Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 267 (2019) 2011–2051

Journal of Differential Equations

www.elsevier.com/locate/jde

Dynamics of a nonlocal dispersal SIS epidemic model with Neumann boundary conditions

Fei-Ying Yang [a], Wan-Tong Li [a,*], Shigui Ruan [b]

[a] School of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu 730000, PR China
[b] Department of Mathematics, University of Miami, Coral Gables, FL 33146, USA

Received 11 September 2018; revised 11 February 2019
Available online 13 March 2019

Abstract

In this paper we study a nonlocal dispersal susceptible-infected-susceptible (SIS) epidemic model with Neumann boundary condition, where the spatial movement of individuals is described by a nonlocal (convolution) diffusion operator, the transmission rate and recovery rate are spatially heterogeneous, and the total population number is constant. We first define the basic reproduction number R_0 and discuss the existence, uniqueness and stability of steady states of the nonlocal dispersal SIS epidemic model in terms of R_0. Then we consider the impacts of the large diffusion rates of the susceptible and infectious populations on the persistence and extinction of the disease. The obtained results indicate that the nonlocal movement of the susceptible or infectious individuals will enhance the persistence of the infectious disease. In particular, our analytical results suggest that the spatial heterogeneity tends to boost the spread of the infectious disease.

MSC: 35B40; 45A05; 45F05; 47G20

Keywords: Nonlocal dispersal; Basic reproduction number; Disease-free equilibrium; Endemic equilibrium; Stability

* Corresponding author.
E-mail address: wtli@lzu.edu.cn (W.-T. Li).

https://doi.org/10.1016/j.jde.2019.03.001

1. Introduction

This paper is concerned with the following nonlocal dispersal SIS epidemic model with

$$\begin{cases} \frac{\partial S}{\partial t} = d_S \int_\Omega J(x-y)\,[S(y,t)-S(x,t)]\,dy - \frac{\beta(x)SI}{S+I} + \gamma(x)I, & x \in \Omega,\ t>0, \\ \frac{\partial I}{\partial t} = d_I \int_\Omega J(x-y)\,[I(y,t)-I(x,t)]\,dy + \frac{\beta(x)SI}{S+I} - \gamma(x)I, & x \in \Omega,\ t>0, \\ S(x,0)=S_0(x),\ I(x,0)=I_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ (n denotes the dimension) is a bounded domain; $S(x,t)$ and $I(x,t)$ denote the density of susceptible and infectious individuals at location x and time t, respectively; d_S and d_I are positive diffusion coefficients for the susceptible and infectious individuals; $\beta(x)$ and $\gamma(x)$ are positive continuous functions on $\bar{\Omega}$ that represent the transmission rate of susceptible individuals and the recovery of infectious individuals at x, respectively. The integral operator $\int_{\mathbb{R}^n} J(x-y)(u(y,t)-u(x,t))dy$ describes diffusion processes. As in [18], if $u(x,t)$ is thought of as the density at a point x at time t, and $J(x-y)$ is thought of as the probability distribution of jumping from location y to location x, then $\int_{\mathbb{R}^n} J(x-y)u(y,t)dy$ is the rate at which individuals are arriving at position x from all other places and $-\int_{\mathbb{R}^n} J(x-y)u(x,t)dy$ is the rate at which they are leaving location x to travel to all other sites. Since integrals are taken over the domain Ω, we assume that diffusion takes places only in Ω. Individuals may not enter or leave the domain Ω. This is analogous to the homogeneous Neumann boundary condition in the literature, we also call it *Neumann boundary condition*, meaning that all the involved integrals are taken over the domain Ω (see the definition in Andreu-Vaillo et al. [4]).

It is known from Allen, Bolker and Lou [2] that the term $\frac{SI}{S+I}$ is a Lipschitz continuous function of S and I in the open first quadrant, we can extend its definition to the entire first quadrant by defining it to be zero when either $S=0$ or $I=0$. Throughout the paper, we assume that the total number of initial infectious individuals is positive; that is,

$$\int_\Omega I(x,0)dx > 0 \text{ with } S_0(x) \geq 0 \text{ and } I_0(x) \geq 0 \text{ for } x \in \Omega$$

and the dispersal kernel function J satisfies

(J) $J(\cdot) \in C(\bar{\Omega})$, $J(0)>0$, $J(x)=J(-x)\geq 0$, $\int_{\mathbb{R}^n} J(x)dx = 1$ and $\int_\Omega J(x-y)dy \not\equiv 1$ for any $x \in \Omega$.

Note that (1.1) is the nonlocal counterpart of the following SIS epidemic reaction-diffusion model

$$\begin{cases} \frac{\partial S}{\partial t} = d_S \Delta S - \frac{\beta(x)SI}{S+I} + \gamma(x)I, & x\in\Omega, t>0, \\ \frac{\partial I}{\partial t} = d_I \Delta I + \frac{\beta(x)SI}{S+I} - \gamma(x)I, & x\in\Omega, t>0, \\ \partial_\nu S = \partial_\nu I = 0, & x \in \partial\Omega, t>0, \end{cases} \tag{1.2}$$

in which ν is the outward unit normal vector on $\partial\Omega$. System (1.2) was first proposed by Allen, Bolker and Lou [2], who mainly discussed the impact of spatial heterogeneity of environment and movement of individuals on the persistence and extinction of a disease. Their results were extended by Peng and Liu [35], in which they proved that the endemic equilibrium is globally asymptotically stable if it exists and this result confirms a conjecture proposed in [2]. Peng [34] provided further understanding regarding the roles of large or small migration rates of the susceptible and infectious populations on the spatial persistence and extinction of the infectious diseases, see also Peng and Yi [37]. Moreover, Peng and Zhao [36] considered (1.2) when the transmission rate and recovery rate are assumed to be spatially heterogeneous and temporally periodic. Recently, Cui and Lou [16] and Cui, Lam and Lou [15] studied the dynamics of (1.2) with advection and found that advection can help to speed up the elimination of infectious diseases. For other results about SIS epidemic models with spatial heterogeneity, we refer to Allen, Bolker and Lou [1,3], Huang, Han and Liu [23], Li, Peng and Wang [28] and Wu and Zou [51].

The nonlocal dispersal as a long range process can better describe some natural phenomena in many situations (Andreu-Vaillo et al. [4], Fife [18]). In fact nonlocal dispersal equations have attracted great attention and have been used to model different dispersal phenomena in population ecology (Hutson et al. [24], Kao, Lou and Shen [25]), material science (Bates [7], Wang [50]), neurology (Sun, Yang and Li [43]), etc. For the study of nonlocal problems, we refer to Chasseigne, Chaves and Rossi [10], Cortázar, Coville and Elgueta [11], Sun, Li and Yang [42] and Zhang, Li and Sun [54] about the asymptotic behavior, Bates et al. [8], Coville, Dávila and Martínez [14], Li, Sun and Wang [29], Li, Zhang and Zhang [30], Pan, Li and Lin [32], Shen and Zhang [40] and Sun, Li and Wang [45] about the traveling waves and entire solutions when $\Omega = \mathbb{R}$, and Bates and Zhao [9], Coville, Dávila and Martínez [13], Sun, Li and Wang [45] and Yang, Li and Sun [53] about the stationary solutions. In particular, the spectrum properties of nonlocal dispersal operators and their essential difference comparing with the random operators are studied in Coville [12], Coville, Dávila and Martínez [14], García-Melián and Rossi [19], Shen and Zhang [40] and Sun, Yang and Li [43].

Nonlocal epidemic models have also been extensively studied since the pioneer work of Kendall [26,27], in which he generalized the Kermack-McKendrick model to a space-dependent integro-differential equation and used the integral term $\beta S(x,t)\int_{-\infty}^{\infty} K(x-y)I(y,t)dy$ to describe how infectious individuals at location y disperse to infect susceptible individuals at location x. Kendall [27], Mollison [31] and Aronson [5] studied the existence of traveling wave solutions in the Kendall model. For further results on nonlocal epidemic models we refer to the monograph of Rass and Radcliffe [38] and a survey by Ruan [39].

Recently, in [52], we studied the following nonlocal dispersal SIS epidemic model with Dirichlet boundary condition

$$\begin{cases} \dfrac{\partial S}{\partial t} = d_S\left[\int_\Omega J(x-y)S(x,t)dy - S(x,t)\right] + \Lambda(x) - \dfrac{\beta(x)SI}{S+I} + \gamma(x)I & \text{in } \Omega\times\mathbb{R}^+, \\ \dfrac{\partial I}{\partial t} = d_I\left[\int_\Omega J(x-y)I(x,t)dy - I(x,t)\right] + \dfrac{\beta(x)SI}{S+I} - \gamma(x)I & \text{in } \Omega\times\mathbb{R}^+, \\ S(x,0)=S_0(x),\ I(x,0)=I_0(x) & \text{in } \Omega, \\ S(x,t)=I(x,t)=0 & \text{in } \mathbb{R}^N\backslash\Omega\times\mathbb{R}^+. \end{cases}$$

The basic reproduction number R_0 was introduced and threshold-type results on the global dynamic in terms of R_0 were established. In this model, the individuals can move in the whole $\mathbb{R}^n$ but vanish outside Ω. In the biological interpretation, there is a hostile environment outside Ω and any individual that jumps outside dies instantaneously, which is similar to the so-called **homogeneous Dirichlet boundary conditions** for the random diffusion equations. Zhao and Ruan [55] proposed a nonlocal model of within-host viral dynamics on a bounded domain in which virus movement is described by a nonlocal (convolution) diffusion operator, investigated the principal eigenvalue of a perturbation of the nonlocal diffusion operator, and showed that if the principal eigenvalue is less or equal to zero, then the infection-free steady state is asymptotically stable while there is an infection steady state which is stable provided that the principal eigenvalue is greater than zero. The present paper is devoted to the dynamic behavior of system (1.1) with **Neumann boundary conditions**.

It is well-known that the basic reproduction number R_0 is an important threshold to determine the dynamic behavior of epidemic models. For system (1.1), it is natural to ask what the basic reproduction number is and how it decides the dynamic behavior of (1.1). As one of the important quantities in epidemiology, the basic reproduction number R_0 of an infectious disease is defined as the expected number of secondary cases produced, in a completely susceptible population, by a typical infective individual (see, e.g., Heffernan, Smith and Wahl [22] and the reference therein). For autonomous epidemic models, Diekmann et al. [17] introduced R_0 by using the next generation operators. Van der Driessche and Watmough [47] established the theory of R_0 for compartment ODE models. Thieme [46] further developed a general theory of spectral bounds and reproduction numbers for the infinite-dimensional population structure and time heterogeneity. For a nonlocal and time-delayed reaction-diffusion model of dengue fever, Wang and Zhao [48] gave the definition of R_0 via a next generation operator and proved the threshold dynamics in terms of R_0. Wang and Zhao [49] presented the theory of R_0 for reaction-diffusion epidemic models with compartment structure and in particular, characterized R_0 by means of the principal eigenvalue of an elliptic eigenvalue problem. As we know that the nonlocal eigenvalue problems may not have principal eigenvalues generally. Naturally, we want to know how to characterize the basic reproduction number of nonlocal dispersal problems. Motivated by the works in [46,48,49], we intend to introduce the basic reproduction number R_0 for model (1.1) and give its characterization. We further prove that $R_0 - 1$ has the same sign as

$$\mu_p(d_I) = \sup_{\substack{\varphi \in L^2(\Omega) \\ \varphi \neq 0}} \frac{-\frac{d_I}{2} \int_\Omega \int_\Omega J(x-y)(\varphi(y)-\varphi(x))^2 dy dx + \int_\Omega (\beta(x)-\gamma(x))\varphi^2(x)dx}{\int_\Omega \varphi^2(x)dx},$$

which is of interest by itself. In general, $\mu_p(d_I)$ may not be the principal eigenvalue of the nonlocal operator

$$\mathcal{M}[u](x) := d_I \int_\Omega J(x-y)(u(y)-u(x))dy + (\beta(x)-\gamma(x))u(x),$$

which may lead to some essential differences between the nonlocal dispersal problems and the reaction-diffusion problems, see Coville [12], Hutson et al. [24], Shen and Zhang [40], Sun, Li and Yang [43], and Sun, Li and Wang [44]. To overcome this difficulty, we attempt to use the basic theory developed in [46] to give the definition of the basic reproduction number of system (1.1).

