}(t, z)}{|z + s^* - \delta|^\iota k_1 e^{\lambda_c (s^* - \delta)} e^{\lambda_c z} \phi_d(t)} - 1 \right| \leq \varepsilon$$

and

$$\left| \frac{u_2(t, z)}{|z|^l k_2 e^{\lambda c z} \phi(t)} - 1 \right| + \left| \frac{v_2(t, z)}{|z|^l k_2 e^{\lambda c z} \phi_d(t)} - 1 \right| \leq \varepsilon$$

whenever $z \leq -K_\varepsilon$. Then it is easy to see that $(u_1^{s^*-l}, v_1^{s^*-l}) \geq (u_2, v_2)$ for each $l \in (0, \delta]$ provided $z \leq -K_\varepsilon - K_\theta$. On the other hand, for each $M > 0$ with $M \geq K_\varepsilon + K_\theta$, there is $l_M > 0$ such that $(u_1^{s^*-l}, v_1^{s^*-l}) \geq (u_2, v_2)$ for all $(t, z, l) \in \mathbb{R} \times [-M, M] \times (0, l_M]$ since $(u_1^{s^*}, v_1^{s^*}) > (u_2, v_2)$ for any $(t, z) \in \mathbb{R} \times \mathbb{R}$. Let M be sufficiently large such that

$$|u_1^{s^*-\delta}(t, z) - 1| + |v_1^{s^*-\delta}(t, z) - 1| + |u_2(t, z) - 1| + |v_2(t, z) - 1| \leq \rho^0, \quad \forall (t, z) \in \mathbb{R} \times [M, \infty),$$

where ρ^0 is given in Proposition 3.9. Without loss of generality, assume that $l_M \leq \delta$. Then we have $(u_1^{s^*-l_M}, v_1^{s^*-l_M}) \geq (u_2, v_2)$ for all $(t, z) \in \mathbb{R} \times (-\infty, M]$. Moreover, by Proposition 3.9, we infer that $(u_1^{s^*-l_M}, v_1^{s^*-l_M}) \geq (u_2, v_2)$ for all $(t, z) \in \mathbb{R} \times [M, \infty)$. Thus, there exists at least a positive number, denoted by l^* , such that $(u_1^{s^*-l^*}, v_1^{s^*-l^*}) \geq (u_2, v_2)$ for all $(t, z) \in \mathbb{R} \times \mathbb{R}$. This however contradicts the definition of s^* . Therefore, $k_1 e^{\lambda c s^*} = k_2$.

Step 3. Define $s_* = \sup\{s \in \mathbb{R} \mid (u_1^s(t, z), v_1^s(t, z)) \leq (u_2(t, z), v_2(t, z)), \forall (t, z) \in \mathbb{R} \times \mathbb{R}\}$. Clearly, s_* is bounded and $k_1 e^{\lambda c s_*} \leq k_2$. To complete the proof, it is sufficient to show that $s^* = s_*$. Indeed, note that

$$-s_* = \inf\{-s \in \mathbb{R} \mid (u_2^{-s}(t, z), v_2^{-s}(t, z)) \geq (u_1(t, z), v_1(t, z)) \forall (t, z) \in \mathbb{R} \times \mathbb{R}\}.$$

By interchanging the roles of (u_1, v_1) and (u_2, v_2) and following the same lines in Step 2, we can conclude that $k_2 e^{-\lambda c s_*} = k_1$, namely, $k_1 e^{\lambda c s_*} = k_2$. It immediately follows that $s^* = s_*$. Therefore, by the definitions of s^* and s_* , we have $(u_1^{s^*}(t, z), v_1^{s^*}(t, z)) = (u_2(t, z), v_2(t, z))$ for all $(t, z) \in \mathbb{R} \times \mathbb{R}$. This completes the proof. \square

Theorem 3.12. Suppose that (H1)–(H3) are satisfied and $d > 0$. Then for any $c \in]c^*, 0[$, (3.1) has no solutions that are nondecreasing with respect to z .

Proof. We argue by contradiction. Assume to the contrary that there exist $(u, v) \in C_b^{1,2}(\mathbb{R}^2)$ and some $c \in]c^*, 0[$ solve (3.1). Let \hat{u} and \hat{v} be defined by (3.10). Then

$$\begin{cases} \hat{u}_{zz} + c\hat{u}_z + \kappa\hat{u} + \int_0^T \frac{g(t, u, v) - g_u(t, 0, 0)u}{\phi} dt = 0, \\ d\hat{v}_{zz} + c\hat{v}_z + \alpha\hat{v} + \int_0^T \frac{h_u(t, 0, 0)u}{\tilde{\phi}} dt + \int_0^T \frac{h(t, u, v) - h_u(t, 0, 0)u - h_v(t, 0, 0)v}{\tilde{\phi}} dt = 0, \end{cases}$$

where α is given by (3.10). Write

$$\hat{g}(z) = \frac{\int_0^T [g(t, u, v) - g_u(t, 0, 0)u] \phi^{-1} dt}{\hat{u}}.$$

That is,

$$\hat{u}_{zz} + c\hat{u}_z + \kappa\hat{u} + \hat{g}(z)\hat{u} = 0.$$

Write $\vartheta = \min\{\frac{-c + \sqrt{c^2 - 2\alpha}}{4d}, \frac{-c}{2}\}$. Let $\delta \in]0, \vartheta[$ be a sufficiently small real number. Since $\lim_{z \rightarrow -\infty} \hat{g}(z) = 0$, we find that $\hat{u} = O(e^{(-c-\delta)z})$ as $z \rightarrow -\infty$. Since $m e^{\vartheta z}$ satisfies (3.15) provided that m is sufficiently large and z is negative. Using (3.13) again, we can conclude that $\hat{v} = O(e^{\vartheta z})$ as $z \rightarrow -\infty$. Thus, $\hat{g}(z)\hat{u}(z) = O(e^{(-c-\delta+\vartheta)z})$ as $z \rightarrow -\infty$. Moreover, it follows from Proposition 6.1 of [33] that

$$\hat{u}(z) = e^{-cz} [n_1 \cos(z\sqrt{4\kappa - c} + l) + n_2 \sin(z\sqrt{4\kappa - c} + l)] + O(e^{(-c-\delta-\epsilon+\vartheta)z}), \quad z \rightarrow -\infty,$$

for every $\epsilon > 0$, where n_1, n_2 , and l are certain constants. According to Theorem 4.3 of [8], there holds that either $n_1 n_2 \neq 0$ or $\hat{u} \equiv 0$. Notice that $\hat{u} > 0$. However, if $n_1 n_2 \neq 0$, then we reach a contradiction since \hat{u} is not monotone. Therefore, (3.1) has no solutions that are nondecreasing in z for $c \in]c^*, 0[$. \square

Corollary 3.13. *Suppose that all the assumptions of Theorem 2.5 are fulfilled. Then for each $c \in \mathbb{R}$ with $|c| \geq 2\sqrt{a_1p - b_1q}$, (1.8) admits a time periodic traveling wave solution $(U(t, z), W(t, z))$ with $z = x + |c|t$ connecting the equilibria $(0, 0)$ and $(1, 1)$ such that it is unique modulo translation and is monotone with respect to z . In addition, for each c with $0 < |c| < 2\sqrt{a_1p - b_1q}$, (1.8) has no time periodic traveling wave solutions connecting $(0, 0)$ and $(1, 1)$ that are monotone in z .*

Proof. Invoking Theorems 2.5, 3.11, and 3.12, it suffices to verify that (H4) and (H5) are valid for (1.8) under the assumptions given in Theorem 2.5. It is shown in the proof of Theorem 2.5 that $\bar{w} = (me^{\lambda_c z} \varphi(t), me^{\lambda_c z} \varphi(t))$ is a super-solution of (1.8), where $\lambda_c = \frac{-c - \sqrt{c^2 - 4a_1p - b_1q}}{2}$ for $c < -2\sqrt{a_1p - b_1q}$, $m > 0$ is an arbitrary constant, and $\varphi(t) = e^{\int_0^t [(a_1p - b_1q) - a_1p - b_1q] ds}$. If $c = -2\sqrt{a_1p - b_1q}$, set $\lambda_c = \frac{-c}{2}$. Let $w(t, z) = (m - nz)e^{\lambda_c z} \varphi(t)$ and $\bar{w} = (w, w)$, where $m > 0$ and $n > 0$ are arbitrary constants. Then it is easy to see that \bar{w} is a (regular) super-solution of (1.8) in $\mathbb{R} \times (-\infty, \frac{m}{n} - \frac{2}{\sqrt{a_1p - b_1q}}]$. In fact, note that $w \geq 0$, $w_z \geq 0$, and $w_{zz} \geq 0$ in $\mathbb{R} \times (-\infty, \frac{m}{n} - \frac{2}{\sqrt{a_1p - b_1q}}]$. Moreover, we have that

$$\begin{aligned} w_{zz} + cw_z + w[a_1p(1 - w) - b_1q(1 - w)] - w_t &\leq w_{zz} + cw_z + w(a_1p - b_1q) - w_t \leq 0, \\ dw_{zz} + cw_z + (1 - w)(a_2pw - b_2qw) - w_t &\leq w_{zz} + cw_z + w(a_2p - b_2q) - w_t \leq 0. \end{aligned}$$

Concerning (H5), let $g(t, u, v) = u(a_1p(1 - u) - b_1q(1 - v))$, $h(t, u, v) = (1 - v)(a_2pu - b_2qv)$. Then, $g_u(t, 1, 1) = -a_1p$, $g_v(t, 1, 1) = b_1q$, $h_u(t, 1, 1) = 0$, and $h_v(t, 1, 1) = b_2q - a_2p$. Set $v = b_2q - a_2p$. Due to the assumption, we have $v < 0$. Moreover, since $a_1p - b_1q \geq a_2p - b_2q$, it follows that $-b_1q \geq -a_1p - (b_2q - a_2p)$, and hence $-\frac{a_1p}{v} - v < 0$. In other words,

$$\int_0^T (-a_1(t)p(t) - v) dt < 0.$$

Now let $\varphi_2(t) = e^{\int_0^t [(b_2q - a_2p) - v] ds}$ and $\varphi_1(t) = \varphi_1(0)e^{\int_0^t (-a_1p - v) d\tau} + \int_0^t e^{\int_s^t (-a_1p - v) d\tau} b_1q\varphi_2 ds$, where $\varphi_1(0) = (1 - e^{\int_0^T (-a_1p - v) d\tau})^{-1} \int_0^T e^{\int_s^T (-a_1p - v) d\tau} b_1q\varphi_2 ds$. Clearly, both φ_1 and φ_2 are strictly positive periodic functions of t . Furthermore, it is easy to see that

$$v \begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix} = \begin{pmatrix} -a_1(t)p(t) & b_1(t)q(t) \\ 0 & b_2(t)q(t) - a_2(t)p(t) \end{pmatrix} \begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix} - \begin{pmatrix} \varphi_1'(t) \\ \varphi_2'(t) \end{pmatrix}.$$

This completes the proof. \square

4. Stability of periodic traveling wave solutions

In this section, we study the asymptotic stability of a periodic traveling wave solution of

$$\begin{cases} u_t = u_{xx} + g(t, u, v), \\ v_t = dv_{xx} + h(t, u, v), \end{cases} \tag{4.1}$$

where $0 < d \leq 1$, $g, h \in C^{\theta, 2}(\mathbb{R} \times \mathbb{R}^2, \mathbb{R})$, $g(t + T, u, v) = g(t, u, v)$, and $h(t + T, u, v) = h(t, u, v)$ for any $(t, u, v) \in \mathbb{R} \times \mathbb{R}^2$, $g(t, 0, 0) = g(t, 1, 1) = h(t, 0, 0) = h(t, 1, 1) \equiv 0$ for all $t \in \mathbb{R}$. Let $(u^*(t, x), v^*(t, x))$ be a periodic traveling wave solution of (4.1) that connects $(u, v) = (0, 0)$ and $(u, v) = (1, 1)$ and is monotonically increasing along the moving coordinate frame, i.e., $(u^*(t, x), v^*(t, x)) = (U(t, x - ct), W(t, x - ct)) = (U(t, z), W(t, z))$ with $z = x - ct$. Here $(U(t, z), W(t, z)) \in C^{1, 2}(\mathbb{R} \times \mathbb{R}, \mathbb{R}^2)$, (U, W) and c solve (3.1), and $(U_z, W_z) > (0, 0)$. To establish the asymptotic stability of (U, W) , we use the same type of methods employed in [19]. We first consider the initial value problem:

$$\begin{cases} u_t = u_{xx} + g(t, u, v), \\ v_t = dv_{xx} + h(t, u, v); \\ (u(0, x), v(0, x)) = (u_0(x), v_0(x)), \\ (0, 0) \preceq (u_0(x), v_0(x)) \preceq (1, 1), \end{cases} \tag{4.2}$$

where $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$. Throughout this section, (H1)–(H5) given in the last section will remain valid. In addition, we always assume that $(u_0(x), v_0(x)) \in C^\gamma(\mathbb{R}, [0, 1] \times [0, 1])$ with $(0, 0) \preceq (u_0(x), v_0(x)) \preceq (1, 1)$, where $\gamma \in]0, 1[$. We often denote by $(u(t, x, u_0), v(t, x, v_0))$ the solution of (4.2) with the initial value $(u_0(x), v_0(x))$. It is a standard practice to show the global existence of $(u(t, x, u_0), v(t, x, v_0))$.

Proposition 4.1. *Assume that (H1) and (H2) are satisfied. Let $(u(t, x, u_\tau), v(t, x, v_\tau)) \in C([\tau, \infty) \times \mathbb{R}, [0, 1]^2) \cap C_b^{1+\gamma/2, 2+\gamma}((\tau, \infty) \times \mathbb{R})$ be a solution of*

$$\begin{cases} u_t = d_1 u_{xx} + c_1 u_x + g(t, u, v), \\ v_t = d_2 v_{xx} + c_2 v_x + h(t, u, v), \end{cases} \quad (4.3)$$

with $(u(\tau, x, u_\tau), v(\tau, x, v_\tau)) = (u_\tau(x), v_\tau(x))$. Here $\tau \in \mathbb{R}$, $\gamma \in (0, 1)$, d_i, c_i ($i = 1, 2$) are constants; and $d_i \geq k$ for some positive constant k .

- (i) Let $(\bar{u}, \bar{v}), (\underline{u}, \underline{v}) \in C_b^{1+\gamma/2, 2+\gamma}([\tau, \infty) \times \mathbb{R})$ be respectively the regular super- and sub-solutions of (4.3). If $(\underline{u}(\tau, x), \underline{v}(\tau, x)) \leq (u_\tau(x), v_\tau(x)) \leq (\bar{u}(\tau, x), \bar{v}(\tau, x))$ for all $x \in \mathbb{R}$, and $(1, 1) \geq (\underline{u}(t, x), \underline{v}(t, x))$ for all $(t, x) \in [\tau, \infty) \times \mathbb{R}$, then

$$(\underline{u}(t, x), \underline{v}(t, x)) \leq (u(t, x, u_\tau), v(t, x, v_\tau)) \leq (\bar{u}(t, x), \bar{v}(t, x))$$

for all $(t, x) \in [\tau, \infty) \times \mathbb{R}$.

- (ii) Let (\bar{u}, \bar{v}) and $(\underline{u}, \underline{v}) \in C_b^{1+\gamma/2, 2+\gamma}((\tau, \infty) \times \{x < z^* + ct\}) \cap C_b([\tau, \infty) \times \{x \leq z^* + ct\})$ be respectively the regular super- and sub-solutions of (4.3) in $(\tau, \infty) \times \{x < z^* + ct\}$, where $z^* \in \mathbb{R}$ and $c \in \mathbb{R}$ are certain constants. Assume that

$$(\underline{u}(t, x), \underline{v}(t, x)) \leq (u(t, x, u_\tau), v(t, x, v_\tau)) \leq (\bar{u}(t, x), \bar{v}(t, x))$$

for all $(t, x) \in [\tau, \infty) \times \{x = z^* + ct\}$, $(\underline{u}(\tau, x), \underline{v}(\tau, x)) \leq (u_\tau(x), v_\tau(x)) \leq (\bar{u}(\tau, x), \bar{v}(\tau, x))$ for all $x \leq z^* + c\tau$, and $(\underline{u}(t, x), \underline{v}(t, x)) \leq (1, 1)$ for all $(t, x) \in [\tau, \infty) \times \{x \leq z^* + ct\}$. Then

$$(\underline{u}(t, x), \underline{v}(t, x)) \leq (u(t, x, u_\tau), v(t, x, v_\tau)) \leq (\bar{u}(t, x), \bar{v}(t, x))$$

for all $(t, x) \in [\tau, \infty) \times \{x \leq z^* + ct\}$.

Proof. We present a proof for the sake of clarity and completeness although it is similar to that of Lemma 2.4. We will need this proposition in several places. Only the first inequality of (ii) will be proved since others can be shown in a similar fashion. Set

$$M = \sup_{(t,x) \in [\tau, \infty) \times \{x \leq z^* + ct\}} (|\bar{u}| + |\bar{v}| + |\underline{u}| + |\underline{v}|).$$

Due to the assumption, $0 < M < \infty$. Let

$$\iota := \max_{i=1,2} \{|c_i| + |c|\}, \quad m := |u - \underline{u}|_\infty + |v - \underline{v}|_\infty, \quad \zeta_\iota(t, z, s, \eta) := \frac{m e^{\omega(t-\eta)}}{\iota^2 + s^2} (z^2 + \iota^2 + N(t - \eta)),$$

where $N > 2a_{\max}$ (here $a_{\max} := \max_{i=1,2} \{|d_i|\}$) and $\omega > 0$ are fixed constants such that

$$\omega \geq 2 \left(1 + \sup_{(t,u,v) \in \mathbb{R} \times [0, M] \times [0, M]} \{|g_u| + |g_v| + |h_u| + |h_v|\} \right).$$

Note that $2a_{\max} + 2\iota|z| - N - \frac{\omega}{2}(z^2 + \iota^2) < 0$ for all $z \in \mathbb{R}$. Set

$$u^s = u - \underline{u} + \zeta_\iota(t, x - ct, s, \tau), \quad v^s = v - \underline{v} + \zeta_\iota(t, x - ct, s, \tau), \quad s > |z^*|.$$

Observe that $(u^s(\tau, x), v^s(\tau, x)) > (0, 0)$ for all $x \in [-s + c\tau, z^* + c\tau]$, and $(u^s(t, x), v^s(t, s)) > (0, 0)$ for all $(t, x) \in \{[\tau, \tau + 1] \times \{x = z^* + ct\}\} \cup \{[\tau, \tau + 1] \times \{x = -s + ct\}\}$. Now we show that $(u^s, v^s) \geq (0, 0)$ for all $(t, s) \in [\tau, \tau + 1] \times [-s + ct \leq x \leq z^* + ct]$. Let

$$\alpha_1 = \int_0^1 g_u(t, \tau \underline{u} + (1 - \tau)u, \tau \underline{v} + (1 - \tau)v) d\tau, \quad \alpha_2 = \int_0^1 g_v(t, \tau \underline{u} + (1 - \tau)u, \tau \underline{v} + (1 - \tau)v) d\tau,$$

$$\beta_1 = \int_0^1 h_u(t, \tau \underline{u} + (1 - \tau)u, \tau \underline{v} + (1 - \tau)v) d\tau, \quad \beta_2 = \int_0^1 h_v(t, \tau \underline{u} + (1 - \tau)u, \tau \underline{v} + (1 - \tau)v) d\tau.$$

A straightforward computation shows that

$$\begin{aligned} d_1 u_{xx}^s + c_1 u_x^s - u_t^s &\leq g(t, u, \underline{v}) - g(t, u, v) + \frac{me^{\omega(t-\tau)}}{l^2 + s^2} [2(a_{\max} + l|z|) - N] - \omega \zeta_l(t, z, s, \tau) \\ &= \alpha_1(\underline{u} - u) + \alpha_2(\underline{v} - v) + \frac{me^{\omega(t-\tau)}}{l^2 + s^2} [2(a_{\max} + l|z|) - N] - \omega \zeta_l(t, z, s, \tau) \\ &= \alpha_1(\underline{u} - u) + \alpha_2(\underline{v} - v) - \frac{\omega}{2} \zeta_l(t, z, s, \tau) \\ &\quad + \frac{me^{\omega(t-\tau)}}{l^2 + s^2} \left[2(a_{\max} + l|z|) - N - \frac{\omega}{2}(z^2 + l^2) - \frac{\omega}{2}N(t - \tau) \right] \\ &< -\alpha_1 u^s - \alpha_2 v^s - \left(\frac{\omega}{2} - \alpha_1 - \alpha_2 \right) \zeta_l(t, z, s, \tau), \end{aligned}$$

where $z = x - ct$. Similarly

$$d_2 v_{xx}^s + c_2 v_x^s - v_t^s < -\beta_1 u^s - \beta_2 v^s - \left(\frac{\omega}{2} - \beta_1 - \beta_2 \right) \zeta_l(t, z, s, \tau).$$

Define $t^* = \sup\{t \in (\tau, \tau + 1] \mid (u^s, v^s) > (0, 0) \text{ for all } (\eta, x) \in (\tau, t] \times [-s + c\eta, z^* + c\eta]\}$. Clearly, $t^* > \tau$. If $t^* < \tau + 1$, then there exists $x^* \in]-s + ct^*, z^* + ct^*[$ for which either $u^s(t^*, x^*) = 0$ or $v^s(t^*, x^*) = 0$. Assume without loss of generality that $v^s(t^*, x^*) = 0$, that is,

$$v(t^*, x^*, v_\tau) + \zeta_l(t^*, x^* - ct^*, s, \tau) = \underline{v}(t^*, x^*).$$

As $\underline{v}(t, x) \leq 1$ and $v(t, x, v_\tau) \geq 0$ by the assumption, we must have $\underline{v}(t^*, x^*) > 0$ and $v(t^*, x^*, v_\tau) < 1$. This together with (H2) implies that $\beta_1 \geq 0$ in a neighborhood $[\tau + \varepsilon, t^*] \times [x^* - \varepsilon, x^* + \varepsilon]$ of (t^*, x^*) , where $\varepsilon > 0$ is sufficiently small. Consequently,

$$\begin{aligned} d_2 v_{xx}^s + c_2 v_x^s - v_t^s &< -\beta_1 u^s - \beta_2 v^s - \left(\frac{\omega}{2} - \beta_1 - \beta_2 \right) \zeta_l(t, x - ct, s, \tau) \\ &\leq -\beta_2 v^s \quad \text{in } (\tau + \varepsilon, t^*] \times [x^* - \varepsilon, x^* + \varepsilon]. \end{aligned}$$

Since v^s attains its local minimum at (t^*, x^*) , we find that

$$0 \leq (d_2 v_{xx}^s + c_2 v_x^s - v_t^s)(t^*, x^*) < -\beta_2 v^s(t^*, x^*) = 0,$$

which is impossible. Therefore, $(u^s, v^s) \geq (0, 0)$ for all $(t, x) \in [\tau, \tau + 1] \times [-s + ct, z^* + ct]$. Since $s > |z^*|$ is arbitrary, arguing in the manner similar to that shown in Theorem 2.4, we infer that $(u(t, x, u_\tau), v(t, x, v_\tau)) \geq (\underline{u}(t, x), \underline{v}(t, x))$ for all $(t, x) \in [\tau, \tau + 1] \times \{x \leq z^* + ct\}$. By using $\zeta_l(t, z, s, \tau + n)$ with $n \in \mathbb{N}^+$, we can inductively show that $(u(t, x, u_\tau), v(t, x, v_\tau)) \geq (\underline{u}(t, x), \underline{v}(t, x))$ for all $(t, s) \in [\tau + n, \tau + n + 1] \times \{x \leq z^* + ct\}$, $n \in \mathbb{N}^+$. Therefore, $(u(t, x, u_\tau), v(t, x, v_\tau)) \geq (\underline{u}(t, x), \underline{v}(t, x))$ for all $(t, x) \in [\tau, \infty) \times \{x \leq z^* + ct\}$. The proof is completed. \square

In the sequel, we let $\chi(s)$ be a smooth function such that $\chi(s) = 0$ for $s \leq \underline{s}$, $\chi(s) = 1$ for $s \geq \bar{s}$, and $0 \leq \chi'$ and $|\chi'| + |\chi''| \leq 1$, where \underline{s} and \bar{s} with $\underline{s} < \bar{s}$ are fixed constants.

We set:

$$\begin{cases} \xi_c(t, s) = [1 - \chi(s)]e^{(\lambda_c + \varepsilon)s} \phi(t) + \chi(s)\varphi_1(t), \\ \zeta_c(t, s) = [1 - \chi(s)]e^{(\lambda_c + \varepsilon)s} \phi_1(t) + \chi(s)\varphi_2(t), \end{cases} \quad (4.4)$$

where $c < c^* = -2\sqrt{\kappa}$, $\phi_1(t)$ is given by (3.29) with $d = 1$, $\epsilon \in]0, \sqrt{c^2 - 4\kappa}[$ is a fixed constant such that

$$0 < \beta := -\frac{[(\lambda_c + \epsilon)^2 + c(\lambda_c + \epsilon) + \kappa]}{2} \leq \frac{|\nu|}{2}. \tag{4.5}$$

We also set:

$$\ell^+ := \min \left\{ \min_{t \in [0, T]} \frac{1}{\phi_1(t)}, \min_{t \in [0, T]} \frac{1}{\phi_2(t)} \right\}. \tag{4.6}$$

Proposition 4.2. Assume that (H1)–(H5) are satisfied. Let (U, W) and c solve (3.1) with $c < c^* = -2\sqrt{\kappa}$. Then

$$\limsup_{s \rightarrow \infty} \sup_{(t, z) \in \mathbb{R} \times \mathbb{R}, \ell \in (0, \ell^+]} \frac{U(t, z) - \ell \xi_c(t, z + s) - 1}{\ell \phi_1(t)} \leq -1, \tag{4.7}$$

$$\limsup_{s \rightarrow \infty} \sup_{(t, z) \in \mathbb{R} \times \mathbb{R}, \ell \in (0, \ell^+]} \frac{W(t, z) - \ell \zeta_c(t, z + s) - 1}{\ell \phi_2(t)} \leq -1, \tag{4.8}$$

where ℓ^+ is given by (4.6).