Then we are concerned with the global stability of the disease-free equilibrium and the endemic equilibrium of system (1.1). It is shown that the disease-free equilibrium is unique and globally stable when $R_0 < 1$, which implies that the disease will die out. We establish the existence of an endemic equilibrium by using the sub-super solutions method and obtain the uniqueness of the endemic equilibrium following the method in Berestycki et al. [6] when $R_0 > 1$. Generally, it is difficult to prove the stability of the endemic equilibrium of system

(1.1). However, when $d_S = d_I$, the global stability of the endemic equilibrium can be shown by constructing some auxiliary problems. This result implies that the disease will be persistent when $R_0 > 1$. Finally, we discuss the effect of the diffusion rates d_S and d_I on the disease transmission. Necessarily, we find that the nonlocal movement of the susceptible or infectious individuals tends to enhance the persistence of the disease. We would like to mention that there are some difficulties to be overcome when we prove these results due to the lack of the regularity of solutions of (1.1) or the stationary solutions corresponding to system (1.1).

The paper is organized as follows. In Section 2, we characterize the basic reproduction number of system (1.1). Section 3 is devoted to the existence, uniqueness and global stability of the disease-free equilibrium and the endemic equilibrium. In Section 4, we discuss the effect of the diffusion rates of the susceptible and infectious individuals on the disease transmission. Finally, we give a brief discussion to complete the paper.

2. The basic reproduction number

In this section, we will give the definition of the basic reproduction number for system (1.1) and provide its analytical properties. Let $X = C(\bar{\Omega})$ be the Banach space of real continuous functions on $\bar{\Omega}$. Throughout this section, X is considered as an ordered Banach space with a positive cone $X_+ = \{u \in X |\ u \geq 0\}$. It is well-known that X_+ is generating, normal and has nonempty interior. Additionally, an operator $T : X \to X$ is called positive if $TX_+ \subseteq X_+$.

2.1. Preliminaries

Definition 2.1. A closed linear operator $\mathscr{A}$ in X is said to be *resolvent-positive* if the resolvent set of $\mathscr{A}$, $\rho(\mathscr{A})$, contains a ray (ω, ∞) and the resolvent $(\lambda I - \mathscr{A})^{-1}$ is a positive bounded linear operator for all $\lambda > \omega$.

Definition 2.2. The *spectral bound* of $\mathscr{A}$ is defined by

$$S(\mathscr{A}) = \sup\{\mathrm{Re}\lambda |\ \lambda \in \sigma(\mathscr{A})\},$$

where $\sigma(\mathscr{A})$ denotes the *spectrum* of $\mathscr{A}$. The *spectral radius* of $\mathscr{A}$ is defined as

$$r(\mathscr{A}) = \sup\{|\lambda|;\ \lambda \in \sigma(\mathscr{A})\}.$$

Theorem 2.3 *(Thieme [46]). Let $\mathscr{A}$ be the generator of a C_0-semigroup $\mathcal{S}$ on the ordered Banach space X with a normal and generating cone X_+. Then, $\mathscr{A}$ is a resolvent-positive operator if and only if $\mathcal{S}$ is a positive semigroup, i.e., $\mathcal{S}(t)X_+ \subset X_+$ for all $t \geq 0$. If $\mathscr{A}$ is resolvent-positive, then*

$$(\lambda I - \mathscr{A})^{-1}x = \lim_{b\to\infty} \int_0^b e^{\lambda t}\mathcal{S}(t)x dt,\ \lambda > S(\mathscr{A}),\ x \in X.$$

Theorem 2.4 *(Thieme [46]). Let $\mathscr{B}$ be a resolvent-positive operator on X, $S(\mathscr{B}) < 0$ and $\mathscr{A} = \mathscr{C} + \mathscr{B}$ a positive perturbation of $\mathscr{B}$ with $\mathscr{C}$ a bounded linear operator. If $\mathscr{A}$ is resolvent-positive, $S(\mathscr{A})$ has the same sign as $r(-\mathscr{C}\mathscr{B}^{-1}) - 1$.*

In order to apply the basic theory in Thieme [46] to discuss the basic reproduction number of system (1.1), we first consider the eigenvalue problem

$$\mathcal{M}[u](x) := d_I \int_\Omega J(x-y)(u(y)-u(x))dy + \beta(x)u(x) - \gamma(x)u(x) = -\lambda u(x) \ \text{ in } \Omega, \quad (2.1)$$

which will be also used to obtain the main result in this section. Define

$$\lambda_p(d_I) = \inf_{\substack{\varphi\in L^2(\Omega)\\ \varphi\neq 0}} \frac{\frac{d_I}{2}\int_\Omega\int_\Omega J(x-y)(\varphi(y)-\varphi(x))^2dydx + \int_\Omega(\gamma(x)-\beta(x))\varphi^2(x)dx}{\int_\Omega \varphi^2(x)dx}.$$

It is well-known that $\lambda_p(d_I)$ may be the unique principal eigenvalue of (2.1), see [12,24,40,43].

Lemma 2.5. *Set $m(x) = -d_I\int_\Omega J(x-y)dy + \beta(x) - \gamma(x)$. Suppose there is some $x_0 \in Int(\Omega)$ satisfying that $m(x_0) = \max_{\bar\Omega} m(x)$, and the partial derivatives of $m(x)$ up to order $n-1$ at x_0 are zero. Then $\lambda_p(d_I)$ is the unique principal eigenvalue of (2.1) and its corresponding eigenfunction φ is positive and continuous on $\bar\Omega$.*

Lemma 2.6. *$\lambda_p(d_I)$ is the principal eigenvalue of (2.1) if and only if*

$$\lambda_p(d_I) < \min_{\bar\Omega}\left\{ d_I \int_\Omega J(x-y)dy + \gamma(x) - \beta(x)\right\}.$$

The proof of Lemma 2.6 is the same as Proposition 3.2 in Coville et al. [14].

Remark 2.7. Note that $\lambda_p(d_I)$ is continuous on J, $\beta(x)$ and $\gamma(x)$, see the proof in Coville [12].

Below, we always assume $\beta(x) - \gamma(x)$ is non-constant without other description.

Theorem 2.8. *Assume that $\lambda_p(d_I)$ is the principal eigenvalue of (2.1). Then the following alternatives hold:*

(i) *$\lambda_p(d_I)$ is a strictly monotone increasing function of d_I;*
(ii) *$\lambda_p(d_I) \to \min_{\bar\Omega}\{\gamma(x) - \beta(x)\}$ as $d_I \to 0$;*
(iii) *$\lambda_p(d_I) \to \frac{1}{|\Omega|}\int_\Omega(\gamma(x) - \beta(x))dx$ as $d_I \to +\infty$;*
(iv) *If $\int_\Omega \beta(x)dx \geq \int_\Omega \gamma(x)dx$, then $\lambda_p(d_I) < 0$ for all $d_I > 0$;*
(v) *If $\beta(x_*) > \gamma(x_*)$ for some $x_* \in \Omega$ and $\int_\Omega \beta(x)dx < \int_\Omega \gamma(x)dx$, then the equation $\lambda_p(d_I) = 0$ has a unique positive root denoted by d_I^*. Furthermore, if $d_I < d_I^*$, then $\lambda_p(d_I) < 0$ and if $d_I > d_I^*$, then $\lambda_p(d_I) > 0$.*

Proof. Let $\varphi(x)$ be the corresponding eigenfunction to $\lambda_p(d_I)$ and normalize it as $\|\varphi\|_{L^2(\Omega)} = 1$. Then

$$\lambda_p(d_I) = \frac{d_I}{2}\int_\Omega\int_\Omega J(x-y)(\varphi(y)-\varphi(x))^2 dydx + \int_\Omega (\gamma(x)-\beta(x))\varphi^2(x)dx.$$

Obviously, $\varphi(x)$ is not a constant. Otherwise, $\lambda_p(d_I) = \gamma(x) - \beta(x)$ according to (2.1), which is a contradiction. Assume $d_I > d_{I_1}$. Then, following the variational characterization of $\lambda_p(d_I)$, we have

$$\lambda_p(d_I) > \frac{d_{I_1}}{2}\int_\Omega\int_\Omega J(x-y)(\varphi(y)-\varphi(x))^2 dydx + \int_\Omega (\gamma(x)-\beta(x))\varphi^2(x)dx \geq \lambda_p(d_{I_1}).$$

This proves (i).

Now let $\theta(x) = \gamma(x) - \beta(x)$ and $\theta_{\min} = \min_{\bar\Omega} \theta(x)$. Consider the eigenvalue problem

$$d_I \int_\Omega J(x-y)(u(y)-u(x))dy - \theta_{\min} u(x) = -\lambda u(x) \text{ in } \Omega. \tag{2.2}$$

Thus, the principal eigenvalue of (2.2) is $\lambda_p^* = \theta_{\min}$, see [20]. Hence, we have $\lambda_p(d_I) \geq \theta_{\min}$. Now, if we can prove that $\limsup_{d_I\to 0} \lambda_p(d_I) \leq \theta_{\min}$, then the result is obtained. On the contrary, assume there exists some $\varepsilon > 0$ such that

$$\limsup_{d_I\to 0} \lambda_p(d_I) \geq \theta_{\min} + \varepsilon.$$

By the definition of lim sup, there exists some $\hat{d}_I > 0$ such that if $d_I \leq \hat{d}_I$, then

$$\lambda_p(d_I) \geq \theta_{\min} + \frac{\varepsilon}{2}.$$

Additionally, the continuity of $\theta(x)$ gives that there are some $x_0 \in \Omega$ and $r > 0$ such that

$$\theta_{\min} \geq \theta(x) - \frac{\varepsilon}{4} \quad \text{for } x \in B_r(x_0) \subset \Omega.$$

Hence, $\lambda_p(d_I) \geq \theta(x) + \frac{\varepsilon}{4}$ for $d_I \leq \hat{d}_I$ and $x \in B_r(x_0)$. Let $\varphi(x)$ be the eigenfunction to $\lambda_p(d_I)$. Then, it follows from (2.1) that

$$\int_\Omega J(x-y)(\varphi(x)-\varphi(y))dy = \frac{\lambda_p(d_I)-\theta(x)}{d_I}\varphi(x) \geq \frac{\varepsilon}{4d_I}\varphi(x) \quad \text{in} \quad B_r(x_0).$$

Let λ_1 be the principal eigenvalue of the linear problem

$$\begin{cases} \int_{\mathbb{R}^N} J(x-y)(u(y)-u(x))dy = -\lambda u(x) & \text{in } B_r(x_0), \\ u(x) = 0 & \text{in } \mathbb{R}^N \setminus B_r(x_0). \end{cases}$$

It is well-known that $0 < \lambda_1 < 1$, see [19]. Let $\psi(x)$ be the corresponding eigenfunction to λ_1 normalized by $\|\psi\|_{L^\infty(B_r(x_0))} = 1$. Set

$$\overline{\Phi}(x) = \frac{\varphi(x)}{\inf_{B_r(x_0)} \varphi(x)}, \quad \underline{\Phi}(x) = \psi(x) \le 1.$$

Consider the following linear problem

$$\begin{cases} \int_{\mathbb{R}^N} J(x-y)(u(y)-u(x))dy = -\frac{\varepsilon}{4d_I} u(x) & \text{in } B_r(x_0), \\ u(x) = 0 & \text{in } \mathbb{R}^N \backslash B_r(x_0). \end{cases} \tag{2.3}$$

By the direct computation, we have

$$\begin{aligned} &\int_{B_r(x_0)} J(x-y)\overline{\Phi}(y)dy - \overline{\Phi}(x) + \frac{\varepsilon}{4d_I}\overline{\Phi}(x) \\ &\le \int_{\Omega} J(x-y)(\overline{\Phi}(y) - \overline{\Phi}(x))dy + \frac{\varepsilon}{4d_I}\overline{\Phi}(x) \le 0 \end{aligned}$$

and

$$\begin{aligned} &\int_{B_r(x_0)} J(x-y)\underline{\Phi}(y)dy - \underline{\Phi}(x) + \frac{\varepsilon}{4d_I}\underline{\Phi}(x) \\ &= \int_{B_r(x_0)} J(x-y)\psi(y)dy - \psi(x) + \frac{\varepsilon}{4d_I}\psi(x) \ge 0 \end{aligned}$$

provided $d_I \le \min\{\hat{d}_I, \frac{\varepsilon}{4\lambda_1}\}$. Then, by the super-sub solution method (see [21]), (2.3) admits a positive solution between $\underline{\Phi}(x)$ and $\overline{\Phi}(x)$, which implies that $\lambda_1 = \frac{\varepsilon}{4d_I}$. This contradicts to the independence of d_I about λ_1. Thus, $\lim_{d_I \to 0} \lambda_p(d_I) = \theta_{\min}$ and (ii) holds.

Now taking $\varphi^2 = \frac{1}{|\Omega|}$, the definition of $\lambda_p(d_I)$ yields that

$$\begin{aligned} \lambda_p(d_I) &\le \frac{\frac{d_I}{2}\int_\Omega\int_\Omega J(x-y)(\varphi(y)-\varphi(x))^2 dydx + \int_\Omega(\gamma(x)-\beta(x))\varphi^2(x)dx}{\int_\Omega \varphi^2(x)dx} \\ &= \frac{1}{|\Omega|}\int_\Omega (\gamma(x)-\beta(x))dx \le \max_{\bar{\Omega}}(\gamma(x)-\beta(x)). \end{aligned}$$

Since $\lambda_p(d_I)$ is strictly increasing on d_I, the limit of $\lambda_p(d_I)$ exists as $d_I \to +\infty$. Assume $\lim_{d_I \to +\infty} \lambda_p(d_I) = \lambda_\infty$. Then, $\lambda_\infty \le \max_{\bar{\Omega}}(\gamma(x) - \beta(x))$. Letting $\psi_{d_I}(x)$ be the corresponding eigenfunction to $\lambda_p(d_I)$ and normalizing it by $\|\psi_{d_I}\|_{L^\infty(\Omega)} = 1$, then we have

$$d_I \int_\Omega J(x-y)(\psi_{d_I}(y) - \psi_{d_I}(x))dy + (\beta(x) - \gamma(x))\psi_{d_I}(x) = -\lambda_p(d_I)\psi_{d_I}(x) \ \text{ in } \Omega. \tag{2.4}$$

Note that there exists some $d_0 > 0$ such that

$$\int_\Omega J(x-y)dy + \frac{\gamma(x)-\beta(x)-\lambda_p(d_I)}{d_I} > 0$$

for any $d_I \geq d_0$. Then, for each $d_I \geq d_0$, (2.4) implies that

$$\psi_{d_I}(x) = \frac{\int_\Omega J(x-y)\psi_{d_I}(y)dy}{\int_\Omega J(x-y)dy + \frac{\gamma(x)-\beta(x)-\lambda_p(d_I)}{d_I}} \in C(\bar{\Omega}). \tag{2.5}$$

Choose a sequence $\{d_{I,n}\}_{n=1}^{\infty}$ satisfying $d_{I,n} \to +\infty$ as $n \to +\infty$. Thus, the eigenfunction sequence $\{\psi_{d_{I,n}}\}$ weakly converges to some $\psi(x)$ in $L^2(\Omega)$. Hence, we have

$$\int_\Omega J(x-y)\psi_{d_{I,n}}(y)dy \to \int_\Omega J(x-y)\psi(y)dy \quad \text{uniformly on } \bar{\Omega} \text{ as } n \to +\infty.$$

Note that

$$\int_\Omega J(x-y)dy + \frac{\gamma(x)-\beta(x)-\lambda_p(d_{I,n})}{d_{I,n}} \to \int_\Omega J(x-y)dy \quad \text{uniformly on } \bar{\Omega} \text{ as } n \to +\infty.$$

Following (2.5), there is

$$\psi_{d_{I,n}}(x) \to \psi(x) \text{ uniformly on } \bar{\Omega} \text{ as } n \to +\infty.$$

Since

$$\int_\Omega J(x-y)(\psi_{d_{I,n}}(y)-\psi_{d_{I,n}}(x))dy = \frac{\gamma(x)-\beta(x)-\lambda_p(d_{I,n})}{d_{I,n}}\psi_{d_{I,n}}(x) \quad \text{in } \Omega,$$

we have

$$\int_\Omega J(x-y)(\psi_{d_{I,n}}(y)-\psi_{d_{I,n}}(x))dy \to 0 \quad \text{uniformly on } \bar{\Omega} \text{ as } n \to +\infty.$$

According to Proposition 3.3 in [4], we know $\psi(x)$ is a constant. Integrating both sides of (2.4) over Ω, we have

$$\int_\Omega (\beta(x)-\gamma(x))\psi_{d_{I,n}}(x)dx = -\lambda_p(d_{I,n})\int_\Omega \psi_{d_{I,n}}(x)dx.$$

Thus, there is

$$\lim_{n\to+\infty} \lambda_p(d_{I,n}) = \frac{1}{|\Omega|}\int_\Omega (\gamma(x)-\beta(x))dx.$$

Additionally, by the definition of $\lambda_p(d_I)$, (iv) is obvious. Meanwhile, (v) is the direct conclusion of (i)-(iii). The proof is complete. $\square$

Define an operator as follows

$$A[u](x) := d_I \int_\Omega J(x-y)(u(y)-u(x))dy - \gamma(x)u(x). \tag{2.6}$$

We have the following result.

Proposition 2.9. *If the operator A is defined by* (2.6)*, then A is a resolvent-positive operator on X and $S(A) < 0$.*

Proof. By the definition of the operator A, we know that A is a bounded linear operator on X. It is known that the operator A can generate a positive C_0-semigroup, see [25]. Then, following from Theorem 2.3, we have that A is a resolvent-positive operator on X.

Let

$$\sigma_p := \sup_{\substack{\varphi\in L^2(\Omega)\\ \varphi\neq 0}} \frac{-\frac{d_I}{2}\int_\Omega\int_\Omega J(x-y)(\varphi(y)-\varphi(x))^2dydx - \int_\Omega \gamma(x)\varphi^2(x)dx}{\int_\Omega \varphi^2(x)dx}.$$

Obviously, $\sigma_p < 0$. Let $h(x) = -d_I \int_\Omega J(x-y)dy - \gamma(x)$. We may choose some function sequence $\{h_n(x)\}_{n=1}^\infty$ with $\|h_n - h\|_{L^\infty(\Omega)} \to 0$ as $n \to +\infty$ such that the eigenvalue problem

$$A_n[\varphi](x) := d_I \int_\Omega J(x-y)\varphi(y)dy + h_n(x)\varphi(x) = \lambda\varphi(x) \text{ in } \Omega$$

admits a principal eigenpair denoted by $(\sigma_p^n, \varphi_n(x))$, where $\sigma_p^n \to \sigma_p$ as $n \to +\infty$. Note that $\sigma_p^n = S(A_n)$ for each given n (see Bates and Zhao [9]). Since $\sigma_p < 0$, there exists some $\delta > 0$ such that $\sigma_p^n < -\delta$ provided $n \geq n_0$ for some $n_0 > 0$. Thus, we have $S(A_n) < -\delta$ for $n \geq n_0$. Due to $h_n \to h$ as $n \to +\infty$, we can obtain that $S(A_n) \to S(A)$ as $n \to +\infty$, see Lemma 3.1 in [41]. This implies that $S(A) < 0$. The proof is complete. $\square$

2.2. *The basic reproduction number*

Consider the nonlocal dispersal problem

$$\frac{\partial u_I(x,t)}{\partial t} = d_I \int_\Omega J(x-y)(u_I(y,t)-u_I(x,t))dy - \gamma(x)u_I(x,t), \tag{2.7}$$

where $x \in \Omega$ and $t > 0$. If $u_I(x,t)$ is thought of as a density of the infected individuals at a point x at time t, $J(x-y)$ is thought of as the probability distribution of jumping from location y to location x, then $\int_\Omega J(y-x)u(y,t)dy$ is the rate at which the infected individuals are arriving at position x from all other places, and $-\int_\Omega J(y-x)u(x,t)dy$ is the rate at which they are leaving location x to travel to all other sites. By the theory of semigroups of linear operators, we know that the operator A can generate a uniformly continuous semigroup, denoted by $T(t)$. Suppose that $\phi(x)$ is the distribution of initial infection at location x. Then the distribution of those infective members is at time t (as time evolution) is $(T(t)\phi)(x)$. Set $\mathscr{F}[\varphi](x) := \beta(x)\varphi(x)$ for $\varphi \in X$. Hence, the distribution of new infection at time t is $\mathscr{F}[T(t)\phi](x)$ and the total new infections are

$$\int_0^\infty \mathscr{F}[T(t)\phi](x)dt.$$

Define

$$L[\phi](x) := \int_0^\infty \mathscr{F}[T(t)\phi](x)dt = \beta(x)\int_0^\infty T(t)\phi dt.$$

Then, inspired by the ideas of next generation operators (see [17,47–49]), we may define the spectral radius of L,

$$R_0 = r(L),$$

as the basic reproduction number of system (1.1). We have the following result.

Theorem 2.10. $R_0 - 1$ *has the same sign as* $\lambda_* := S(A + \mathscr{F})$.

Proof. Since A is the generator of the semigroup $T(t)$ on X and A is resolvent-positive, it then follows from Theorem 2.3 that

$$(\lambda I - A)^{-1}\phi = \int_0^\infty e^{-\lambda t}T(t)\phi dt \quad \text{for any } \lambda > S(A),\ \phi \in X. \tag{2.8}$$

Choosing $\lambda = 0$ in (2.8), we obtain

$$-A^{-1}\phi = \int_0^\infty T(t)\phi dt \text{ for all } \phi \in X. \tag{2.9}$$

Then, the definition of the operator L implies that $L = -\mathscr{F}A^{-1}$. Let $\mathcal{M} := A + \mathscr{F}$. We know that $\mathcal{M}$ can generate a uniformly continuous positive semigroup, then $\mathcal{M}$ is resolvent-positive. Meanwhile, $S(A) < 0$. Thus, following from Theorem 2.4, we have $S(\mathcal{M})$ has the same sign as $r(-\mathscr{F}A^{-1}) - 1 = R_0 - 1$. The proof is complete. □

Note that if $\lambda_p(d_I)$ is the principal eigenvalue of (2.1), then $-\lambda_p(d_I) = S(A + \mathscr{F})$. In this case, $-\lambda_p(d_I)$ has the same sign as $R_0 - 1$ according to Theorem 2.10. However, we still have the following result no matter $\lambda_p(d_I)$ is the principal eigenvalue of (2.1) or not.

Corollary 2.11. $\lambda_p(d_I)$ *has the same sign as* $1 - R_0$.

In fact, this is easily seen from the proof in Proposition 2.9 that $-\lambda_p(d_I) = S(A + \mathscr{F})$. Thus, Corollary 2.11 is obvious from Theorem 2.10.

Additionally, following Theorems 2.8 and 2.10, we have the following corollaries.