Proof. The proof is similar to that of Lemma 3.1 in [19]. Only the first inequality (4.7) will be proved since the other can be shown similarly. Assume to the contrary that the claimed conclusion is not true. Then there exist three sequences $\{(t_n, z_n)\}$, $\{\ell_n\}$, and $\{s_n\}$ and a positive constant ϵ such that

$$s_n \rightarrow \infty \quad \text{as } n \rightarrow \infty \quad \text{and} \quad \frac{U(t_n, z_n) - \ell_n \xi_c(t_n, z_n + s_n) - 1}{\ell_n \phi_1(t_n)} \geq -1 + \epsilon$$

for all $n \in \mathbb{N}^+$. Since $U(\cdot, z)$, $\xi_c(\cdot, z + s)$, and $\phi_1(\cdot)$ are periodic functions with the same period T , we may assume that $t_n \in [0, T]$ for all $n \in \mathbb{N}^+$. Hence there exist a subsequence of $\{t_n\}$, still labeled by $\{t_n\}$, and $t^* \in [0, T]$ such that $t_n \rightarrow t^*$. We also notice that $\ell_n \in (0, \ell^+]$, which implies that there exists an $\ell^* \in [0, \ell^+]$ such that $\ell_n \rightarrow \ell^*$. Furthermore, as $s_n \rightarrow \infty$, two cases may occur, that is, either $z_n + s_n \rightarrow \infty$ or $z_n + s_n$ is bounded from above. If $z_n + s_n \rightarrow \infty$, then we find that

$$-1 = \lim_{n \rightarrow \infty} \frac{-\ell_n \xi_c(t_n, z_n + s_n)}{\ell_n \phi_1(t_n)} \geq \frac{U(t_n, z_n) - \ell_n \xi_c(t_n, z_n + s_n) - 1}{\ell_n \phi_1(t_n)} \geq -1 + \epsilon.$$

This contradiction excludes the possibility that $z_n + s_n \rightarrow \infty$ and leads us to the case that $z_n + s_n$ is bounded from above, which implies that $z_n \rightarrow -\infty$. As $\lim_{z \rightarrow -\infty} U(t, z) = 0$ uniformly in t and $\xi_c(t_n, z_n + s_n) \geq 0$ for all $n \in \mathbb{N}^+$, it follows that

$$-\frac{1}{\phi_1(t^*)} = \lim_{n \rightarrow \infty} \frac{U(t_n, z_n) - 1}{\phi_1(t_n)} \geq \lim_{n \rightarrow \infty} -(1 - \epsilon)\ell_n \geq -(1 - \epsilon)\ell^+,$$

which contradicts the definition of ℓ^+ . Therefore, (4.7) holds for $c < c^*$. \square

In what follows, we fix $s_0 \in \mathbb{R}$ such that

$$\sup_{(t, s) \in \mathbb{R} \times \mathbb{R}} \frac{U(t, s) - \ell \xi_c(t, s + s_0) - 1}{\phi_1(t)} \leq -\frac{\ell}{2} \quad \text{for all } \ell \in (0, \ell^+] \tag{4.9}$$

and

$$\sup_{(t, s) \in \mathbb{R} \times \mathbb{R}} \frac{W(t, s) - \ell \zeta_c(t, s + s_0) - 1}{\phi_2(t)} \leq -\frac{\ell}{2} \quad \text{for all } \ell \in (0, \ell^+]. \tag{4.10}$$

Lemma 4.3. Suppose that (H1)–(H5) are satisfied. Let (U, W) and c solve (3.1) with $c < c^*$. Let β and ℓ^+ be given by (4.5) and (4.6), respectively. Then there exists $\delta^* \in (0, \ell^+]$ such that for each $z_0 \in \mathbb{R}$ and each $\sigma \geq \max\{1/\beta, 1/|c|\beta\}$, $(u^\pm(t, x), v^\pm(t, x))$ are respectively the super- and sub-solutions of (4.1) in $\mathbb{R}^+ \times \mathbb{R}$ whenever $\delta \in (0, \delta^*]$. Here

$$\begin{aligned} u^\pm(t, x) &= U(t, x - ct + z_0 \pm \sigma(1 - e^{-\beta t})) \pm \delta \xi_c(t, x - ct + z_0 + s_0 \pm \sigma(1 - e^{-\beta t})) e^{-\beta t}, \\ v^\pm(t, x) &= W(t, x - ct + z_0 \pm \sigma(1 - e^{-\beta t})) \pm \delta \zeta_c(t, x - ct + z_0 + s_0 \pm \sigma(1 - e^{-\beta t})) e^{-\beta t}. \end{aligned}$$

Proof. We only show that (u^+, v^+) is a super-solution of (4.1) since the other case can be proved similarly. Let $z = x - ct + z_0 + \sigma(1 - e^{-\beta t})$ and $z' = x - ct + z_0 + \sigma(1 - e^{-\beta t}) + s_0$. A direct calculation yields that

$$\begin{aligned} &g(t, u^+, v^+) + u_{xx}^+ - u_t^+ \\ &= g(t, U(t, z) + e^{-\beta t} \delta \xi_c(t, z'), W(t, z) + e^{-\beta t} \delta \zeta_c(t, z')) - g(t, U(t, z), W(t, z)) + \delta e^{-\beta t} \beta \xi_c \\ &\quad + \delta e^{-\beta t} \left\{ -\frac{\sigma \beta}{\delta} U_z + (1 - \chi) e^{(\lambda_c + \epsilon)z'} [((\lambda_c + \epsilon)^2 + c(\lambda_c + \epsilon))\phi - \phi'] - \chi \phi'_1 + \Delta_1(t, z') \right\} \\ &= \delta e^{-\beta t} [(g_1(t, z)\xi_c(t, z') + g_2(t, z)\zeta_c(t, z')) + \beta \xi_c] \\ &\quad + \delta e^{-\beta t} \left\{ -\frac{\sigma \beta}{\delta} U_z - (1 - \chi) e^{(\lambda_c + \epsilon)z'} [g_u(t, 0, 0)\phi + g_v(t, 0, 0)\phi_1 + 2\beta\phi] - \chi \phi'_1 + \Delta_1(t, z') \right\} \\ &= \delta e^{-\beta t} \left\{ -\frac{\sigma \beta}{\delta} U_z + e^{(\lambda_c + \epsilon)z'} (1 - \chi) [(g_1 - g_u(t, 0, 0))\phi + (g_2 - g_v(t, 0, 0))\phi_1 - \beta\phi] \right\} \\ &\quad + \delta e^{-\beta t} \Delta_1(t, z') + \delta e^{-\beta t} \chi [(g_1 - g_u(t, 1, 1))\phi_1 + (g_2 - g_v(t, 1, 1))\phi_2 + v\phi_1 + \beta\phi_1], \end{aligned}$$

where

$$\begin{aligned} \Delta_1(t, z') &= e^{(\lambda_c + \epsilon)z'} [-\chi''\phi - 2(\lambda_c + \epsilon)\chi'\phi + \chi''\phi_1 - (1 - \chi)\sigma\beta e^{-\beta t}(\lambda_c + \epsilon)\phi] \\ &\quad - (\sigma\beta e^{-\beta t} - c)\chi'(\phi_1 - e^{(\lambda_c + \epsilon)z'}\phi), \\ g_1(t, z) &= \int_0^1 [g_u(t, \tau(U + \delta\xi_c) + (1 - \tau)U, \tau(W + \delta\zeta_c) + (1 - \tau)W)] d\tau, \\ g_2(t, z) &= \int_0^1 [g_v(t, \tau(U + \delta\xi_c) + (1 - \tau)U, \tau(W + \delta\zeta_c) + (1 - \tau)W)] d\tau. \end{aligned}$$

Similarly,

$$\begin{aligned} &h(t, u^+, v^+) + dv_{xx}^+ - v_t^+ \\ &= \delta e^{-\beta t} [h_1(t, z)\xi_c(t, z') + h_2(t, z)\zeta_c(t, z') + \beta\zeta] + (d - 1)(\lambda_c + \epsilon)^2(1 - \chi)e^{(\lambda_c + \epsilon)z'} \\ &\quad + \delta e^{-\beta t} \left\{ -\frac{\sigma \beta}{\delta} W_z - (1 - \chi) e^{(\lambda_c + \epsilon)z'} [h_u(t, 0, 0)\phi + h_v(t, 0, 0)\phi_1 + 2\beta\phi_1] - \chi \phi'_2 + \Delta_2(t, z') \right\} \\ &\leq \delta e^{-\beta t} \left\{ -\frac{\sigma \beta}{\delta} W_z + e^{(\lambda_c + \epsilon)z'} (1 - \chi) [(h_1 - h_u(t, 0, 0))\phi + (h_2 - h_v(t, 0, 0))\phi_1 - \beta\phi_1] \right\} \\ &\quad + \delta e^{-\beta t} \Delta_2(t, z') + \delta e^{-\beta t} \chi [(h_1 - h_u(t, 1, 1))\phi_1 + (h_2 - h_v(t, 1, 1))\phi_2 + v\phi_2 + \beta\phi_2], \end{aligned}$$

where

$$\begin{aligned} \Delta_2(t, z) &= e^{(\lambda_c + \epsilon)z'} [-d\chi''\phi_1 - 2d(\lambda_c + \epsilon)\chi'\phi_1 + d\chi''\phi_2 - (1 - \chi)\sigma\beta e^{-\beta t}(\lambda_c + \epsilon)\phi_1] \\ &\quad - (\sigma\beta e^{-\beta t} - c)\chi'(\phi_2 - e^{(\lambda_c + \epsilon)z'}\phi_1), \\ h_1(t, z) &= \int_0^1 [g_u(t, \tau(U + \delta\xi_c) + (1 - \tau)U, \tau(W + \delta\zeta_c) + (1 - \tau)W)] d\tau, \\ h_2(t, z) &= \int_0^1 [g_v(t, \tau(U + \delta\xi_c) + (1 - \tau)U, \tau(W + \delta\zeta_c) + (1 - \tau)W)] d\tau. \end{aligned}$$

Write

$$\begin{aligned} \Gamma_0(t, z) &= |g_1 - g_u(t, 0, 0)| + |g_2 - g_v(t, 0, 0)| + |h_1 - h_u(t, 0, 0)| + |h_2 - g_v(t, 0, 0)|, \\ \Gamma_1(t, z) &= |g_1 - g_u(t, 1, 1)| + |g_2 - g_v(t, 1, 1)| + |h_1 - h_u(t, 1, 1)| + |h_2 - g_v(t, 1, 1)|. \end{aligned}$$

As $\lim_{z \rightarrow -\infty} \{\sup_{t \in \mathbb{R}} \Gamma_0(t, z)\} = 0$ and $\lim_{z \rightarrow \infty} \{\sup_{t \in \mathbb{R}} \Gamma_1(t, z)\} = 0$, we can choose $M > 0$ such that $\chi(-M) = 0$, $\chi(M) = 1$,

$$\sup_{(t, z) \in \mathbb{R} \times (-\infty, M]} |\Gamma_0| \leq \frac{\beta \min\{\min_t \phi, \min_t \phi_1\}}{\max_t(\phi + \phi_1)}, \quad \sup_{(t, z) \in \mathbb{R} \times [M, -\infty)} |\Gamma_1| \leq \frac{\nu \min\{\min_t \varphi_1, \min_t \varphi_2\}}{2 \max_t(\varphi_1 + \varphi_2)},$$

and $\sup_{(t, z) \in \mathbb{R} \times (-M, M]} |\Gamma_0| + |\Gamma_1| \leq K$, where $K > 0$ depends only on $\|\phi\|$, $\|\phi_1\|$, $\|\varphi_1\|$, $\|\varphi_2\|$, λ_c , ϵ , ν , and $\max_{(t, u, v) \in \mathbb{R} \times [-2, 2]^2} \{|g_u|, |g_v|, |h_u|, |h_v|\}$. Therefore, when $z \leq -M$, it follows that

$$\begin{aligned} g(t, u^+, v^+) + u_{xx}^+ - u_t^+ &\leq \delta e^{-\beta t} \left[-\frac{\sigma\beta}{\delta} U_z - e^{(\lambda_c + \epsilon)z'} (1 - \chi)\sigma\beta e^{-\beta t} (\lambda_c + \epsilon)\phi \right] < 0, \\ h(t, u^+, v^+) + dv_{xx}^+ - v_t^+ &\leq \delta e^{-\beta t} \left[-\frac{\sigma\beta}{\delta} W_z - e^{(\lambda_c + \epsilon)z'} (1 - \chi)\sigma\beta e^{-\beta t} (\lambda_c + \epsilon)\phi_1 \right] < 0. \end{aligned}$$

Whence, when $z \geq M$, we have that

$$\begin{aligned} g(t, u^+, v^+) + u_{xx}^+ - u_t^+ &\leq -\delta e^{-\beta t} \frac{\sigma\beta}{\delta} U_z < 0, \\ h(t, u^+, v^+) + dv_{xx}^+ - v_t^+ &\leq -\delta e^{-\beta t} \frac{\sigma\beta}{\delta} W_z < 0. \end{aligned}$$