Corollary 2.12. *If $\beta(x_0) > \gamma(x_0)$ for some $x_0 \in \Omega$ and $\int_\Omega \beta(x)dx < \int_\Omega \gamma(x)dx$. Then there exists some $d_* > 0$ such that $R_0 > 1$ for all $0 < d_I < d_*$ and $R_0 < 1$ for $d_I > d_*$.*

Proof. Since $\beta(x_0) > \gamma(x_0)$ for some $x_0 \in \Omega$, the continuity of $\beta(x)$ and $\gamma(x)$ gives that $\beta(x) > \gamma(x)$ for any $x \in B_r(x_0)$, which $B_r(x_0)$ is a ball with center x_0 and radius $r > 0$. Let $\Omega_* = B_r(x_0) \cap \Omega$ and denote

$$\tilde{\varphi} := \begin{cases} C, & x \in \Omega_*, \\ 0, & x \in \Omega \backslash \Omega_* \end{cases}$$

for some nonzero constant C. Then, by using the definition of $\lambda_p(d_I)$ and the continuity of $\lambda_p(d_I)$ on d_I and taking $\tilde{\varphi}$ to be the test function, we have

$$\lambda_p(0) < \frac{1}{|\Omega_*|} \int\limits_{\Omega_*} (\gamma(x) - \beta(x))dx < 0.$$

Moreover, it follows from the definition of $\lambda_p(d_I)$ that

$$\lambda_p(d_I) \le \max_{\bar{\Omega}} \{\gamma(x) - \beta(x)\}.$$

Then, there exists some $\hat{d} > 0$ such that

$$\lambda_p(d_I) < \max_{\bar{\Omega}} \left\{ d_I \int\limits_\Omega J(x-y)dy + \gamma(x) - \beta(x) \right\}$$

for any $d_I > \hat{d}$. According to Lemma 2.6, $\lambda_p(d_I)$ is the principal eigenvalue of (2.1) for $d_I > \hat{d}$. Thus, using Theorem 2.8, we have

$$\lim_{d_I \to +\infty} \lambda_p(d_I) = \frac{1}{|\Omega|} \int\limits_\Omega (\gamma(x) - \beta(x))dx.$$

Since $\lambda_p(d_I)$ is nondecreasing on d_I, there is some $d_* > 0$ such that

$$\lambda_p(d_I) \begin{cases} < 0 \ \text{ if } \ 0 < d_I < d_*, \\ > 0 \ \text{ if } \ d_I > d_*. \end{cases}$$

Using Corollary 2.11, we complete the proof. □

Corollary 2.13. *If $\int_\Omega \beta(x)dx > \int_\Omega \gamma(x)dx$, then $R_0 > 1$ for any $d_I > 0$. Further, if $\beta(x) < \gamma(x)$ for $x \in \Omega$, then $R_0 < 1$ for all $d_I > 0$.*

This is easy seen from the definition of $\lambda_p(d_I)$ and Corollary 2.11.

Lemma 2.14. *Assume $(\mu_p, \phi(x))$ with $\phi(x) > 0$ is a principal eigenpair of the weighted eigenvalue problem*

$$-d_I \int_\Omega J(x-y)(\phi(y)-\phi(x))dy + \gamma(x)\phi(x) = \mu\beta(x)\phi(x), \quad x \in \Omega. \tag{2.10}$$

Then, μ_p is a unique positive principal eigenvalue and can be characterized by

$$\mu_p = \inf_{\substack{\varphi \in L^2(\Omega) \\ \varphi \neq 0}} \frac{\frac{d_I}{2}\int_\Omega\int_\Omega J(x-y)(\varphi(y)-\varphi(x))^2 dydx + \int_\Omega \gamma(x)\varphi^2(x)dx}{\int_\Omega \beta(x)\varphi^2(x)dx}.$$

Proof. Let $(\mu_i, \phi_i(x))$ $(i = 1, 2)$ with $\phi_i(x) > 0$ satisfying

$$-d_I \int_\Omega J(x-y)(\phi_i(y)-\phi_i(x))dy + \gamma(x)\phi_i(x) = \mu_i\beta(x)\phi_i(x).$$

Following these equations, it is easy to obtain that

$$(\mu_1 - \mu_2)\int_\Omega \beta(x)\phi_1(x)\phi_2(x)dx = 0.$$

The positivity of $\phi_1(x)$ and $\phi_2(x)$ gives that $\mu_1 = \mu_2$. Further, according to (2.10), there is

$$\mu_p = \frac{\frac{d_I}{2}\int_\Omega\int_\Omega J(x-y)(\phi(y)-\phi(x))^2 dydx + \int_\Omega \gamma(x)\phi^2(x)dx}{\int_\Omega \beta(x)\phi^2(x)dx}. \tag{2.11}$$

Obviously, $\mu_p > 0$.

In the following we prove that

$$\mu_p = \mu_p' := \inf_{\substack{\varphi \in L^2(\Omega) \\ \varphi \neq 0}} \frac{\frac{d_I}{2}\int_\Omega\int_\Omega J(x-y)(\varphi(y)-\varphi(x))^2 dydx + \int_\Omega \gamma(x)\varphi^2(x)dx}{\int_\Omega \beta(x)\varphi^2(x)dx}.$$

In view of (2.11), we have $\mu_p \geq \mu_p'$. Assume that $\mu_p > \mu_p'$. Then, there exists some μ_* such that $\mu_p' < \mu_* < \mu_p$. Set

$$\mathbb{H}(\varphi) = \frac{d_I}{2}\int_\Omega\int_\Omega J(x-y)(\varphi(y)-\varphi(x))^2 dydx + \int_\Omega \gamma(x)\varphi^2(x)dx$$

and define

$$\sigma(\mu) = \sup_{\substack{\varphi\in L^2(\Omega)\\ \varphi\neq 0}} \frac{\mu \int_\Omega \beta(x)\varphi^2(x)dx - \mathbb{H}(\varphi)}{\int_\Omega \varphi^2(x)dx}.$$

Then, it follows from (2.11) that $\sigma(\mu_p) = 0$. Since $\mu_* > \mu_p'$, there is some $v \in L^2(\Omega)$ and $v \neq 0$ satisfying

$$\mu_* > \frac{\frac{d_I}{2}\int_\Omega\int_\Omega J(x-y)(v(y)-v(x))^2 dydx + \int_\Omega \gamma(x)v^2(x)dx}{\int_\Omega \beta(x)v^2(x)dx} > 0.$$

This implies that $\sigma(\mu_*) > 0$. On the other hand, by the definition of $\sigma(\mu)$, it is easy to see that $\sigma(\mu)$ is nondecreasing on μ. Due to $\mu_* < \mu_p$, we have $\sigma(\mu_*) \leq \sigma(\mu_p)$. That is $\sigma(\mu_*) \leq 0$, which is a contradiction. This completes the proof. □

Corollary 2.15. *If $(\mu^*, \phi^*(x))$ with $\phi^*(x) > 0$ satisfies the following linear problem*

$$\begin{cases} \int_{\mathbb{R}^N} J(x-y)(\phi^*(y)-\phi^*(x))dy = -\mu^*\gamma(x)\phi^*(x) & \text{in } \Omega, \\ \phi^*(x) = 0 & \text{on } \mathbb{R}^N\backslash\Omega, \end{cases}$$

then μ^* is unique and positive.

Theorem 2.16. *If the nonlocal weighted eigenvalue problem*

$$-d_I \int_\Omega J(x-y)(\phi(y)-\phi(x))dy + \gamma(x)\phi(x) = \mu\beta(x)\phi(x), \quad x \in \Omega$$

admits a unique positive principal eigenvalue μ_p with positive eigenfunction and there exists some positive function $\psi_{d_I}(x) \in L^2(\Omega)$ satisfying

$$L[\psi_{d_I}](x) = R_0\psi_{d_I}(x),$$

then $R_0 = r(-\mathscr{F}A^{-1}) = \frac{1}{\mu_p}$ and the following two conclusions hold:

(i) $R_0 \to \max_{\bar{\Omega}}\{\frac{\beta(x)}{\gamma(x)}\}$ *as $d_I \to 0$;*
(ii) $R_0 \to \frac{\int_\Omega \beta(x)dx}{\int_\Omega \gamma(x)dx}$ *as $d_I \to +\infty$.*

Proof. Note that

$$\beta(x)\int_0^\infty T(t)\psi_{d_I}dt = R_0\psi_{d_I}(x).$$

In view of (2.9), we have $-A^{-1}\psi_{d_I} = \int_0^\infty T(t)\psi_{d_I}dt$. Accordingly,

$$-\beta(x)A^{-1}[\psi_{d_I}](x) = R_0\psi_{d_I}(x). \tag{2.12}$$

Let $\varphi = -A^{-1}\psi_{d_I}$. Obviously, φ is positive. It follows from (2.12) that $-A\varphi = \frac{1}{R_0}\beta(x)\varphi$. That is, $(\frac{1}{R_0}, \varphi)$ satisfies

$$-d_I \int_\Omega J(x-y)(\varphi(y)-\varphi(x))dy + \gamma(x)\varphi(x) = \frac{1}{R_0}\beta(x)\varphi(x).$$

Following Lemma 2.14, it is clear that $(\frac{1}{R_0}, \varphi)$ is the principal eigenpair of (2.10). Hence, $R_0 = r(-\mathscr{F}A^{-1}) = \frac{1}{\mu_p}$. Meanwhile, R_0 can be characterized by

$$R_0 = \sup_{\substack{\varphi \in L^2(\Omega) \\ \varphi \neq 0}} \frac{\int_\Omega \beta(x)\varphi^2(x)dx}{\frac{d_I}{2}\int_\Omega\int_\Omega J(x-y)(\varphi(y)-\varphi(x))^2dydx + \int_\Omega \gamma(x)\varphi^2(x)dx}. \tag{2.13}$$

To prove (i) and (ii). Let $R_0 = R_0(d_I)$ and $\eta(x) = \frac{\beta(x)}{\gamma(x)}$. For any $v \in L^2(\Omega)$ and $v \neq 0$, we have

$$\begin{aligned}
&\frac{\int_\Omega \beta(x)v^2(x)dx}{\frac{d_I}{2}\int_\Omega\int_\Omega J(x-y)(v(y)-v(x))^2dydx + \int_\Omega \gamma(x)v^2(x)dx} \\
&\leq \frac{\max_{\bar\Omega}\eta(x)\int_\Omega \gamma(x)v^2(x)dx}{\frac{d_I}{2}\int_\Omega\int_\Omega J(x-y)(v(y)-v(x))^2dydx + \int_\Omega \gamma(x)v^2(x)dx} \\
&\leq \max_{\bar\Omega}\eta(x).
\end{aligned}$$

Hence, $R_0(d_I) \leq \max_{\bar\Omega}\eta(x) := \eta_*$. To our goal, we only need to prove that $\liminf_{d_I\to 0} R_0(d_I) \geq \eta_*$. On the contrary, assume that there exists some $\varepsilon > 0$ such that

$$\liminf_{d_I\to 0} R_0(d_I) \leq \eta_* - \varepsilon.$$

By the definition of liminf, there is some $d_0 > 0$ such that

$$R_0(d_I) \leq \eta_* - \frac{\varepsilon}{2}$$

for any $d_I \leq d_0$. Additionally, the continuity of $\eta(x)$ gives that there exists some $x_* \in \bar\Omega$ so that

$$\eta_* \leq \eta(x) + \frac{\varepsilon}{4} \text{ for any } x \in B_\rho(x_*),$$

in which $B_\rho(x_*)$ is a ball with center x_* and radius ρ. Hence,

$$R_0(d_I) \leq \eta(x) - \frac{\varepsilon}{4} \quad \text{for all } x \in B_\rho(x_*).$$

It is noticed that for any $x \in B_\rho(x_*)$

$$\begin{aligned} d_I \int_\Omega J(x-y)(\psi_{d_I}(x) - \psi_{d_I}(y))dy &= \left(\frac{\beta(x)}{R_0(d_I)} - \gamma(x)\right)\psi_{d_I}(x) \\ &\geq \left(\frac{\beta(x)}{\eta(x) - \frac{\varepsilon}{4}} - \gamma(x)\right)\psi_{d_I}(x) \\ &= \frac{\varepsilon\gamma(x)}{4(\eta(x) - \frac{\varepsilon}{4})}\psi_{d_I}(x) \\ &\geq \frac{\varepsilon\gamma(x)}{4\eta_*}\psi_{d_I}(x). \end{aligned}$$

On the other hand, it follows from García-Melián and Rossi [19] that the problem

$$\begin{cases} \int_{\mathbb{R}^N} J(x-y)(v(y) - v(x))dy = -\mu \max_{\bar{\Omega}}\{\gamma(x)\}v(x) & \text{in } B_\rho(x_*), \\ v(x) = 0 & \text{on } \mathbb{R}^N \backslash B_\rho(x_*) \end{cases}$$

admits a principal eigenpair $(\tilde{\mu}, \varphi^*(x))$ and $0 < \tilde{\mu} < \frac{1}{\max_{\bar{\Omega}} \gamma(x)}$. Now let

$$\underline{\Psi}(x) = \frac{\varphi^*(x)}{\inf_{B_\rho(x_*)} \varphi^*(x)}, \quad \overline{\Psi}(x) = K\psi_{d_I}(x) \ \text{ for constant } K > 1.$$

For the simple calculation, $\underline{\Psi}(x)$ and $\overline{\Psi}(x)$ are a pair of sub-super solutions of the following linear problem

$$\begin{cases} \int_{\mathbb{R}^N} J(x-y)(u(y) - u(x))dy = -\frac{\varepsilon\gamma(x)}{4d_I\eta_*}u(x) & \text{in } B_\rho(x_*), \\ u(x) = 0 & \text{on } \mathbb{R}^N \backslash B_\rho(x_*) \end{cases} \tag{2.14}$$

when $d_I \leq \min\{d_0, \frac{\varepsilon}{4\tilde{\mu}\eta_*}\}$. Then, there is a positive solution of (2.14). Following Corollary 2.15, it is obtained that $\mu_p' := \frac{\varepsilon}{4d_I\eta_*}$ is a principal eigenvalue of (2.14) which depends on the parameter d_I and this is a contradiction.

Next, we prove (ii). By the variational characterization of $R_0(d_I)$, it is easily seen that

$$R_0(d_I) \geq \frac{\int_\Omega \beta(x)dx}{\int_\Omega \gamma(x)dx}.$$

Since $R_0(d_I)$ is non-increasing on d_I, the limit of $R_0(d_I)$ exists as $d_I \to +\infty$. Notice that $(R_0(d_I), \psi_{d_I}(x))$ satisfies

$$-d_I \int_\Omega J(x-y)(\psi_{d_I}(y) - \psi_{d_I}(x))dy + \gamma(x)\psi_{d_I}(x) = \frac{1}{R_0(d_I)}\beta(x)\psi_{d_I}(x). \tag{2.15}$$

Choose some sequence $\{d_{I,n}\}_n^\infty$ satisfying $d_{I,n} \to +\infty$ as $n \to +\infty$ and normalize $\psi_{d_{I,n}}(x)$ as $\|\psi_{d_{I,n}}\|_{L^\infty(\Omega)} = 1$. Since there is some $n_0 > 0$ such that

$$\Delta(x) := \int_\Omega J(x-y)dy + \frac{\gamma(x) - \frac{\beta(x)}{R_0(d_{I,n})}}{d_{I,n}} > 0$$

for all $n \geq n_0$, we have

$$\psi_{d_{I,n}}(x) = \frac{\int_\Omega J(x-y)\psi_{d_{I,n}}(y)dy}{\Delta(x)}$$

for all $n \geq n_0$. Thus, $\psi_{d_{I,n}}(x) \to \psi^*$ strongly in $L^2(\Omega)$ as $n \to +\infty$. This implies that ψ^* satisfies

$$\int_\Omega J(x-y)(\psi^*(y) - \psi^*(x))dy = 0.$$

Hence, ψ^* is a positive constant. Integrating both sides of (2.15) with $d_{I,n}$ and $\psi_{d_{I,n}}(x)$ on Ω yields that

$$\int_\Omega \gamma(x)\psi_{d_{I,n}}(x)dx = \frac{1}{R_0(d_{I,n})}\int_\Omega \beta(x)\psi_{d_{I,n}}(x)dx.$$

Letting $n \to +\infty$, one has

$$\lim_{n\to+\infty} R_0(d_{I,n}) = \frac{\int_\Omega \beta(x)dx}{\int_\Omega \gamma(x)dx}.$$

This completes the proof. $\square$

Remark 2.17. Comparing to the corresponding elliptic problem, the operator

$$L[\varphi](x) = \beta(x)\int_0^\infty T(t)\varphi dt$$

is not a compact operator. Thus, $r(L)$ may not be a principal eigenvalue of L and the basic reproduction number R_0 cannot be characterized as (2.13) in general. However, (2.13) can still be used to determine the dynamic behavior of system (1.1) as a threshold value.

3. The dynamic behavior of system (1.1)

By the standard semigroup theory of linear bounded operator (Pazy [33]), we know from Kao, Lou and Shen [25] that (1.1) admits a unique nonnegative solution $(S_*(x,t), I_*(x,t))$ for all $x \in \Omega$ and $t \in (0, T_{max})$ with T_{max} the maximal existence time for solutions of (1.1), which is continuous with respect to x and t. That is, we have the following result.

Proposition 3.1. *Assume that* $(S_0(\cdot), I_0(\cdot)) \in X \times X$. *Then there exists a* $T_{max} > 0$ *such that system* (1.1) *has a unique solution* $(S_*(x,t), I_*(x,t))$. *Moreover, either* $T_{\max} = +\infty$ *or* $\lim\limits_{t \to T_{\max}-0} \|(S_*(\cdot,t), I_*(\cdot,t))\|_{X\times X} = +\infty$.

Note that by the maximum principle, it is easy to get that both $S_*(x,t)$ and $I_*(x,t)$ are bounded on $\bar{\Omega} \times (0, T_{\max})$. Thus, Proposition 3.1 implies that $T_{max} = +\infty$. In fact, it follows from the second equation of (1.1) that $I_*(x,t)$ satisfies

$$\frac{\partial I_*(x,t)}{\partial t} \le d_I \int_\Omega J(x-y)(I_*(y,t) - I_*(x,t))dy + (\beta(x) - \gamma(x))I_*(x,t). \tag{3.1}$$

Consider the following initial value problem:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = d_I \int_\Omega J(x-y)(u(y,t) - u(x,t))dy + (\beta(x) - \gamma(x))u(x,t), & x \in \Omega,\ t > 0, \\ u(x,0) = I_0(x), & x \in \Omega. \end{cases} \tag{3.2}$$

By the maximum principle, the linear initial value problem (3.2) admits a unique solution $u(x,t)$ for all $t > 0$. According to the comparison principle, we have $I_*(x,t) \le u(x,t)$ for $x \in \Omega$ and $t > 0$. On the other hand, following the first equation of (1.1), one can get that $S_*(x,t)$ satisfies

$$\frac{\partial S_*(x,t)}{\partial t} \le d_S \int_\Omega J(x-y)(S_*(y,t) - S_*(x,t))dy + \gamma(x)I_*(x,t).$$

The analogous discussion can give that $S_*(x,t)$ exists for all $t > 0$.

Additionally, by the strong maximum principle and the same discussion as above, we can obtain that $S_*(x,t) > 0$ and $I_*(x,t) > 0$ for $x \in \Omega$ and $t > 0$ for the assumption of $S_0(x)$ and $I_0(x)$ in the Introduction.

Let

$$N := \int_\Omega (S_0(x) + I_0(x))dx.$$

If we add up the first equation and the second equation of (1.1) and integrate it on Ω, then

$$\frac{\partial}{\partial t} \int_\Omega (S_*(x,t) + I_*(x,t))dx = 0 \ \text{ for all } t \ge 0.$$

This implies that the total population size is constant, that is

$$\int_\Omega (S_*(x,t) + I_*(x,t))dx = N \ \text{ for all } t \ge 0.$$

Definition 3.2. We say that a steady state $(\tilde{S}(x), \tilde{I}(x))$ of system (1.1) is *globally stable* if the solutions $(S_*(x,t), I_*(x,t))$ of (1.1) satisfy

$$\lim_{t \to +\infty} (S_*(x,t), I_*(x,t)) = (\tilde{S}(x), \tilde{I}(x)) \text{ in } X \times X$$

for any initial data $(S_0(\cdot), I_0(\cdot)) \in \{X_+ \times X_+\} \backslash \left\{ \left(\frac{N}{|\Omega|}, 0 \right) \right\}$.

In next subsection, we will consider the stationary problem of system (1.1):

$$\begin{cases} d_S \int_\Omega J(x-y)(S(y)-S(x))dy = \frac{\beta(x)SI}{S+I} - \gamma(x)I, & x \in \Omega, \\ d_I \int_\Omega J(x-y)(I(y)-I(x))dy = -\frac{\beta(x)SI}{S+I} + \gamma(x)I, & x \in \Omega. \end{cases} \tag{3.3}$$

3.1. The disease-free equilibrium

In this subsection, we discuss the existence and stability for the solution $(S(x), I(x))$ of (3.3) with $S(x) > 0$ and $I(x) = 0$, which is called as the disease-free equilibrium of (1.1).

Lemma 3.3. *System (3.3) admits a disease-free equilibrium $(\hat{S}, 0)$, which is unique and given by $\hat{S} = \frac{N}{|\Omega|}$ on $\bar{\Omega}$.*

Proof. Let $(\tilde{S}, 0)$ be any disease-free equilibrium. Then, following (3.3), we obtain that

$$\int_\Omega J(x-y)(\tilde{S}(y) - \tilde{S}(x))dy = 0 \text{ in } \Omega.$$

It is well-known from [4, Proposition 3.3] that $\tilde{S}(x)$ is a constant. And since $\int_\Omega \tilde{S}(x)dx = N$, we have $\tilde{S}(x) = \frac{N}{|\Omega|}$ on $\bar{\Omega}$. The proof is complete. □

Then, we have the following globally stability result.