Now let

$$\begin{aligned} \Delta(t, z) &= |e^{(\lambda_c + \epsilon)z'} [-\chi''\phi - 2(\lambda_c + \epsilon)\chi'\phi + \chi''\varphi_1 - (1 - \chi)\sigma\beta e^{-\beta t} (\lambda_c + \epsilon)\phi]| \\ &\quad + |e^{(\lambda_c + \epsilon)z'} [-d\chi''\phi_1 - 2d(\lambda_c + \epsilon)\chi'\phi_1 + d\chi''\varphi_2 - (1 - \chi)\sigma\beta e^{-\beta t} (\lambda_c + \epsilon)\phi_1]| \\ &\quad + |-c\chi'(\varphi_1 - e^{(\lambda_c + \epsilon)z'}\phi) - c\chi'(\varphi_2 - e^{(\lambda_c + \epsilon)z'}\phi_1)| + e^{(\lambda_c + \epsilon)z'} |\Gamma_0| + |\Gamma_1|. \end{aligned}$$

Then, for $z \in [-M, M]$, it follows that

$$\begin{aligned} g(t, u^+, v^+) + u_{xx}^+ - u_t^+ &\leq \delta e^{-\beta t} \left[-\frac{\sigma\beta}{\delta} U_z + \sigma\beta e^{-\beta t} \chi'(\varphi_1 - e^{(\lambda_c + \epsilon)z'}\phi) + \Delta(t, z) \right] \\ &\leq \delta\sigma\beta e^{-\beta t} \left[-\frac{U_z}{\delta} + e^{-\beta t} \chi'(\varphi_1 - e^{(\lambda_c + \epsilon)z'}\phi) + \frac{\Delta(t, z)}{\sigma\beta} \right], \\ h(t, u^+, v^+) + dv_{xx}^+ - v_t^+ &\leq \delta e^{-\beta t} \left[-\frac{\sigma\beta}{\delta} W_z + \sigma\beta e^{-\beta t} \chi'(\varphi_2 - e^{(\lambda_c + \epsilon)z'}\phi_1) + \Delta(t, z) \right] \\ &\leq \delta\sigma\beta e^{-\beta t} \left[-\frac{W_z}{\delta} + e^{-\beta t} \chi'(\varphi_2 - e^{(\lambda_c + \epsilon)z'}\phi_1) + \frac{\Delta(t, z)}{\sigma\beta} \right]. \end{aligned}$$

Since $(U_z, W_z) > (0, 0)$, there exists $\gamma > 0$ such that $\gamma \leq \min\{\inf_{(t, z) \in \mathcal{E}} U_z, \inf_{(t, z) \in \mathcal{E}} W_z\}$, where $\mathcal{E} = \mathbb{R} \times [-M, M]$.

Let

$$\begin{aligned} \Delta^{as}(t, z) &= e^{(\lambda_c + \epsilon)z'} [|\chi''\phi| + |2(\lambda_c + \epsilon)\chi'\phi| + |\chi''\varphi_1| + |(1 - \chi)e^{-\beta t} (\lambda_c + \epsilon)\phi|] \\ &\quad + e^{(\lambda_c + \epsilon)z'} [|d\chi''\phi_1| + |2d(\lambda_c + \epsilon)\chi'\phi_1| + |d\chi''\varphi_2| + |(1 - \chi)e^{-\beta t} (\lambda_c + \epsilon)\phi_1|] \\ &\quad + |\chi'(\varphi_1 - e^{(\lambda_c + \epsilon)z'}\phi)| + |\chi'(\varphi_2 - e^{(\lambda_c + \epsilon)z'}\phi_1)| + e^{(\lambda_c + \epsilon)z'} |\Gamma_0| + |\Gamma_1|, \\ \delta^* &= \min \left\{ \ell^+, \frac{\gamma}{2 \max_{(t, z) \in \mathbb{R} \times [-M, M]} \{\varphi_1 + \varphi_2 + e^{(\lambda_c + \epsilon)(z+s_0)}(\phi + \phi_1) + \Delta^{as}(t, z)\}} \right\}. \end{aligned} \tag{4.11}$$

Note that $z' = z + s_0$ and $|\frac{\Delta}{\sigma\beta}| \leq \Delta^{as}$ since $\sigma\beta \geq \max\{1, |c|\}$. Then we readily see that (u^+, v^+) is a super-solution of (4.1) for any $\delta \in (0, \delta^*)$. The proof is completed. \square

In what follows, we set:

$$\begin{aligned} u_{\sigma}^{\pm}(t, x, z_0) &= U(t, x - ct + z_0 \pm \sigma(1 - e^{-\beta t})) \pm \delta^* \xi_c(t, x - ct + z_0 + s_0 \pm \sigma(1 - e^{-\beta t})) e^{-\beta t}, \\ v_{\sigma}^{\pm}(t, x, z_0) &= W(t, x - ct + z_0 \pm \sigma(1 - e^{-\beta t})) \pm \delta^* \zeta_c(t, x - ct + z_0 + s_0 \pm \sigma(1 - e^{-\beta t})) e^{-\beta t}, \end{aligned}$$

where β and δ^* are given by (4.5) and (4.11), respectively.

Lemma 4.4. *Suppose that (H1)–(H5) are satisfied. Assume that*

$$\lim_{x \rightarrow -\infty} \frac{u_0(x)}{k\phi(0)e^{\lambda_c x}} = 1, \quad \lim_{x \rightarrow -\infty} \frac{v_0(x)}{k\phi_d(0)e^{\lambda_c x}} = 1 \tag{4.12}$$

for some positive constant k . Furthermore, assume that

$$\liminf_{x \rightarrow \infty} (u_0(x) - 1) \geq -\varepsilon_0, \quad \liminf_{x \rightarrow \infty} (v_0(x) - 1) \geq -\varepsilon_0 \tag{4.13}$$

for some $\varepsilon_0 \in [0, \frac{\delta^*}{2\ell^+})$. Then there exist $z_0 \in \mathbb{R}$, $\sigma^* \geq 1$, and $t^* > 0$ such that

$$(u_{\sigma}^-(t, x, z_0), v_{\sigma}^-(t, x, z_0)) \leq (u(t, x, u_0), v(t, x, v_0)) \leq (u_{\sigma}^+(t, x, z_0), v_{\sigma}^+(t, x, z_0)) \tag{4.14}$$

for all $(t, x) \in [t^*, \infty) \times \mathbb{R}$ and $\sigma \geq \sigma^*$.

Proof. We first show that there exists $t^* > 0$ such that

$$\sup_{(t,s) \in \mathbb{R} \times \mathbb{R}} \frac{U(t, s) - \delta^* \xi_c(t, s + s_0) e^{-\beta t^*} - 1}{\varphi_1(t)} < \liminf_{x \rightarrow \infty} \left\{ \inf_{t \in \mathbb{R}} \frac{u(t^*, x, u_0) - 1}{\varphi_1(t)} \right\}. \tag{4.15}$$

By virtue of the assumptions, there exists $\gamma > 1$ for which $\gamma \varepsilon_0 \leq \frac{\delta^*}{2\ell^+}$. Moreover, thanks to (4.13), the fact that $|u(t, x, u_0)| + |v(t, x, v_0)|$ is bounded for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$ and $(u_0, v_0) \in C^{\alpha}(\mathbb{R}, \mathbb{R}^2)$, and the basic properties for the heat potentials, there exists $t^* > 0$ such that

$$\liminf_{x \rightarrow \infty} \left\{ \inf_{t \in \mathbb{R}} \frac{u(t^*, x) - 1}{\varphi_1(t)} \right\} > -\ell^+ \gamma \varepsilon_0 e^{-\beta t^*}, \quad \liminf_{x \rightarrow \infty} \left\{ \inf_{t \in \mathbb{R}} \frac{v(t^*, x) - 1}{\varphi_2(t)} \right\} > -\ell^+ \gamma \varepsilon_0 e^{-\beta t^*}.$$

This together with (4.9) and (4.10) yields (4.15).

We next show that there exist $z_0 \in \mathbb{R}$ and $\sigma^* \geq 1$ such that

$$u_{\sigma}^-(t^*, x, z_0) \leq u(t^*, x, u_0) \tag{4.16}$$

whenever $\sigma \geq \sigma^*$. In view of (3.25), we can fix $z_0 \in \mathbb{R}$ such that

$$\lim_{x \rightarrow -\infty} \frac{U(0, x + z_0)}{k\phi(0)e^{\lambda_c x}} = 1, \quad \lim_{x \rightarrow -\infty} \frac{W(0, x + z_0)}{k\phi_d(0)e^{\lambda_c x}} = 1.$$

Note that such a z_0 is unique. In addition, since $(u(0, x, u_0), v(0, x, v_0)) = (u_0, v_0)$, for any compact subset $I \subset [0, \infty)$, it follows from Proposition A.1 in Appendix A that

$$\lim_{x \rightarrow -\infty} \frac{u(t, x, u_0)}{U(t, x - ct + z_0)} = 1, \quad \lim_{x \rightarrow -\infty} \frac{v(t, x, v_0)}{W(t, x - ct + z_0)} = 1 \quad \text{uniformly in } t \in I. \tag{4.17}$$

Now assume to the contrary that (4.16) is not true. Then there exist two sequences $\{x_n\}$ and $\{\sigma_n\}$ such that

$$\sigma_n \rightarrow \infty \quad \text{as } n \rightarrow \infty \quad \text{and} \quad u_{\sigma_n}^-(t^*, x_n, z_0) > u(t^*, x_n, u_0). \tag{4.18}$$

Notice that $u_{\sigma_n}^-(t^*, x_n, z_0) = U(t^*, z_n) - \delta^* \xi_c(t^*, z_n + s_0) e^{-\beta t^*}$, where $z_n = x_n - ct^* + z_0 - \sigma_n(1 - e^{-\beta t^*})$. Up to extraction of a subsequence, two cases may occur: either the sequence $\{z_n\}$ is bounded from below or $\lim_{n \rightarrow \infty} z_n = -\infty$. If $\{z_n\}$ is bounded from below, then $x_n \rightarrow \infty$ as $n \rightarrow \infty$. Hence, it follows from (4.18) that

$$\begin{aligned} \sup_{(t,s) \in \mathbb{R} \times \mathbb{R}} \frac{U(t, s) - \delta^* \xi_c(t, s + s_0) e^{-\beta t} - 1}{\varphi_1(t)} &\geq \sup_n \frac{U(t^*, z_n) - \delta^* \xi_c(t^*, z_n + s_0) e^{-\beta t^*} - 1}{\varphi_1(t^*)} \\ &\geq \liminf_{n \rightarrow \infty} \left\{ \frac{u(t^*, x_n, u_0) - 1}{\varphi_1(t^*)} \right\} \geq \liminf_{x \rightarrow \infty} \left\{ \inf_{t \in \mathbb{R}} \frac{u(t^*, x, u_0) - 1}{\varphi_1(t)} \right\}, \end{aligned}$$

which contradicts (4.15).

Therefore, we have the case that $\lim_{n \rightarrow \infty} z_n = -\infty$. If this occurs, we only need to consider two possibilities: either $\{x_n\}$ is bounded or $x_n \rightarrow -\infty$. If $\{x_n\}$ is bounded, without loss of generality, we may assume that $x_n \rightarrow x_\infty$ for some $x_\infty \in \mathbb{R}$. Meanwhile, note that $\lim_{n \rightarrow \infty} u_{\sigma_n}^-(t^*, x_n, z_0) = 0$. Moreover, since $(u_0, v_0) \succeq (0, 0)$, the comparison principle implies $(u(t^*, x, u_0), v(t^*, x, v_0)) > (0, 0)$ for all $x \in \mathbb{R}$. It follows from the continuity of $u(t, x, u_0)$ with respect to x that

$$0 = \lim_{n \rightarrow \infty} u_{\sigma_n}^-(t^*, x_n, z_0) \geq \lim_{n \rightarrow \infty} u(t^*, x_n, u_0) = u(t^*, x_\infty, u_0) \geq 0.$$

This shows that $u(t^*, x_\infty, u_0) = 0$, which is a contradiction. This contradiction rules out the possibility that $\{x_n\}$ is bounded. In case that $x_n \rightarrow -\infty$ as $n \rightarrow \infty$, in view of (4.17) and (4.18), we find that

$$0 = \lim_{n \rightarrow \infty} \frac{u_{\sigma_n}^-(t^*, x_n, z_0)}{U(t^*, x_n - ct^* + z_0)} \geq \lim_{n \rightarrow \infty} \frac{u(t^*, x_n, u_0)}{U(t^*, x_n - ct^* + z_0)} = 1.$$

The contradiction yields that there exists a $\sigma_1 \geq 1$ for which $u_{\sigma}^-(t^*, x, z_0) \leq u(t^*, x, u_0)$ for all $\sigma \geq \sigma_1$. With the same reasoning, we can show that there exists a $\sigma_2 \geq 1$ such that $v_{\sigma}^-(t^*, x, z_0) \leq v(t^*, x, v_0)$ for all $\sigma \geq \sigma_2$.

Next we prove that

$$u(t^*, x, u_0) \leq u_{\sigma}^+(t^*, x, z_0), \quad \sigma \geq \sigma_3 \tag{4.19}$$

for some $\sigma_3 \geq 1$. Again, we shall argue by contradiction. Assume that this not true, then there exist two sequences $\{x_n\}$ and $\{\sigma_n\}$ such that $\lim_{n \rightarrow \infty} \sigma_n = \infty$ and

$$u_{\sigma_n}^+(t^*, x_n, z_0) < u(t^*, x_n, u_0). \tag{4.20}$$

Thus, up to extraction of a subsequence, two cases may occur: either $\lim_{n \rightarrow \infty} z_n = \infty$ or $\{z_n\}$ is bounded from above, where $z_n = x_n - ct^* + z_0 + \sigma_n(1 - e^{-\beta t^*})$. If $\lim_{n \rightarrow \infty} z_n = \infty$, then we find:

$$1 + \delta^* e^{-\beta t^*} \varphi_1(t) = \lim_{n \rightarrow \infty} u_{\sigma_n}^+(t^*, x_n, z_0) \leq \sup_n u(t^*, x_n, u_0).$$

This is a contradiction since the comparison principle implies that $(u(t, x, u_0), v(t, x, v_0)) \leq (1, 1)$ for any $t > 0$. Thus it is impossible that $\lim_{n \rightarrow \infty} z_n = \infty$.

In case that $\{z_n\}$ is bounded from above. Then $x_n \rightarrow -\infty$, and it follows from (4.17) and (4.20) that

$$\infty = \lim_{n \rightarrow \infty} \frac{u_{\sigma_n}^+(t^*, x_n, z_0)}{U(t^*, x_n - ct^* + z_0)} \leq \lim_{n \rightarrow \infty} \frac{u(t^*, x_n, u_0)}{U(t^*, x_n - ct^* + z_0)} = 1.$$

This contradiction confirms (4.19). Similarly we can show that there exists a $\sigma_4 \geq 1$ such that $v(t^*, x, v_0) \leq v_{\sigma}^+(t^*, x)$ once $\sigma \geq \sigma_4$. Therefore, let $\sigma^* = \max_{i=1,2,3,4} \{\sigma_i\}$, we have:

$$(u_{\sigma}^-(t^*, x, z_0), v_{\sigma}^-(t^*, x, z_0)) \leq (u(t^*, x, u_0), v(t^*, x, v_0)) \leq (u_{\sigma}^+(t^*, x, z_0), v_{\sigma}^+(t^*, x, z_0)),$$

provided that $\sigma \geq \sigma^*$. To prove (4.14), we observe that $(u_{\sigma}^-(t, x, z_0), v_{\sigma}^-(t, x, z_0)) \leq (1, 1)$ and $(u_{\sigma}^+(t, x, z_0), v_{\sigma}^+(t, x, z_0)) > (0, 0)$. Furthermore, the comparison principle implies that

$$(0, 0) < (u(t, x, u_0), v(t, x, v_0)) < (1, 1) \quad \text{for any } t \geq t^*.$$

Thus, (4.14) follows from Proposition 4.1. The proof is completed. \square

Proposition 4.5. *Suppose that all assumptions of Lemma 4.4 are satisfied. Let $(U, W) \in C_b^{1,2}(\mathbb{R} \times \mathbb{R})$ and c solve (3.1) with $c < c^*$. Let ϵ be specified by (4.5). Let z_0 be the number such that*

$$\lim_{x \rightarrow -\infty} \frac{U(0, x + z_0)}{k\phi(0)e^{\lambda_c x}} = 1, \quad \lim_{x \rightarrow -\infty} \frac{W(0, x + z_0)}{k\phi_d(0)e^{\lambda_c x}} = 1.$$

Then for each $\eta > 0$, there exist $\sigma_\eta \in \mathbb{R}$ and $D_\eta > 0$ such that

$$\begin{aligned} U(t, x - ct + z_0 - \eta) - D_\eta \phi(t) e^{(\lambda_c + \epsilon)(x - ct)} &\leq u(t, x, u_0), \\ W(t, x - ct + z_0 - \eta) - D_\eta \phi_1(t) e^{(\lambda_c + \epsilon)(x - ct)} &\leq v(t, x, v_0) \end{aligned} \tag{4.21}$$

for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$, and

$$\begin{aligned} u(t, x, u_0) &\leq U(t, x - ct + z_0 + \eta) + D_\eta \phi(t) e^{(\lambda_c + \epsilon)(x - ct)}, \\ v(t, x, v_0) &\leq W(t, x - ct + z_0 + \eta) + D_\eta \phi_1(t) e^{(\lambda_c + \epsilon)(x - ct)} \end{aligned} \tag{4.22}$$

whenever $x - ct \leq \sigma_\eta$.

Proof. We shall start with (4.21). Assume without loss of generality that $z_0 = 0$. We first show that there exists $D_\eta^0 > 0$ such that

$$(U(0, x - \eta) - D\phi(0)e^{(\lambda_c + \epsilon)x}, W(0, x - \eta) - D\phi_1(0)e^{(\lambda_c + \epsilon)x}) \leq (u_0(x), v_0(x)) \tag{4.23}$$

for all $x \in \mathbb{R}$ when $D \geq D_\eta^0$. Indeed, there exists $M_\eta > 0$ such that

$$U(0, x - \eta) - \phi(0)e^{(\lambda_c + \epsilon)x} \leq u_0(x), \quad W(0, x - \eta) - \phi_1(0)e^{(\lambda_c + \epsilon)x} \leq v_0(x) \quad \text{for all } |x| \geq M_\eta.$$

Since $(u_0, v_0) \geq (0, 0)$ for all $|x| \leq M_\eta$, there exists $D_\eta^0 \geq 1$ for which

$$U(0, x - \eta) - D\phi(0)e^{(\lambda_c + \epsilon)x} \leq u_0(x), \quad W(0, x - \eta) - D\phi_1(0)e^{(\lambda_c + \epsilon)x} \leq v_0(x) \quad \text{for all } |x| \leq M_\eta$$

when $D \geq D_\eta^0$. Hence (4.23) follows. Let $m^* := \frac{\min\{\min_t \phi, \min_t \phi_1\}}{\max_t \{\phi + \phi_1\}}$ for the duration of the proof. Since $g, h \in C^{0,1}$, there exists $\bar{\epsilon} > 0$ such that

$$\begin{aligned} \left| h_u(t, u, v) - h_u(t, 0, 0) \right| + \left| h_v(t, u, v) - h_v(t, 0, 0) \right| &\leq \beta m^* \\ \left| g_u(t, u, v) - g_u(t, 0, 0) \right| + \left| g_v(t, u, v) - g_v(t, 0, 0) \right| &\leq \beta m^* \end{aligned} \quad \text{when } |u| + |v| \leq \bar{\epsilon},$$

where β is given by (4.5). In view of Theorem 3.8, for any $\epsilon \in]0, 1[$, there exists $z_\epsilon^\eta \leq 0$ such that

$$\begin{aligned} (1 - \epsilon)k e^{\lambda_c z} \phi(t) &\leq U(t, z - \eta) \leq (1 + \epsilon)k e^{\lambda_c z} \phi(t) \\ (1 - \epsilon)k e^{\lambda_c z} \phi_d(t) &\leq W(t, z - \eta) \leq (1 + \epsilon)k e^{\lambda_c z} \phi_d(t) \end{aligned} \quad \text{for all } z \leq z_\epsilon^\eta. \tag{4.24}$$

Now set $m_+ = \max_t \{1, \frac{\phi_1}{\phi_d}\}$ and $m_- = \min_t \{1, \frac{\phi_1}{\phi_d}\}$ and fix ϵ such that ϵ is sufficiently small, and

$$\left(1 + \frac{m_+}{m_-}\right) (1 + \epsilon)k_1 e^{\lambda_c z} \max_{t \in \mathbb{R}} \{\phi + \phi_d\} \leq \frac{\bar{\epsilon}}{2} \quad \text{for all } z \leq z_\epsilon^\eta.$$

Let $D_\eta^- = \frac{1}{m_-} (1 + k) (D_\eta^0 + \frac{e^{-(\lambda_c + \epsilon)z_\epsilon^\eta}}{\min_t \{1, \phi, \phi_1\}})$. Since $(u(t, x, u_0), v(t, x, v_0)) \geq (0, 0)$ and $(U, W) < (1, 1)$, it is readily seen that

$$\begin{aligned} \underline{u}_\eta(t, x) &:= U(t, x - ct - \eta) - D_\eta^- \phi(t) e^{(\lambda_c + \epsilon)(x - ct)} \leq 0 \leq u(t, x, u_0), \\ \underline{v}_\eta(t, x) &:= W(t, x - ct - \eta) - D_\eta^- \phi_1(t) e^{(\lambda_c + \epsilon)(x - ct)} \leq 0 \leq v(t, x, v_0) \end{aligned}$$

for all $(t, x) \in \{(t, x) \in \mathbb{R}^+ \times \mathbb{R} \mid x - ct \geq z_\epsilon^\eta\}$.

Define $z_* := \frac{1}{\epsilon} \ln \frac{k(1 + \epsilon)}{D_\eta^- m_-}$. Clearly, $z_* \leq z_\epsilon^\eta$ as long as ϵ is sufficiently small. It then follows from (4.24) that

$$\begin{aligned} -(1 + \epsilon) \left(1 + \frac{m_+}{m_-}\right) k \phi e^{\lambda_c z} &\leq e^{\lambda_c z} \phi [k(1 - \epsilon) - D_\eta^- e^{\epsilon z}] \leq \underline{u}_\eta(t, x) \leq (1 + \epsilon) k \phi e^{\lambda_c z}, \\ -(1 + \epsilon) \left(1 + \frac{m_+}{m_-}\right) k \phi_d e^{\lambda_c z} &\leq e^{\lambda_c z} \phi_d \left[k(1 - \epsilon) - D_\eta^- \frac{\phi_1}{\phi_d} e^{\epsilon z} \right] \leq \underline{v}_\eta(t, x) \leq (1 + \epsilon) k \phi_d e^{\lambda_c z} \end{aligned} \tag{4.25}$$

for all $z = x - ct \leq z_*$. In addition, $(\underline{u}_\eta(t, x), \underline{v}_\eta(t, x)) \leq (0, 0)$ for all $(t, x) \in \{(t, x) \mid z_* \leq x - ct \leq z_\epsilon^\eta\}$. To summarize, we have that

$$\begin{aligned} (\underline{u}_\eta(t, x), \underline{v}_\eta(t, x)) &\leq (u(t, x, u_0), v(t, x, v_0)) \quad \text{for all } (t, x) \in \mathbb{R}^+ \times \{x \geq z_* + ct\}, \\ \{(t, x) \mid \underline{u}_\eta(t, x) \geq 0\} \cup \{(t, x) \mid \underline{v}_\eta(t, x) \geq 0\} &\subseteq \mathbb{R} \times \{x \leq z_* + ct\}, \\ (\underline{u}_\eta(0, x), \underline{v}_\eta(0, x)) &\leq (u_0(x), v_0(x)) \quad \text{for all } x \in \mathbb{R}. \end{aligned}$$

In order to prove that $(\underline{u}_\eta(t, x), \underline{v}_\eta(t, x)) \leq (u(t, x, u_0), v(t, x, v_0))$ for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$, we show that $(\underline{u}_\eta, \underline{v}_\eta)$ is a sub-solution of (4.1) in $\mathbb{R}^+ \times \{x \leq z^* + ct\}$. In fact, due to (4.24) and (4.25), we have that $|U| + |W| \leq \bar{\varepsilon}$ and $|\underline{u}_\eta| + |\underline{v}_\eta| \leq \bar{\varepsilon}$ whenever $x - ct \leq z^*$. In light of the calculation made in the proof of Lemma 4.3, we readily see that

$$\begin{aligned} &g(t, \underline{u}_\eta, \underline{v}_\eta) + (\underline{u}_\eta)_{xx} - (\underline{u}_\eta)_t \\ &= D_\eta^- e^{(\lambda_c + \epsilon)(x - ct)} \left\{ \int_0^1 \phi(t) [g_u(t, 0, 0) - g_u(t, \tau \underline{u}_\eta + (1 - \tau)U, \tau \underline{v}_\eta + (1 - \tau)W)] d\tau \right. \\ &\quad \left. - \int_0^1 \phi_1(t) g_v(t, \tau \underline{u}_\eta + (1 - \tau)U, \tau \underline{v}_\eta + (1 - \tau)W) d\tau + 2\beta\phi(t) \right\} \geq 0 \end{aligned}$$

and

$$\begin{aligned} &h(t, \underline{u}_\eta, \underline{v}_\eta) + d(\underline{v}_\eta)_{xx} - (\underline{v}_\eta)_t \\ &\geq D_\eta^- e^{(\lambda_c + \epsilon)(x - ct)} \left\{ \int_0^1 \phi(t) [h_u(t, 0, 0) - h_u(t, \tau \underline{u}_\eta + (1 - \tau)U, \tau \underline{v}_\eta + (1 - \tau)W)] d\tau \right. \\ &\quad \left. + \int_0^1 \phi_1(t) [h_v(t, 0, 0) - h_v(t, \tau \underline{u}_\eta + (1 - \tau)U, \tau \underline{v}_\eta + (1 - \tau)W)] d\tau + 2\beta\phi_1(t) \right\} \geq 0 \end{aligned}$$

for all $(t, z) \in \{x - ct \leq z^*\}$. Therefore, it follows from Proposition 4.1 that

$$(\underline{u}_\eta(t, x), \underline{v}_\eta(t, x)) \leq (u(t, x, u_0), v(t, x, v_0)) \quad \text{for all } (t, x) \in \mathbb{R}^+ \times \mathbb{R}.$$