Theorem 3.4. *If $R_0 < 1$, then all positive solutions of (1.1) converge to the disease-free equilibrium $\left(\frac{N}{|\Omega|}, 0 \right)$ as $t \to +\infty$ in $X \times X$.*

Proof. Since $R_0 < 1$, we have $-\lambda_p(d_I) = \lambda_* < 0$ (λ_* is defined in Theorem 2.10) according to Corollary 2.11. That is $\lambda_p(d_I) > 0$. Recall that $m(x) = -d_I \int_\Omega J(x-y)dy + \beta(x) - \gamma(x)$. Moreover, since $m(x)$ is continuous on $\bar{\Omega}$, there exists some $x_0 \in \bar{\Omega}$ such that $m(x_0) = \max_{x \in \bar{\Omega}} m(x)$. Define a function sequence as follows:

$$m_n(x) = \begin{cases} m(x_0), & x \in B_{x_0}(\frac{1}{n}), \\ m_{n,1}(x), & x \in (B_{x_0}(\frac{2}{n}) \backslash B_{x_0}(\frac{1}{n})), \\ m(x), & x \in \Omega \backslash B_{x_0}(\frac{2}{n}), \end{cases}$$

where $B_{x_0}(\frac{1}{n}) = \{x \in \Omega | \, |x - x_0| < \frac{1}{n}\}$, $m_{n,1}(x)$ satisfies $m_{n,1} \le m(x_0)$, and $m_{n,1}(x)$ is continuous in Ω. Indeed, $m_{n,1}(x)$ exists if only we take n is large enough, denoted by $n \ge n_0 > 0$. Thus, Lemma 2.5 implies that the eigenvalue problem

$$d_I \int_\Omega J(x-y)\phi(y)dy + m_n(x)\phi(x) = -\lambda\phi(x)$$

admits a principal eigenpair, denoted by $(\lambda_p^n(d_I), \phi_n)$. According to Remark 2.7, there exists some $n_1 \ge n_0$ such that for any $n \ge n_1$

$$\lambda_p^n(d_I) \ge \frac{1}{2}\lambda_p(d_I) + \|m_n - m\|_{L^\infty}.$$

Normalizing $\phi_n(x)$ as $\|\phi_n\|_{L^\infty(\Omega)} = 1$ and letting $\overline{u}(x,t) = Me^{-\frac{1}{2}\lambda_p(d_I)t}\phi_n(x)$, the direct calculation yields that

$$\begin{aligned}
&\frac{\partial \overline{u}(x,t)}{\partial t} - d_I \int_\Omega J(x-y)(\overline{u}(y,t) - \overline{u}(x,t))dy - \frac{\beta(x)\overline{u}S_*}{\overline{u}+S_*} + \gamma(x)\overline{u} \\
&\ge -\frac{1}{2}\lambda_p(d_I)Me^{-\frac{1}{2}\lambda_p(d_I)t}\phi_n(x) - Me^{-\frac{1}{2}\lambda_p(d_I)t}\left[d_I \int_\Omega J(x-y)\phi_n(y)dy + m_n(x)\phi_n(x)\right] \\
&\quad +(m_n(x) - m(x))Me^{-\frac{1}{2}\lambda_p(d_I)t}\phi_n(x) \\
&\ge \left[\lambda_p^n(d_I) - \frac{1}{2}\lambda_p(d_I) + (m_n(x) - m(x))\right]Me^{-\frac{1}{2}\lambda_p(d_I)t}\phi_n(x) \ge 0,
\end{aligned}$$

provided $n \ge n_1$. Take M large enough such that $\overline{u}(x,0) \ge I_0(x)$. Then, the comparison principle [54, Lemma 2.2] yields that $I_*(x,t) \le \overline{u}(x,t)$ for $x \in \Omega$ and $t > 0$. Consequently, we get that $I_*(x,t) \to 0$ uniformly on $\bar{\Omega}$ as $t \to +\infty$.

It remains to prove that $S_*(x,t) \to \frac{N}{|\Omega|}$ uniformly on $\bar{\Omega}$ as $t \to +\infty$. By the above discussion and the continuity of $\beta(x)$ and $\gamma(x)$, there exists some $C_0 > 0$ such that

$$\left\|\gamma I_* - \frac{\beta S_* I_*}{S_* + I_*}\right\|_{L^\infty(\Omega)} \le C_0 e^{-\frac{1}{2}\lambda_p(d_I)t}. \tag{3.4}$$

Define

$$\alpha = \alpha(J,\Omega) = \inf_{u \in L^2(\Omega), \int_\Omega u = 0, u \not\equiv 0} \frac{\frac{d_S}{2}\int_\Omega\int_\Omega J(x-y)(u(y)-u(x))^2 dydx}{\int_\Omega u^2(x)dx}. \tag{3.5}$$

Following Proposition 3.4 and Lemma 3.5 in [4], we get

$$0 < \alpha \le d_S \min_{x \in \bar{\Omega}} \int_\Omega J(x-y)dy.$$

Meanwhile, by the same method of the proof of Lemma 3.5 in [4], there holds

$$\lambda_p(d_I) \le \min_{\bar{\Omega}} \left(d_I \int_\Omega J(x-y)dy + \gamma(x) - \beta(x) \right).$$

Set

$$S_*(x,t) = \hat{S}_1(x,t) + \frac{1}{|\Omega|} \int_\Omega S_*(x,t)dx.$$

Due to $I_*(x,t) \to 0$ uniformly on $\bar{\Omega}$ as $t \to +\infty$, we know $\int_\Omega S_*(x,t)dx \to N$ as $t \to +\infty$. Thus, one can get

$$\frac{1}{|\Omega|} \int_\Omega S_*(x,t)dx \to \frac{N}{|\Omega|} \text{ as } t \to +\infty.$$

Note that $\int_\Omega \hat{S}_1(x,t)dx = 0$ and $\hat{S}_1(x,t)$ satisfies

$$\frac{\partial \hat{S}_1(x,t)}{\partial t} = d_S \int_\Omega J(x-y)(\hat{S}_1(y,t) - \hat{S}_1(x,t))dy + f(x,t), \quad x \in \Omega, \ t > 0, \tag{3.6}$$

where

$$f(x,t) = \gamma(x)I_* - \frac{\beta(x)S_*I_*}{S_*+I_*} - \frac{1}{|\Omega|} \int_\Omega \left(\gamma(x)I_* - \frac{\beta(x)S_*I_*}{S_*+I_*} \right) dx.$$

According to (3.4), there exists some positive constant $c_* > 0$ such that

$$|f(x,t)| \le c_* e^{-\frac{1}{2}\lambda_p(d_I)t}.$$

Now, let $W(t) = \int_\Omega \hat{S}_1^2(x,t)dx$. Hence, the direct calculation yields that

$$\begin{aligned}
\frac{dW(t)}{dt} &= 2\int_\Omega \hat{S}_1(x,t) \frac{\partial \hat{S}_1(x,t)}{\partial t} dx \\
&= 2\int_\Omega \hat{S}_1(x,t) \left[d_S \int_\Omega J(x-y)(\hat{S}_1(y,t) - \hat{S}_1(x,t))dy + f(x,t) \right] dx \\
&= -d_S \int_\Omega \int_\Omega J(x-y)(\hat{S}_1(y,t) - \hat{S}_1(x,t))^2 dydx + 2\int_\Omega \hat{S}_1(x,t) f(x,t)dx \\
&\le -2\alpha W(t) + 4c_* N e^{-\frac{1}{2}\lambda_p(d_I)t}.
\end{aligned}$$

This implies that

$$\begin{aligned} W(t) &\le W(0)e^{-2\alpha t} + ce^{-2\alpha t}\int_0^t e^{(2\alpha-\frac{1}{2}\lambda_p(d_I))s}ds \\ &= \begin{cases} (W(0)+ct)e^{-2\alpha t} & \text{if } \lambda_p(d_I)=4\alpha, \\ c_1e^{-2\alpha t}+c_2e^{-\frac{1}{2}\lambda_p(d_I)t} & \text{if } \lambda_p(d_I)\neq 4\alpha \end{cases} \end{aligned} \tag{3.7}$$

for some positive constants c, c_1 and c_2. On the other hand, it follows from (3.6) that

$$\hat{S}_1(x,t) = \hat{S}_1(x,0)e^{-a(x)t} + e^{-a(x)t}\int_0^t e^{a(x)s}\left[d_S\int_\Omega J(x-y)\hat{S}_1(y,s)dy + f(x,s)\right]ds, \tag{3.8}$$

where $a(x) = d_S\int_\Omega J(x-y)dy$. By Hölder inequality, we have

$$\int_\Omega J(x-y)\hat{S}_1(y,s)dy \le c_3W^{\frac{1}{2}}(t) \tag{3.9}$$

for some positive constant c_3. Then, combining (3.7)-(3.9), it can be obtained that

$$|\hat{S}_1(x,t)| \to 0 \ \text{ as } \ t \to +\infty.$$

Consequently, we have

$$S_*(x,t) \to \frac{N}{|\Omega|} \ \text{uniformly on } \bar{\Omega} \text{ as } t \to +\infty.$$

This completes the proof. □

Remark 3.5. Theorem 3.4 implies that when $R_0 < 1$, the disease will die out.

3.2. The endemic equilibrium

In this subsection, we consider the existence and uniqueness of the positive solutions of (3.3) which is the so-called endemic equilibrium of (1.1). Also, the long-time behavior of positive solutions of (1.1) will be discussed.

Lemma 3.6. *The pair of $(\tilde{S}(x), \tilde{I}(x))$ is a solution of (3.3) if and only if $(\tilde{S}(x), \tilde{I}(x))$ is a solution of*

$$k = d_S\tilde{S} + d_I\tilde{I}, \ x \in \Omega,$$

$$d_I \int_\Omega J(x-y)(\tilde{I}(y) - \tilde{I}(x))dy + \frac{\beta(x)\tilde{S}\tilde{I}}{\tilde{S}+\tilde{I}} - \gamma(x)\tilde{I} = 0, \ x \in \Omega,$$

$$N = \int_\Omega (\tilde{S}(x) + \tilde{I}(x))dx,$$

where k is some positive constant.

Proof. Suppose $(\tilde{S}(x), \tilde{I}(x))$ is a solution of (3.3). Then, adding the two equations of (3.3) yields that

$$\int_\Omega J(x-y)[(d_S\tilde{S}(y) + d_I\tilde{I}(y)) - (d_S\tilde{S}(x) + d_I\tilde{I}(x))]dy = 0, \quad x \in \Omega.$$

Thus, there is some constant k according to Proposition 3.3 in [4] such that

$$d_S\tilde{S}(x) + d_I\tilde{I}(x) = k, \ x \in \Omega.$$

Meanwhile, the other cases are obvious.

In turn, if $d_S\tilde{S}(x) + d_I\tilde{I}(x) = k$, we have

$$d_S \int_\Omega J(x-y)(\tilde{S}(y) - \tilde{S}(x))dy + d_I \int_\Omega J(x-y)(\tilde{I}(y) - \tilde{I}(x))dy = 0, \ x \in \Omega.$$

Then,

$$\begin{aligned} &d_S \int_\Omega J(x-y)(\tilde{S}(y) - \tilde{S}(x))dy \\ &= -d_I \int_\Omega J(x-y)(\tilde{I}(y) - \tilde{I}(x))dy = \frac{\beta(x)\tilde{S}\tilde{I}}{\tilde{S}+\tilde{I}} - \gamma(x)\tilde{I}, \end{aligned}$$

which implies that $(\tilde{S}, \tilde{I})$ satisfies (3.3). This ends the proof. □

Let $S(x) := \frac{\tilde{S}(x)}{k}$ and $I(x) := \frac{d_I\tilde{I}(x)}{k}$, where k is defined as in Lemma 3.6. Then, we have the following result.