We now proceed to prove (4.22). Since $(U(t, \cdot), W(t, \cdot))$ is nondecreasing, by virtue of Lemma 4.4, it suffices to show that (4.22) holds for each $\eta \leq \sigma^*$, where σ^* is specified by Lemma 4.4. By means of the same arguments used at the beginning, we can show that for each $\eta > 0$, there exists $D_\eta^1 > 0$ such that

$$(u(0, x, u_0), v(0, x, v_0)) \leq (U(0, x + \eta) + D\phi(0)e^{(\lambda_c + \epsilon)x}, W(0, x + \eta) + D\phi_d(0)e^{(\lambda_c + \epsilon)x})$$

for all $x \in \mathbb{R}$ whenever $D \geq D_\eta^1$. In terms of Lemma 4.4, we have:

$$\begin{aligned} u(t, x, u_0) &\leq U(t, x - ct + \sigma^*(1 - e^{-\beta t})) + \delta^* \phi e^{(\lambda_c + \epsilon)(x - ct + s_0 + \sigma^*(1 - e^{-\beta t}))} e^{-\beta t}, \\ v(t, x, v_0) &\leq W(t, x - ct + \sigma^*(1 - e^{-\beta t})) + \delta^* \phi_1 e^{(\lambda_c + \epsilon)(x - ct + s_0 + \sigma^*(1 - e^{-\beta t}))} e^{-\beta t}, \end{aligned}$$

when $(t, x) \in [t^*, \infty) \times \{x \leq \underline{z} - s_0 - \sigma^* + ct\}$. Moreover, it follows from (4.17) that

$$\lim_{x \rightarrow -\infty} (u(t, x, u_0), v(t, x, v_0)) = (0, 0) \quad \text{uniformly in } [0, t^*].$$

Therefore, there exists $\sigma_0 \in \mathbb{R}$ such that

$$(u(t, x, u_0), v(t, x, v_0)) \leq \frac{m^* \bar{\varepsilon}}{4} (1, 1) \quad \text{and} \quad (U(t, x - ct + \eta), W(t, x - ct + \eta)) \leq \frac{m^* \bar{\varepsilon}}{4} (1, 1)$$

as long as $(t, x) \in \mathbb{R}^+ \times \{x \leq \sigma_0 + ct\}$, where $\eta \leq \sigma^*$. Let $\sigma_\eta = \sigma_0 + \frac{\ln \frac{\bar{\varepsilon}}{4D_\eta^1 \max_{t \in [0, T]} (\phi + \phi_1)}}{\lambda_c + \epsilon}$ and $D_\eta^+ = \frac{\bar{\varepsilon} e^{-(\lambda_c + \epsilon)\sigma_\eta}}{4 \max_{t \in [0, T]} (\phi + \phi_1)}$. Here $\bar{\varepsilon}$ is the same as above and D_η^1 has been selected sufficiently large such that $2D_\eta^1 \geq \bar{\varepsilon}$. Thus, we have $D_\eta^+ \geq D_\eta^1$. In addition, for each $\eta \in (0, \sigma^*]$, it is easy to see that

$$(\bar{u}_\eta(t, x), \bar{v}_\eta(t, x)) := (U(t, z + \eta) + D_\eta^+ \phi e^{(\lambda_c + \epsilon)z}, W(t, z + \eta) + D_\eta^+ \phi_1 e^{(\lambda_c + \epsilon)z}) \leq \frac{\bar{\varepsilon}}{2} (1, 1)$$

for all $z = x - ct \leq \sigma_\eta$. In particular, we have:

$$(u(t, x, u_0), v(t, x, v_0)) \leq \frac{m^* \bar{\epsilon}}{4} (1, 1) \leq (\bar{u}_\eta(t, x), \bar{v}_\eta(t, x))$$

for all $(t, x) \in \mathbb{R}^+ \times \{x = \sigma_\eta + ct\}$. Moreover, using the arguments similar to those presented in the proof of Lemma 4.3, we can show that $(\bar{u}_\eta, \bar{v}_\eta)$ is a super-solution of (4.1) in $\{x - ct \leq \sigma_\eta\}$. Then, by using Proposition 4.1 again, we obtain that

$$(\bar{u}_\eta(t, x), \bar{v}_\eta(t, x)) \geq (u(t, x, u_0), v(t, x, v_0)) \quad \text{for all } (t, x) \in \mathbb{R}^+ \times \{x \leq \sigma_\eta + ct\}. \tag{4.26}$$

For each $\eta > \sigma^*$, choose $\sigma_\eta = \sigma_{\sigma^*}$ and $D_\eta^+ = D_{\sigma^*}^+$, then we see that (4.26) is still true. Now set $D_\eta = \max\{D_\eta^-, D_\eta^+\}$. Then (4.21) and (4.22) follow. The proof is completed. \square

Lemma 4.6. *Suppose that (H1)–(H5) are satisfied. Assume that $(U, W) \in C_b^{1,2}(\mathbb{R} \times \mathbb{R})$ and c solve (3.1) with $c < c^*$. Assume that $(u(t, x), v(t, x)) \in C_b^{1,2}(\mathbb{R} \times \mathbb{R})$ is a solution of (4.1) for all $(t, x) \in \mathbb{R} \times \mathbb{R}$ such that*

$$(U(t, z + z_0 + \alpha), W(t, z + z_0 + \alpha)) \leq (u(t, x), v(t, x)) \leq (U(t, z + z_0 + \beta), W(t, z + z_0 + \beta))$$

for certain constants α, β and z_0 with $\alpha \leq 0 \leq \beta$ and $z_0 \in \mathbb{R}$, where $z = x - ct$. Assume that for each $\eta > 0$, there exist $\sigma_\eta \in \mathbb{R}$ and $D_\eta \in \mathbb{R}^+$ such that

$$\begin{aligned} U(t, x - ct + z_0 - \eta) - D_\eta \phi e^{(\lambda_c + \epsilon)(x - ct)} &\leq u(t, x) \leq U(t, x - ct + z_0 + \eta) + D_\eta \phi e^{(\lambda_c + \epsilon)(x - ct)}, \\ W(t, x - ct + z_0 - \eta) - D_\eta \phi_1 e^{(\lambda_c + \epsilon)(x - ct)} &\leq v(t, x) \leq W(t, x - ct + z_0 + \eta) + D_\eta \phi_1 e^{(\lambda_c + \epsilon)(x - ct)} \end{aligned}$$

for all $(t, x) \in \{(t, x) \mid x - ct \leq \sigma_\eta\}$, where ϵ is specified by (4.5). Then

$$(u(t, x), v(t, x)) = (U(t, x - ct + z_0), W(t, x - ct + z_0)) \quad \text{for all } (t, x) \in \mathbb{R} \times \mathbb{R}.$$

Proof. The proof shall be divided into four steps for the sake of clarity. Assume again that $z_0 = 0$.

Step 1. Define:

$$\eta^* = \inf \left\{ \eta \in [0, +\infty) \mid \begin{pmatrix} u(t, x) \\ v(t, x) \end{pmatrix} \leq \begin{pmatrix} U(t, x - ct + \eta) \\ W(t, x - ct + \eta) \end{pmatrix}, \forall (t, x) \in \mathbb{R} \times \mathbb{R} \right\}.$$

Notice that η^* is bounded and satisfies $0 \leq \eta^* \leq \beta$ since $(U(t, \cdot), W(t, \cdot))$ is monotonically increasing. Our goal is to prove that $(u(t, x), v(t, x)) \leq (U(t, x - ct), W(t, x - ct))$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}$. Namely, $\eta^* = 0$. Assume to the contrary that $\eta^* > 0$. Then we claim that there exists $\underline{\sigma} \in (-\infty, \sigma_{\frac{\eta^*}{2}}]$ such that

$$(u(t, x), v(t, x)) \leq \left(U\left(t, x - ct + \frac{\eta^*}{2}\right), W\left(t, x - ct + \frac{\eta^*}{2}\right) \right) \tag{4.27}$$

for all $(t, x) \in \{x - ct \leq \underline{\sigma}\}$. If this is not true, then there exist two sequences $\{t_n\}$ and $\{x_n\}$ such that

$$\lim_{n \rightarrow \infty} (x_n - ct_n) = -\infty \quad \text{and} \quad (u(t_n, x_n), v(t_n, x_n)) > \left(U\left(t_n, x_n - ct_n + \frac{\eta^*}{2}\right), W\left(t_n, x_n - ct_n + \frac{\eta^*}{2}\right) \right).$$

On the other hand, we have:

$$\lim_{n \rightarrow \infty} \frac{U(t_n, z_n + \frac{\eta^*}{4}) + D_{\frac{\eta^*}{4}} \phi(t_n) e^{(\lambda_c + \epsilon)z_n}}{U(t_n, z_n + \frac{\eta^*}{2})} = 0, \quad \lim_{n \rightarrow \infty} \frac{W(t_n, z_n + \frac{\eta^*}{4}) + D_{\frac{\eta^*}{4}} \phi_1(t_n) e^{(\lambda_c + \epsilon)z_n}}{W(t_n, z_n + \frac{\eta^*}{2})} = 0,$$

where $z_n = x_n - ct_n$. Therefore, it follows from the assumption that

$$(u(t_n, x_n), v(t_n, x_n)) \leq \left(U\left(t_n, x_n - ct_n + \frac{\eta^*}{2}\right), W\left(t_n, x_n - ct_n + \frac{\eta^*}{2}\right) \right)$$

when $x_n - ct_n \leq \underline{\sigma}_*$ for some $\underline{\sigma}_* \in (-\infty, \sigma_{\frac{\eta^*}{4}}]$. This is a contradiction, hence (4.27) follows.

Step 2. We now show that

$$\inf_{\underline{\sigma} \leq x - ct \leq \sigma} U(t, x - ct + \eta^*) - u(t, x) > 0, \quad \inf_{\underline{\sigma} \leq x - ct \leq \sigma} W(t, x - ct + \eta^*) - v(t, x) > 0, \tag{4.28}$$

for any $\sigma \geq \underline{\sigma}$. We only prove the first inequality as the other can be proved in the exactly same way. Assume to the contrary that $\inf_{\underline{\sigma} \leq x-ct \leq \sigma} U(t, x-ct + \eta^*) - u(t, x) = 0$. Then there exist two sequences $\{t_n\}$ and $\{x_n\}$ such that

$$\underline{\sigma} \leq x_n - ct_n \leq \sigma \quad \text{and} \quad \lim_{n \rightarrow \infty} [U(t_n, x_n - ct_n + \eta^*) - u(t_n, x_n)] = 0. \tag{4.29}$$

We need to take two cases into consideration, i.e., either $\{t_n\}$ is bounded or $\{t_n\}$ is unbounded. If $\{t_n\}$ is unbounded, up to extraction of a subsequence, we may assume without loss of generality that $\lim_{n \rightarrow \infty} t_n = \infty$. Thus, there exists a sequence $\{j_n\}$ with $j_n \in \mathbb{N}^+$ such that $\lim_{n \rightarrow \infty} j_n = \infty$ and $t_n \in [j_n T, (j_n + 1)T]$. Let $t'_n = t_n - j_n T$. Clearly, $t'_n \in [0, T]$. Write $z_n = x_n - ct_n$. Notice that $\{z_n\}$ is bounded and $cj_n T = x_n - ct'_n - z_n$. Now set:

$$(u_n(t, x), v_n(t, x)) = (u(t + j_n T, x + cj_n T), v(t + j_n T, x + cj_n T)).$$

As both g and h are periodic in t , $(u_n(t, x), v_n(t, x))$ are the solutions of (4.1) as well. Due to the regularities of $\{u_n\}$ and $\{v_n\}$ with respect to t and x , up to extraction of a subsequence, $\{(u_n, v_n)\}$ converges uniformly in any compact set of $\mathbb{R} \times \mathbb{R}$ to a solution of (4.1) in $\mathbb{R} \times \mathbb{R}$, denoted by $(u_\infty(t, x), v_\infty(t, x))$. Note that

$$(u_n(t, x), v_n(t, x)) = (u(t + j_n T, x + cj_n T), v(t + j_n T, x + cj_n T)) \leq (U(x - ct + \eta^*), W(x - ct + \eta^*)).$$

Consequently, we see that $(u_\infty(t, x), v_\infty(t, x)) \leq (U(x - ct + \eta^*), W(x - ct + \eta^*))$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}$. Moreover, since both $\{t'_n\}$ and $\{z_n\}$ are bounded, there exist t_∞ and z_∞ , and a subsequence of $\{(t'_n, z_n)\}$ (still denoted by $\{(t'_n, z_n)\}$ for convenience) such that $\lim_{n \rightarrow \infty} (t'_n, z_n) = (t_\infty, z_\infty)$. Thus, it follows from (4.29) that

$$\begin{aligned} & U(t_\infty, z_\infty + \eta^*) - u_\infty(t_\infty, z_\infty) \\ &= \lim_{n \rightarrow \infty} [U(t'_n, z_n + \eta^*) - u_n(t'_n, z_n)] \\ &= \lim_{n \rightarrow \infty} [U(t'_n + j_n T, x_n - ct_n + \eta^*) - u(t'_n + j_n T, x_n - ct_n + \eta^*)] \\ &= \lim_{n \rightarrow \infty} [U(t_n, x_n - ct_n + \eta^*) - u(t_n, x_n)] = 0. \end{aligned}$$