Lemma 3.7. *The pair $(\tilde{S}(x), \tilde{I}(x))$ is a solution of (3.3) if and only if $(\tilde{S}(x), \tilde{I}(x))$ is a solution of*

$$\begin{cases} 1 = d_S S(x) + I(x), \ x \in \Omega, \\ 0 = d_I \int\limits_{\Omega} J(x-y)(I(y) - I(x))dy + (\beta(x) - \gamma(x))I - \dfrac{d_S \beta(x) I^2}{d_S I + d_I(1-I)}, \ x \in \Omega, \\ k = \dfrac{d_I N}{\int_\Omega (d_I S(x) + I(x))dx}. \end{cases} \tag{3.10}$$

Theorem 3.8. *Suppose $R_0 > 1$. Then (3.3) has a nonnegative solution $(\tilde{S}(x), \tilde{I}(x))$ which satisfies $\tilde{S}(\cdot), \tilde{I}(\cdot) \in C(\bar{\Omega})$ and $\tilde{I}(x) \not\equiv 0$ on $\bar{\Omega}$. Moreover, $(\tilde{S}(x), \tilde{I}(x))$ is a unique solution of (3.3), $0 < \tilde{S}(x) < \frac{k}{d_S}$ and $0 < \tilde{I}(x) < \frac{k}{d_I}$ for some positive constant k dependent on d_S, d_I.*

Proof. Since $R_0 > 1$, we obtain that $\lambda_p(d_I) < 0$ according to Corollary 2.11. Without loss of generality, letting $m(x) = -d_I \int_\Omega J(x-y)dy + \beta(x) - \gamma(x)$, we can find a function sequence $\{m_n\}_{n=1}^{\infty}$ such that $\|m_n - m\|_{L^\infty(\Omega)} \to 0$ as $n \to +\infty$ and the eigenvalue problem

$$d_I \int\limits_{\Omega} J(x-y)\varphi_n(y)dy + m_n(x)\varphi_n(x) = -\lambda\varphi_n(x) \quad \text{in } \Omega$$

admits a principal eigenpair $(\lambda_p^n(d_I), \varphi_n(x))$. Furthermore, taking n large enough, provided $n \geq n_0$, we have

$$\lambda_p^n(d_I) \leq \frac{1}{2}\lambda_p(d_I) - \|m_n - m\|_{L^\infty(\Omega)}.$$

Now, constructing $\underline{I}(x) = \delta\varphi_n(x)$ for some $\delta > 0$ and a direct computation yields that

$$\begin{aligned} & d_I \int\limits_{\Omega} J(x-y)(\underline{I}(y) - \underline{I}(x))dx + (\beta(x) - \gamma(x))\underline{I} - \frac{d_S\beta(x)\underline{I}^2}{d_S\underline{I} + d_I(1-\underline{I})} \\ & = -\delta\lambda_p^n(d_I)\varphi_n(x) + \delta\varphi_n(x)(m(x) - m_n(x)) - \frac{d_S\beta(x)\delta^2\varphi_n^2(x)}{d_S\delta\varphi_n(x) + d_I(1-\delta\varphi_n(x))} \\ & \geq -\frac{1}{2}\lambda_p(d_I)\delta\varphi_n(x) - \frac{d_S\beta(x)\delta^2\varphi_n^2(x)}{d_S\delta\varphi_n(x) + d_I(1-\delta\varphi_n(x))} \\ & \geq 0, \end{aligned}$$

provided δ small enough. Denote $\overline{I}(x) = 1$. Then, it is easy to verify that

$$d_I \int\limits_{\Omega} J(x-y)(\overline{I}(y) - \overline{I}(x))dx + (\beta(x) - \gamma(x))\overline{I} - \frac{d_S\beta(x)\overline{I}^2}{d_S\overline{I} + d_I(1-\overline{I})} = -\gamma(x) < 0,$$

which implies that $\overline{I}$ is a super solution. We can take $\delta > 0$ sufficiently small such that $\underline{I} \leq \overline{I}$ on $\bar{\Omega}$. By the standard iteration method [12,21,54], there exists some $I(\cdot) \in L^2(\Omega)$ satisfying (3.10) and $0 < I(x) \leq 1$.

Now, we prove $I(x)$ is continuous on $\bar{\Omega}$. Denote

$$\Phi(x, I) = \beta(x) - \gamma(x) - \frac{d_S \beta(x) I}{d_S I + d_I(1-I)} \quad \text{and} \quad \Theta(x, I) = \Phi(x, I) I.$$

By a direct computation, there is $\partial_I \Phi(x, s) < 0$ for all $s > 0$. For any $x_1, x_2 \in \bar{\Omega}$, we find that

$$\begin{aligned} & d_I \int_\Omega (J(x_1 - y) - J(x_2 - y)) I(y) dy - d_I I(x_1) \int_\Omega (J(x_1 - y) - J(x_2 - y)) dy \\ & + [\Theta(x_1, I(x_1)) - \Theta(x_2, I(x_1))] \\ = & - \left[-d_I \int_\Omega J(x_2 - y) dy + \partial_I \Theta(x_2, \tau I(x_1) + (1-\tau) I(x_2)) \right] (I(x_1) - I(x_2)), \end{aligned} \tag{3.11}$$

in which $0 \le \tau \le 1$. Assume $I(x_1) \ge I(x_2)$ without loss of generality. Since $\partial_I \Phi(x, s) < 0$ for all $s > 0$, one can get

$$\partial_I \Theta(x, s) = \Phi(x, s) + \partial_I \Phi(x, s) s < \Phi(x, s)$$

for all $s > 0$. Thus,

$$\partial_I \Theta(x_2, \tau I(x_1) + (1-\tau) I(x_2)) \le \Phi(x_2, \tau I(x_1) + (1-\tau) I(x_2)) < \Phi(x_2, I(x_2)). \tag{3.12}$$

Note that $I(x_2)$ satisfies

$$d_I \int_\Omega J(x_2 - y) I(y) dy + \left[-d_I \int_\Omega J(x_2 - y) dy + \Phi(x_2, I(x_2)) \right] I(x_2) = 0.$$

Since $I > 0$, there exists some $\delta > 0$ such that

$$-d_I \int_\Omega J(x_2 - y) dy + \Phi(x_2, I(x_2)) < -\delta. \tag{3.13}$$

Hence, it follows from (3.12) and (3.13) that

$$-d_I \int_\Omega J(x_2 - y) dy + \partial_I \Theta(x_2, \tau I(x_1) + (1-\tau) I(x_2) < -\delta. \tag{3.14}$$

Therefore, applying (3.11) and (3.14) yields that $I(x)$ is continuous on $\bar{\Omega}$.

We claim that $I(x) \neq 1$ for all $x \in \bar{\Omega}$. On the contrary, assume that there is $x_0 \in Int(\Omega)$ such that $I(x_0) = 1$. Thus, (3.10) yields that $\gamma(x_0) = d_I \int_\Omega J(x_0 - y)(I(y) - I(x_0)) dy \le 0$, which is a contradiction. On the other hand, if $x_0 \in \partial\Omega$, we can find a point sequence $\{x_n\} \subset \Omega$ such that $x_n \to x_0$ and $I(x_n) = 1$, $I(x_n) \to I(x_0)$ as $n \to +\infty$. The same arguments can lead to a contradiction.

Next we prove the uniqueness of positive solutions of (3.3). Assume $I_1(x)$ is another solution of (3.3) and $I(x) \le I_1(x) \le 1$ on $\bar{\Omega}$ without loss of generality. The other case can be obtained by the same method. Define

$$\tau^* = \inf\{\tau > 0 |\ I(x) \ge \tau I_1(x),\ x \in \bar{\Omega}\}.$$

By the boundedness of $I(x)$ and $I_1(x)$, τ^* is well defined. We claim that $\tau^* \ge 1$. On the contrary, assume $\tau^* < 1$. The direct calculation yields that

$$\begin{aligned} & d_I \int_\Omega J(x-y)(\tau^* I_1(y) - \tau^* I_1(x))dy + (\beta(x) - \gamma(x))\tau^* I_1 - \frac{d_S \beta(x) \tau^{*2} I_1^2}{d_S \tau^* I_1 + d_I(1-\tau^* I_1)} \\ & = \tau^* \beta(x) \left(\frac{d_S I_1}{d_S I_1 + d_I(1-I_1)} - \frac{d_S \tau^* I_1}{d_S \tau^* I_1 + d_I(1-\tau^* I_1)} \right) I_1 > 0. \end{aligned} \tag{3.15}$$

By the definition of τ^*, there is some $x_0 \in \Omega$ such that $I(x_0) = \tau^* I_1(x_0)$. Thus, we have

$$\begin{aligned} & d_I \int_\Omega J(x_0-y)\tau^* I_1(y)dy - d_I \int_\Omega J(x_0-y)dy\tau^* I_1(x_0) + (\beta(x_0) - \gamma(x_0))\tau^* I_1(x_0) \\ & - \beta(x_0) \frac{d_S \tau^{*2} I_1^2(x_0)}{d_S \tau^* I_1(x_0) + d_I(1-\tau^* I_1(x_0))} \\ & = d_I \int_\Omega J(x_0-y)(\tau^* I_1(y) - I(y))dy \le 0. \end{aligned} \tag{3.16}$$

Let $\omega(y) = \tau^* I_1(y) - I(y)$ for $y \in \Omega$. Combining (3.15) and (3.16), we have $d_I \int_\Omega J(x_0 - y)\omega(y)dy = 0$. Thus, this implies that $\omega(y) = 0$ almost everywhere in Ω. That is $I(x) = \tau^* I_1(x)$ almost everywhere in Ω. Hence,

$$\begin{aligned} 0 &= d_I \int_\Omega J(x-y)(I(y) - I(x))dy + \beta(x)\left(1 - \frac{d_S I(x)}{d_S I(x) + d_I(1-I(x))}\right) I(x) - \gamma(x) I(x) \\ &= \tau^* \left[d_I \int_\Omega J(x-y)(I_1(y) - I_1(x))dy + (\beta(x) - \gamma(x)) I_1(x) - \frac{d_S \beta(x) I_1^2(x)}{d_S I_1(x) + d_I(1-I_1(x))} \right] \\ & \quad + \tau^* \beta(x) \left(\frac{d_S I_1(x)}{d_S I_1(x) + d_I(1-I_1(x))} - \frac{d_S \tau^* I_1(x)}{d_S \tau^* I_1(x) + d_I(1-\tau^* I_1(x))} \right) I_1(x) \\ &= \tau^* \beta(x) \left(\frac{d_S I_1(x)}{d_S I_1(x) + d_I(1-I_1(x))} - \frac{d_S \tau^* I_1(x)}{d_S \tau^* I_1(x) + d_I(1-\tau^* I_1(x))} \right) I_1(x) > 0, \end{aligned}$$

which is a contradiction. Thus, $\tau^* \ge 1$ and this implies that $I(x) = I_1(x)$. The uniqueness of positive solutions of (3.3) is obtained.

Note that $S(\cdot) = \frac{1-I(\cdot)}{d_S} \in C(\bar{\Omega})$. Meanwhile, (3.3) admits a unique solution pair $(\tilde{S}, \tilde{I})$ and $\tilde{S}(\cdot) \in C(\bar{\Omega})$, $\tilde{I}(\cdot) \in C(\bar{\Omega})$. Additionally, there are $0 < \tilde{I}(x) < \frac{k}{d_I}$, $0 < \tilde{S}(x) < \frac{k}{d_S}$ for some positive constant k dependent on d_S and d_I. The proof is complete. $\square$

Finally we discuss the stability of stationary solution in the sense of Definition 3.2. First, we introduce a nonlocal dispersal problem as

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = d\int_\Omega J(x-y)(u(y,t)-u(x,t))dy + (r(x)-c(x)u)u, & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{3.17}$$

where $d > 0$ is a positive constant and $u_0(x)$ is a bounded continuous function.

Lemma 3.9. *Assume $r(\cdot), c(\cdot) \in C(\bar{\Omega})$ and $c(x) > 0$ on $\bar{\Omega}$. Then the positive stationary solution u_* of (3.17) is unique if and only if $\lambda_p(d) < 0$, in which*

$$\lambda_p(d) = \inf_{\varphi \in L^2(\Omega), \varphi \not\equiv 0} \frac{\frac{d}{2}\int_\Omega\int_\Omega J(x-y)(\varphi(y)-\varphi(x))^2 dydx - \int_\Omega r(x)\varphi^2(x)dx}{\int_\Omega \varphi^2(x)dx}.$$

Moreover, u_* is globally asymptotically stable.

We can see the proof of Lemma 3.9 in [43,44]. Here, we omit it.