In other words, $u_\infty(t_\infty, x_\infty) = U(t_\infty, x_\infty - ct_\infty + \eta^*)$, where $x_\infty = z_\infty + ct_\infty$. Since

$$\left[\int_0^1 g_u(t, sU^{\eta^*} + (1-s)u_\infty, sW^{\eta^*} + (1-s)v_\infty) ds \right] (U^{\eta^*} - u_\infty) + (U^{\eta^*} - u_\infty)_{xx} - (U^{\eta^*} - u_\infty)_t \leq 0,$$

it follows from the strong maximum principle that $u_\infty(t, x) = U(t, x - ct + \eta^*)$ for all $(t, x) \in (-\infty, t_\infty] \times \mathbb{R}$, where $(U^{\eta^*}(t, s), W^{\eta^*}(t, s)) = (U(t, s + \eta^*), W(t, s + \eta^*))$. On the other hand, thanks to (4.27), we have:

$$\begin{aligned} (u_n(t, x), v_n(t, x)) &= (u(t + j_n T, x + cj_n T), v(t + j_n T, x + cj_n T)) \\ &\leq \left(U\left(t, x - ct + \frac{\eta^*}{2}\right), W\left(t, x - ct + \frac{\eta^*}{2}\right) \right) \end{aligned}$$

as long as $x - ct \leq \underline{\sigma}$. Then, by taking the limit, we find that $u_\infty(t, x) \leq U(t, x - ct + \frac{\eta^*}{2})$ for all $(t, x) \in \{x - ct \leq \underline{\sigma}\}$. This is a contradiction because $U(t, \cdot + \frac{\eta^*}{2}) < U(t, \cdot + \eta^*)$. This contradiction rules out the possibility that $\{t_n\}$ is unbounded, and we are led to the case that $\{t_n\}$ is bounded. However, if $\{t_n\}$ is bounded, then there exists (t^*, x^*) such that $u(t^*, x^*) = U(t^*, x^* - ct^* + \eta^*)$, hence we can again deduce a contradiction with the same reasoning. Therefore, (4.28) holds.

Step 3. In view of the assumption, we find that

$$\lim_{z \rightarrow \infty} \sup_{x-ct \geq z} |u(t, x) - 1| = 0, \quad \lim_{z \rightarrow \infty} \sup_{x-ct \geq z} |v(t, x) - 1| = 0.$$

Hence, there exists $\bar{\sigma} \geq \underline{\sigma}$ such that

$$(u(t, x), v(t, x)) \in [1 - \rho^0, 1]^2, \quad (U(t, x - ct + \eta^*), W(t, x - ct + \eta^*)) \in [1 - \rho^0, 1]^2 \quad \text{when } x - ct \geq \bar{\sigma},$$

where ρ^0 is given in Proposition 3.9. As (U, W) is uniformly continuous in $[0, T] \times [\underline{\sigma}, \bar{\sigma} + \eta^*]$, by virtue of (4.28) and the time periodicity of (U, W) , there exists $\bar{\eta} \in [\frac{\eta^*}{2}, \eta^*]$ for which

$$\inf_{\underline{\sigma} \leq x-ct \leq \bar{\sigma}} U(t, x - ct + \bar{\eta}) - u(t, x) \geq 0, \quad \inf_{\underline{\sigma} \leq x-ct \leq \bar{\sigma}} W(t, x - ct + \bar{\eta}) - v(t, x) \geq 0. \tag{4.30}$$

We next show that

$$\inf_{x-ct \geq \bar{\sigma}} U(t, x - ct + \bar{\eta}) - u(t, x) \geq 0, \quad \inf_{x-ct \geq \bar{\sigma}} W(t, x - ct + \bar{\eta}) - v(t, x) \geq 0. \tag{4.31}$$

To this end, let

$$u^\delta(t, x) = U(t, x - ct + \bar{\eta}) - u(t, x) + \delta\varphi_1(t) \quad \text{and} \quad v^\delta(t, x) = U(t, x - ct + \bar{\eta}) - u(t, x) + \delta\varphi_2(t).$$

Define

$$\bar{\delta} = \inf\{\delta \in [0, \infty) \mid (u^\delta(t, x), v^\delta(t, x)) \geq (0, 0) \text{ for all } (t, x) \in \{x - ct \geq \bar{\sigma}\}\}.$$

We need to show that $\bar{\delta} = 0$. If this is not true, then with the same reasoning as that in Proposition 3.9, we infer that either $\inf_{x-ct \geq \bar{\sigma}} u^\delta = 0$ or $\inf_{x-ct \geq \bar{\sigma}} v^\delta = 0$. Assume again that $\inf_{x-ct \geq \bar{\sigma}} v^\delta = 0$. Then there exist two sequences $\{t_n\}$ and $\{x_n\}$ such that $\{x_n - ct_n\}$ is bounded, $x_n - ct_n \geq \bar{\sigma}$, and $\lim_{n \rightarrow \infty} v^\delta(t_n, x_n) = 0$. Moreover, we have:

$$\left[\int_0^1 h_v(t, sU^{\bar{\eta}} + (1-s)u, sW^{\bar{\eta}} + (1-s)v) ds \right] v^\delta + v_{xx}^\delta - v_t^\delta \leq 0 \quad \text{for all } (t, x) \in \{x - ct > \bar{\sigma}\},$$

where $U^{\bar{\eta}}(t, \cdot) = U(t, \cdot + \bar{\eta})$ and $W^{\bar{\eta}}(t, \cdot) = W(t, \cdot + \bar{\eta})$. If $\{t_n\}$ is bounded, then there exists (t^*, x^*) such that $x^* - ct^* > \bar{\sigma}$ and $v^\delta(t^*, x^*) = 0$. Hence we can readily reach a contradiction by applying the (strong) maximum principle.

In case that $\{t_n\}$ is unbounded. With a slight abuse of notation, we still write that $t_n = j_n T + t'_n$ with an unbounded integer sequence $\{j_n\}$ and bounded sequence $\{t'_n\}$. Set again:

$$u_n^\delta(t, x) = u^\delta(t + j_n T, x + c j_n T) = U(t, x - ct + \bar{\eta}) - u(t + j_n T, x + c j_n T) + \bar{\delta}\varphi_1(t),$$

$$v_n^\delta(t, x) = v^\delta(t + j_n T, x + c j_n T) = W(t, x - ct + \bar{\eta}) - v(t + j_n T, x + c j_n T) + \bar{\delta}\varphi_2(t).$$

Notice that

$$\left[\int_0^1 h_v(t, sU^{\bar{\eta}} + (1-s)u_n, sW^{\bar{\eta}} + (1-s)v_n) ds \right] v_n^\delta + (v_n^\delta)_{xx} - (v_n^\delta)_t \leq 0, \quad \forall (t, x) \in \{x - ct > \bar{\sigma}\},$$

where $u_n(t, x) = u(t + j_n T, x + c j_n T)$ and $v_n(t, x) = v(t + j_n T, x + c j_n T)$. By taking the limit, we can again derive a contradiction in the exactly same way as above and infer that (4.31) holds. As $(U(t, \cdot + \frac{\eta^*}{2}), W(t, \cdot + \frac{\eta^*}{2})) \leq (U(t, \cdot + \bar{\eta}), W(t, \cdot + \bar{\eta}))$, from (4.27), (4.30), and (4.31), it follows that $(u(t, x), v(t, x)) \leq (U(t, x - ct + \bar{\eta}), W(t, x - ct + \bar{\eta}))$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}$. It apparently contradicts the definition of η^* . Thus, we must have $\eta^* = 0$. In other words,

$$(u(t, x), v(t, x)) \leq (U(t, x - ct), W(t, x - ct)) \quad \text{for all } (t, x) \in \mathbb{R} \times \mathbb{R}.$$

Step 4. Define:

$$\eta_* = \inf \left\{ \eta \in [0, +\infty) \mid \begin{pmatrix} u(t, x) \\ v(t, x) \end{pmatrix} \geq \begin{pmatrix} U(t, x - ct - \eta) \\ W(t, x - ct - \eta) \end{pmatrix}, \forall (t, x) \in \mathbb{R} \times \mathbb{R} \right\}.$$

Notice that η_* is bounded and satisfies $0 \leq \eta_* \leq -\alpha$. Arguing in a similar manner, we can show that $\eta_* = 0$, that is,

$$(U(t, x - ct), W(t, x - ct)) \leq (u(t, x), v(t, x)) \quad \text{for all } (t, x) \in \mathbb{R} \times \mathbb{R}.$$

Therefore, it follows that $(u(t, x), v(t, x)) \equiv (U(t, x - ct), W(t, x - ct))$. \square

Theorem 4.7. *Suppose that all the assumptions of Lemma 4.4 are satisfied. Let $(U, W) \in C_b^{1,2}(\mathbb{R} \times \mathbb{R})$ and c solve (3.1) with $c < c^*$. Then*

$$\lim_{t \rightarrow \infty} |u(t, x, u_0) - U(t, x - ct + z_0)| + |v(t, x, v_0) - W(t, x - ct + z_0)| = 0 \tag{4.32}$$

for some $z_0 \in \mathbb{R}$. In particular, z_0 is the unique number such that

$$\lim_{x \rightarrow -\infty} \frac{U(0, x + z_0)}{k\phi(0)e^{\lambda_c x}} = 1 \quad \text{and} \quad \lim_{x \rightarrow -\infty} \frac{W(0, x + z_0)}{k\phi_d(0)e^{\lambda_c x}} = 1.$$

Proof. We assume again that $z_0 = 0$. Assume to the contrary that (4.32) is not true. Then there exist $\varepsilon > 0$ and a sequence $\{(t_n, x_n)\} \in \mathbb{R}^+ \times \mathbb{R}$ such that $\lim_{n \rightarrow \infty} t_n = \infty$, and

$$|u(t_n, x_n, u_0) - U(t_n, x_n - ct_n)| + |v(t_n, x_n, v_0) - W(t_n, x_n - ct_n)| \geq \varepsilon. \tag{4.33}$$

Since $t_n \rightarrow \infty$ as $n \rightarrow \infty$, there exists a sequence $\{j_n\}$ with $j_n \in \mathbb{N}^+$ such that $\lim_{n \rightarrow \infty} j_n = \infty$ and $t_n \in [j_n T, (j_n + 1)T]$. As before, we let $t'_n = t_n - j_n T$ and write $z_n = x_n - ct_n$. If $\{z_n\}$ is bounded, then set

$$(u_n(t, x), v_n(t, x)) = (u(t + j_n T, x + cj_n T, u_0), v(t + j_n T, x + cj_n T, v_0)).$$

Clearly, for each n , $(u_n(t, x), v_n(t, x))$ is a solution of (4.1) in $(-j_n T, \infty) \times \mathbb{R}$ satisfying $(u_n(-j_n T, x), v_n(-j_n T, x)) = (u_0(x + cj_n T), v_0(x + cj_n T))$. Denote again by $(u_\infty(t, x), v_\infty(t, x))$ the solution of (4.1) in $\mathbb{R} \times \mathbb{R}$ to which $\{(u_n, v_n)\}$ converges uniformly in any compact set. Due to Lemma 4.4, we see that

$$\begin{aligned} U(t, x - ct - \sigma^*) - \delta^* \Lambda e^{-\beta(t+j_n T)} &\leq u_n(t, x) \leq U(t, x - ct + \sigma^*) + \delta^* \Lambda e^{-\beta(t+j_n T)}, \\ W(t, x - ct - \sigma^*) - \delta^* \Lambda e^{-\beta(t+j_n T)} &\leq v_n(t, x) \leq W(t, x - ct + \sigma^*) + \delta^* \Lambda e^{-\beta(t+j_n T)}, \end{aligned}$$

for all $(t, x) \in [t^* - j_n T, \infty) \times \mathbb{R}$, where $\Lambda = \max\{\sup_{(t,s) \in \mathbb{R} \times \mathbb{R}} \xi_c(t, s), \sup_{(t,s) \in \mathbb{R} \times \mathbb{R}} \zeta_c(t, s)\}$. It then follows that

$$\begin{aligned} U(t, x - ct - \sigma^*) &\leq u_\infty(t, x) \leq U(t, x - ct + \sigma^*), \\ W(t, x - ct - \sigma^*) &\leq v_\infty(t, x) \leq W(t, x - ct + \sigma^*) \quad \text{for all } (t, x) \in \mathbb{R} \times \mathbb{R}. \end{aligned}$$

Furthermore, Proposition 4.5 implies that for each $\eta > 0$, there exists D_η such that

$$(U(t, x - ct - \eta) - D_\eta \phi e^{(\lambda_c + \epsilon)(x - ct)}, W(t, x - ct - \eta) - D_\eta \phi_1 e^{(\lambda_c + \epsilon)(x - ct)}) \leq (u_n(t, x), v_n(t, x))$$

for all $(t, x) \in [-j_n T, \infty) \times \mathbb{R}$, and

$$(u_n(t, x), v_n(t, x)) \leq (U(t, x - ct + \eta) + D_\eta \phi e^{(\lambda_c + \epsilon)(x - ct)}, W(t, x - ct + \eta) + D_\eta \phi_1 e^{(\lambda_c + \epsilon)(x - ct)})$$

whenever $(t, x) \in [-j_n T, \infty) \times (-\infty, \sigma_\eta + ct]$. By taking the limits in above inequalities, we obtain that

$$(U(t, x - ct - \eta) - D_\eta \phi e^{(\lambda_c + \epsilon)(x - ct)}, W(t, x - ct - \eta) - D_\eta \phi_1 e^{(\lambda_c + \epsilon)(x - ct)}) \leq (u_\infty(t, x), v_\infty(t, x))$$

for all $(t, x) \in \mathbb{R} \times \mathbb{R}$, and

$$(u_\infty(t, x), v_\infty(t, x)) \leq (U(t, x - ct + \eta) + D_\eta \phi e^{(\lambda_c + \epsilon)(x - ct)}, W(t, x - ct + \eta) + D_\eta \phi_1 e^{(\lambda_c + \epsilon)(x - ct)})$$

for $(t, x) \in \mathbb{R} \times (-\infty, \sigma_\eta + ct]$. Consequently, it follows from Lemma 4.6 that

$$(u_\infty(t, x), v_\infty(t, x)) \equiv (U(t, x - ct), W(t, x - ct)). \tag{4.34}$$

On the other hand, since $\{t'_n\}$ and $\{z_n\}$ are both bounded, by taking subsequences if necessary, we find that $\lim_{n \rightarrow \infty} (t'_n, z_n) = (t_\infty, z_\infty)$ for some $(t_\infty, z_\infty) \in [0, T] \times \mathbb{R}$. Since $\{(u_n, v_n)\}$ converges uniformly to (u_∞, v_∞) in compact subsets of $\mathbb{R} \times \mathbb{R}$, we have:

$$\begin{aligned} &(u_\infty(t_\infty, ct_\infty + z_\infty), v_\infty(t_\infty, ct_\infty + z_\infty)) \\ &= \lim_{n \rightarrow \infty} (u_n(t'_n, ct'_n + z_n), v_n(t'_n, ct'_n + z_n)) \\ &= \lim_{n \rightarrow \infty} (u(t'_n + j_n T, ct'_n + z_n + cj_n T, u_0), v(t'_n + j_n T, ct'_n + z_n + cj_n T, v_0)) \\ &= \lim_{n \rightarrow \infty} (u(t_n, x_n, u_0), v(t_n, x_n, v_0)). \end{aligned}$$

Moreover, we observe that

$$\begin{aligned} (U(t_\infty, z_\infty), W(t_\infty, z_\infty)) &= \lim_{n \rightarrow \infty} (U(t'_n, z_n), W(t'_n, z_n)) = \lim_{n \rightarrow \infty} (U(t'_n + j_n T, z_n), W(t'_n + j_n T, z_n)) \\ &= \lim_{n \rightarrow \infty} (U(t_n, x_n - ct_n), W(t_n, x_n - ct_n)). \end{aligned}$$

Hence, it follows from (4.33) that

$$|u_\infty(t_\infty, ct_\infty + z_\infty) - U(t_\infty, z_\infty)| + |v_\infty(t_\infty, ct_\infty + z_\infty) - W(t_\infty, z_\infty)| \geq \varepsilon.$$

However, this contradicts (4.34). Thus $\{z_n\}$ has to be unbounded in terms of (4.33). Assume that $\lim_{n \rightarrow \infty} z_n = -\infty$. Then it follows from Lemma 4.4 that

$$\lim_{n \rightarrow \infty} (u(t_n, x_n, u_0), v(t_n, x_n, v_0)) = \lim_{n \rightarrow \infty} (U(t_n, z_n), W(t_n, z_n)) = (0, 0).$$

Likewise, if $\lim_{n \rightarrow \infty} z_n = \infty$, then we have:

$$\lim_{n \rightarrow \infty} (u(t_n, x_n, u_0), v(t_n, x_n, v_0)) = \lim_{n \rightarrow \infty} (U(t_n, z_n), W(t_n, z_n)) = (1, 1).$$

They both contradict (4.33). Therefore, (4.32) follows. The proof is completed. \square

Remark 4.8. With additional assumptions on g and h , the same type of methods utilized in this section can be adopted to prove the asymptotic stability of time periodic traveling waves with the wave speed c^* . This issue will be addressed in our forthcoming paper.

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Appendix A

In this Appendix, we prove a result on the limiting behavior of solutions of (4.1) with prescribed initial data which was used in the proof of Lemma 4.4.

Proposition A.1. *Suppose that (H1)–(H5) are satisfied. Assume that $(u(t, x, u_0), v(t, x, v_0))$ solves (4.1) with $(u(0, x, u_0), v(0, x, v_0)) = (u_0(x), v_0(x))$ and that*

$$\lim_{x \rightarrow -\infty} \frac{u_0(x)}{k\phi(0)e^{\lambda_c x}} = 1, \quad \lim_{x \rightarrow -\infty} \frac{v_0(x)}{k\phi_d(0)e^{\lambda_c x}} = 1$$

for some positive constant k , where $c < c^*$. Let $I \subset [0, +\infty)$ be any compact subinterval. Then there exists $z_0 \in \mathbb{R}$ such that

$$\lim_{x \rightarrow -\infty} \frac{|u(t, x, u_0) - U(t, x - ct + z_0)|}{U(t, x - ct + z_0)} = 0, \quad \lim_{x \rightarrow -\infty} \frac{|v(t, x, u_0) - W(t, x - ct + z_0)|}{W(t, x - ct + z_0)} = 0$$

uniformly in $t \in I$, where $c < c^*$ and $z_0 \in \mathbb{R}$ is the unique number such that

$$\lim_{x \rightarrow -\infty} \frac{U(0, x + z_0)}{k\phi(0)e^{\lambda_c x}} = 1, \quad \lim_{x \rightarrow -\infty} \frac{W(0, x + z_0)}{k\phi_d(0)e^{\lambda_c x}} = 1.$$

Proof. Once again, we shall assume without loss of generality that $z_0 = 0$ through the proof. Let $\rho(r) \in C^3(\mathbb{R})$ be a real positive function with the following properties: (i) $(|\rho(r)| + |\rho'(r)|) \leq C_1 e^{-\delta|r|}$ for certain positive constants C_1 and δ ; (ii) $|\frac{\rho'''(r)}{\rho(r)}| + |\frac{\rho''(r)}{\rho(r)}| + |\frac{\rho'(r)}{\rho(r)}| \leq C_2$ for some positive constant C_2 . By rescaling, we may assume that $\rho(0) = 1$ and $\delta > 2\sqrt{\kappa}$, in other words, $\delta > 2\lambda_c$ for any $c < c^*$. Such a function can be easily constructed, for instance, $\rho(r) = \frac{1}{\cosh(\delta r)}$ has the desired properties.

Now we write $\rho^x(y) = \rho(y - x)$ and set:

$$\hat{u}(t, y) = \rho^x(y)[u(t, y, u_0) - U(t, y - ct)], \quad \hat{v}(t, y) = \rho^x(y)[v(t, y, u_0) - W(t, y - ct)].$$

Then

$$\begin{aligned} \partial_t \hat{u} - \hat{u}_{yy} &= \rho^x(y)\{[u - U]_t - [u - U]_{yy}\} - \rho^x_{yy}(y)[u - U] - 2\rho^x_y(y)[u - U]_y \\ &= \rho^x(y)[g(t, u, v) - g(t, U, W)] - \frac{\rho^x_{yy}(y)}{\rho^x(y)}\hat{u} - \frac{2\rho^x_y(y)}{\rho^x(y)}[\rho^x(y)(u - U)]_y + \frac{2(\rho^x_y(y))^2}{\rho^x(y)}[\rho^x(y)(u - U)]. \end{aligned}$$

Define

$$(L_1 w)(t, y) := w_t - w_{yy} + \frac{2\rho^x_y(y)}{\rho^x(y)}w_y - \left[\frac{2(\rho^x_y(y))^2 - \rho^x_{yy}(y)}{\rho^x(y)} \right]w.$$

Then, we find that

$$L_1 \hat{u} = \left\{ \int_0^1 g_u(t, su + (1-s)U, sv + (1-s)W) ds \right\} \hat{u} + \left\{ \int_0^1 g_v(t, su + (1-s)U, sv + (1-s)W) ds \right\} \hat{v}.$$

Likewise, we have:

$$L_2 \hat{v} = \left\{ \int_0^1 h_u(t, su + (1-s)U, sv + (1-s)W) ds \right\} \hat{u} + \left\{ \int_0^1 h_v(t, su + (1-s)U, sv + (1-s)W) ds \right\} \hat{v},$$

where

$$(L_2 w)(t, y) := w_t - dw_{yy} + \frac{2d\rho^x_y(y)}{\rho^x(y)}w_y - d \left[\frac{2(\rho^x_y(y))^2 - \rho^x_{yy}(y)}{\rho^x(y)} \right]w.$$

By the variation of constants formula and Gronwall's inequality, we obtain that

$$|\hat{u}(t, y)|_{L_\infty(\mathbb{R})} \leq C e^{Kt} [|\hat{u}(0, y)|_{L_\infty(\mathbb{R})} + |\hat{v}(0, y)|_{L_\infty(\mathbb{R})}], \quad t \in I$$

for certain positive constants C and K , which depend only upon d , C_1 , C_2 , and \tilde{M} , where $\tilde{M} = 2 \sup_{(t,s,s') \in \mathbb{R} \times [-2,2] \times [-2,2]} \{|g_u(t, s, s')| + |g_v(t, s, s')| + |h_u(t, s, s')| + |h_v(t, s, s')|\}$ (see Theorem 2.10 in Chapter III of [14] or Theorem 3.1.3 of [32]).

Without loss of generality, we may assume that $I \subseteq [0, T]$. Thus,

$$\begin{aligned} |u(t, x, u_0) - U(t, x - ct)| &\leq C e^{KT} \sup_{|y-x| \leq \frac{|x|}{2}} \{|\rho^x(y)[u_0(y) - U(0, y)]| + |\rho^x(y)[v_0(y) - W(0, y)]|\} \\ &\quad + C e^{KT} \sup_{|y-x| \geq \frac{|x|}{2}} \{|\rho^x(y)[u_0(y) - U(0, y)]| + |\rho^x(y)[v_0(y) - W(0, y)]|\}. \end{aligned}$$

Since

$$|\rho^x(y)[u_0(y) - U(0, y)]| + |\rho^x(y)[v_0(y) - W(0, y)]| \leq 4C_1 e^{-\frac{\delta|x|}{2}} \quad \text{whenever } |y - x| \geq \frac{|x|}{2}.$$

Furthermore, if $y \in \{s \in \mathbb{R}: |s - x| \leq \frac{|x|}{2}\}$, then

$$\begin{aligned} |\rho^x(y)[u_0(y) - U(0, y)]| &\leq C_1 e^{-\delta|y-x|} \left| U(0, y) \left(1 - \frac{u_0(y)}{U(0, y)} \right) \right| \\ &\leq C_1 C' e^{-\delta|y-x|} e^{\lambda_c y} \left| 1 - \frac{u_0(y)}{U(0, y)} \right| \\ &\leq C_1 C' e^{-\delta|y-x|} e^{\lambda_c |y-x|} e^{\lambda_c x} \left| 1 - \frac{u_0(y)}{U(0, y)} \right| \\ &\leq C_1 C' e^{\lambda_c x} \left| 1 - \frac{u_0(y)}{U(0, y)} \right|. \end{aligned}$$

Here we used the fact that $0 < U(0, x) \leq C'e^{\lambda cx}$ for some positive constant C' (see Theorem 3.8).

Similarly, we have

$$|\rho^x(y)[v_0(y) - W(0, y)]| \leq C_1 C' e^{\lambda cx} \left| 1 - \frac{v_0(y)}{W(0, y)} \right| \quad \text{whenever } |y - x| \leq \frac{|x|}{2}.$$

Consequently, for each $t \in I$, it follows that

$$|u(t, x, u_0) - U(t, x - ct)| \leq \widehat{C} e^{\lambda cx} \sup_{|y-x| \leq \frac{|x|}{2}} \left\{ \left| 1 - \frac{u_0(y)}{U(0, y)} \right| + \left| 1 - \frac{v_0(y)}{W(0, y)} \right| \right\} + \widehat{C} e^{-\frac{\delta|x|}{2}},$$

where \widehat{C} is a positive constant that depends on C, C_1, K , and T .

As

$$\lim_{y \rightarrow -\infty} \left\{ \left| 1 - \frac{u_0(y)}{U(0, y)} \right| + \left| 1 - \frac{v_0(y)}{W(0, y)} \right| \right\} = 0,$$

by Theorem 3.8, we readily infer that

$$\lim_{x \rightarrow -\infty} \frac{|u(t, x, u_0) - U(t, x - ct)|}{U(t, x - ct)} = 0 \quad \text{uniformly in } t \in I.$$

Likewise, we have:

$$\lim_{x \rightarrow -\infty} \frac{|v(t, x, v_0) - W(t, x - ct)|}{W(t, x - ct)} = 0 \quad \text{uniformly in } t \in I.$$

The proof is completed. \square

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