Theorem 3.10. *Suppose $d_S = d_I = d$. The following alternatives hold:*

(i) *If $R_0 < 1$, then all the positive solutions of (1.1) converge to the disease-free equilibrium $(\frac{N}{|\Omega|}, 0)$ as $t \to +\infty$ in $X \times X$.*
(ii) *If $R_0 > 1$, then all the positive solutions of (1.1) converge to $(\tilde{S}(x), \tilde{I}(x))$ as $t \to +\infty$ in $X \times X$.*

Proof. Note that (i) is contained in Theorem 3.4. Thus, we only need to prove (ii). Let $v(x,t) = S_*(x,t) + I_*(x,t)$. Then, it follows from (1.1) that

$$\begin{cases} \frac{\partial v(x,t)}{\partial t} = d\int_\Omega J(x-y)(v(y,t)-v(x,t))dy, & x \in \Omega, t > 0, \\ \int_\Omega v(x,t)dx = N, & t > 0, \\ v(x,0) \geq 0, & x \in \Omega. \end{cases} \tag{3.18}$$

Obviously, $\frac{N}{|\Omega|}$ is the constant stationary solution of (3.18). Define

$$\lambda_0 = \inf_{\psi \in L^2(\Omega), \int_\Omega \psi(x)dx = 0, \psi \not\equiv 0} \frac{\frac{d}{2}\int_\Omega\int_\Omega J(x-y)(\psi(y)-\psi(x))^2 dydx}{\int_\Omega \psi^2(x)dx}.$$

By the same discussion as Theorem 3.6 in [4], we get

$$\left\| v(\cdot,t) - \frac{N}{|\Omega|} \right\|_{L^\infty(\Omega)} \leq \tilde{C}e^{-\lambda_0 t}$$

for some positive constant $\tilde{C}$. Thus, $v(x,t) \to \frac{N}{|\Omega|}$ uniformly on $\bar{\Omega}$ as $t \to +\infty$ due to the fact that $v(\cdot,t) \in C(\bar{\Omega})$. Note that $I_*(x,t)$ satisfies

$$\begin{cases} \frac{\partial I_*}{\partial t} = d\int_\Omega J(x-y)(I_*(y,t) - I_*(x,t))dy + (\beta(x) - \gamma(x))I_* - \frac{\beta(x)}{v} I_*^2, & x \in \Omega, t > 0, \\ \int_\Omega I_*(x,0)dx > 0. \end{cases} \tag{3.19}$$

Since $v(x,t) \to \frac{N}{|\Omega|}$ uniformly on $\bar{\Omega}$ as $t \to +\infty$, for any small $\varepsilon > 0$, we can find a large $T > 0$ such that

$$\frac{N}{|\Omega|} - \varepsilon \le v(x,t) \le \frac{N}{|\Omega|} + \varepsilon \ \text{ for all } x \in \bar{\Omega} \text{ and } t \ge T.$$

Inspired by the idea in Peng and Yi [35], we consider the following two auxiliary problems:

$$\begin{cases} \frac{\partial \overline{I}}{\partial t} = d\int_\Omega J(x-y)(\overline{I}(y,t) - \overline{I}(x,t))dy + (\beta(x) - \gamma(x))\overline{I} - \frac{\beta(x)}{\frac{N}{|\Omega|}+\varepsilon} \overline{I}^2, & x \in \Omega, t > 0, \\ \overline{I}(x,T) = I_*(x,T) > 0, & x \in \Omega \end{cases} \tag{3.20}$$

and

$$\begin{cases} \frac{\partial \underline{I}}{\partial t} = d\int_\Omega J(x-y)(\underline{I}(y,t) - \underline{I}(x,t))dy + (\beta(x) - \gamma(x))\underline{I} - \frac{\beta(x)}{\frac{N}{|\Omega|}-\varepsilon} \underline{I}^2, & x \in \Omega, t > 0, \\ \underline{I}(x,T) = I_*(x,T) > 0, & x \in \Omega. \end{cases} \tag{3.21}$$

The comparison principle implies that $\overline{I}(x,t)$ and $\underline{I}(x,t)$ are respectively the upper and lower solutions of (3.19). Thus, we get

$$\underline{I}(x,t) \le I_*(x,t) \le \overline{I}(x,t) \ \text{ for all } x \in \bar{\Omega} \text{ and } t \ge T.$$

Since $R_0 > 1$, we have $\lambda_p(d_I) < 0$. According to Lemma 3.9, there are two positive functions $\overline{I}_\varepsilon(\cdot)$ and $\underline{I}_\varepsilon(\cdot) \in C(\bar{\Omega})$ such that

$$\overline{I}(x,t) \to \overline{I}_\varepsilon(x) \text{ and } \underline{I}(x,t) \to \underline{I}_\varepsilon(x) \text{ uniformly on } \bar{\Omega} \text{ as } t \to +\infty,$$

and $\overline{I}_\varepsilon(x)$, $\underline{I}_\varepsilon(x)$ are respectively the unique steady states of (3.20) and (3.21). That is, $\overline{I}_\varepsilon(x)$ and $\underline{I}_\varepsilon(x)$ satisfy

$$d\int\limits_\Omega J(x-y)(\overline{I}_\varepsilon(y) - \overline{I}_\varepsilon(x))dy + (\beta(x) - \gamma(x))\overline{I}_\varepsilon - \frac{\beta(x)}{\frac{N}{|\Omega|}+\varepsilon}\overline{I}_\varepsilon^2 = 0, \ x \in \Omega$$

and

$$d\int_{\Omega} J(x-y)(\underline{I}_{\varepsilon}(y)-\underline{I}_{\varepsilon}(x))dy+(\beta(x)-\gamma(x))\underline{I}_{\varepsilon}-\frac{\beta(x)}{\frac{N}{|\Omega|}-\varepsilon}\underline{I}_{\varepsilon}^{2}=0,\ x\in\Omega, \tag{3.22}$$

respectively. By the same arguments as in [44], we know there exists some constant M independent of ε such that $\underline{I}_{\varepsilon}(x)\le M$ and $\overline{I}_{\varepsilon}(x)\le M$ for all $x\in\Omega$. Additionally, $\overline{I}_{\varepsilon}(x)$ and $\underline{I}_{\varepsilon}(x)$ are monotone with respect to ε. In fact, assume $\varepsilon_1<\varepsilon_2$, $\underline{I}_{\varepsilon_1}(x)$ and $\underline{I}_{\varepsilon_2}(x)$ are respectively the solutions of (3.22) as $\varepsilon=\varepsilon_1$ and $\varepsilon=\varepsilon_2$. The direct computation yields that

$$\begin{aligned}&d\int_{\Omega} J(x-y)(\underline{I}_{\varepsilon_1}(y)-\underline{I}_{\varepsilon_1}(x))dy+(\beta(x)-\gamma(x))\underline{I}_{\varepsilon_1}-\frac{\beta(x)}{\frac{N}{|\Omega|}-\varepsilon_2}\underline{I}_{\varepsilon_1}^{2}\\ =&\frac{\beta(x)}{\frac{N}{|\Omega|}-\varepsilon_1}\underline{I}_{\varepsilon_1}^{2}-\frac{\beta(x)}{\frac{N}{|\Omega|}-\varepsilon_2}\underline{I}_{\varepsilon_1}^{2}<0.\end{aligned}$$

By the uniqueness of positive solution of (3.22), we get $\underline{I}_{\varepsilon_2}(x)<\underline{I}_{\varepsilon_1}(x)$ for $x\in\Omega$. Meanwhile, the same arguments lead us to obtain that $\overline{I}_{\varepsilon}(x)$ is strictly increasing on ε. Now, there exists a sequence $\{\varepsilon_n\}$ with $\varepsilon_n\to 0$ as $n\to+\infty$ such that

$$\underline{I}_{\varepsilon_n}(x)\to I_1(x)\ \text{ as } n\to+\infty \text{ uniformly on } \bar{\Omega}$$

and

$$\overline{I}_{\varepsilon_n}(x)\to I_2(x)\ \text{ as } n\to+\infty \text{ uniformly on } \bar{\Omega}$$

for some positive continuous functions $I_1(x)$ and $I_2(x)$. Note that $I_1(x)$ and $I_2(x)$ satisfy the following equation

$$d\int_{\Omega} J(x-y)(u(y)-u(x))dy+(\beta(x)-\gamma(x))u(x)-\frac{\beta(x)}{\frac{N}{|\Omega|}}u^2(x)=0\ \text{ in }\Omega. \tag{3.23}$$

Then, following Lemma 3.9, we know $I_1(x)=I_2(x)$ due to the uniqueness of positive solutions of (3.23). Thus, we get that $I_*(x,t)\to I_1(x)$ uniformly on $\bar{\Omega}$ as $t\to+\infty$ and $S_*(x,t)\to\frac{N}{|\Omega|}-I_1(x)$ as $t\to+\infty$. By the uniqueness of positive solutions of (3.3), we have $\tilde{S}(x)=\frac{N}{|\Omega|}-I_1(x)$ and $\tilde{I}(x)=I_1(x)$. This completes the proof. □

Theorem 3.11. *Assume $\beta(x)=r\gamma(x)$ on $\bar{\Omega}$ for some positive constant $r\in(0,+\infty)$. If $r\le 1$, then the disease-free equilibrium is globally asymptotically stable.*

Proof. If $r<1$, then we can get $\lambda_p(d_I)>0$ by the definition of $\lambda_p(d_I)$. In this case, the result is obtained in Theorem 3.4. Thus, we only need discuss the case $r=1$, that is $\beta(x)=\gamma(x)$. In this case, $\lambda_p(d_I)=0$, see [20]. Consequently, system (1.1) is equivalent to

$$\begin{cases} \frac{\partial S(x,t)}{\partial t} = d_S \int_\Omega J(x-y)(S(y,t)-S(x,t))dy + \frac{\beta(x)I^2(x,t)}{S(x,t)+I(x,t)} & \text{in } \Omega \times (0,+\infty), \\ \frac{\partial I(x,t)}{\partial t} = d_I \int_\Omega J(x-y)(I(y,t)-I(x,t))dy - \frac{\beta(x)I^2(x,t)}{S(x,t)+I(x,t)} & \text{in } \Omega \times (0,+\infty), \\ \int_\Omega (S(x,t)+I(x,t))dx = N & \text{in } (0,+\infty), \\ S(x,0)=S_0(x) \geq 0,\ I(x,0)=I_0(x) \geq 0 & \text{in } \Omega. \end{cases} \tag{3.24}$$

Firstly, we claim that

$$\|S_*(\cdot,t)\|_{L^\infty(\Omega)} \leq C_0 \text{ and } \|I_*(\cdot,t)\|_{L^\infty(\Omega)} \leq C_0 \tag{3.25}$$

for some positive constant C_0 independent on $t \geq 0$. Indeed, applying the second equation of (3.24) yields that

$$\frac{\partial I_*(x,t)}{\partial t} \leq d_I \int_\Omega J(x-y)(I_*(y,t)-I_*(x,t))dy.$$

Hence, we consider the following problem

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = d_I \int_\Omega J(x-y)(u(y,t)-u(x,t))dy, & x \in \Omega, \\ u(x,0) = \max_{\bar\Omega} I_0(x), & x \in \Omega. \end{cases}$$

Thus, by the comparison principle, there is $I_*(x,t) \leq u(x,t) \leq \max_{\bar\Omega} I_0(x)$ for all $x \in \Omega$ and $t \geq 0$. On the other hand, for the first equation of (3.24), one can get that

$$\begin{aligned} \frac{\partial S_*(x,t)}{\partial t} &= -d_S \int_\Omega J(x-y)dy S_*(x,t) + d_S \int_\Omega J(x-y)S_*(y,t)dy + \frac{\beta(x)I_*^2(x,t)}{S_*(x,t)+I_*(x,t)} \\ &=: -a(x)S_*(x,t) + W(S_*,I_*), \end{aligned}$$

where $a(x) = d_S \int_\Omega J(x-y)dy$ and

$$W(S_*,I_*) = d_S \int_\Omega J(x-y)S_*(y,t)dy + \frac{\beta(x)I_*^2(x,t)}{S_*(x,t)+I_*(x,t)}.$$

Consequently, we have

$$S_*(x,t) = S_0(x)e^{-a(x)t} + e^{-a(x)t}\int_0^t W(S_*,I_*)(x,s)e^{a(x)s}ds.$$

Since $a(x) \geq \alpha$ (α is defined as (3.5)) and

$$\begin{aligned} W(S_*,I_*)(x,s) \leq& d_S\|J\|_{L^\infty(\Omega)} \int_\Omega S_*(y,t)dy + \beta(x)I_*(x,t) \\ \leq& d_S N\|J\|_{L^\infty(\Omega)} + \max_{\bar\Omega}\{\beta(x)I_0(x)\}, \end{aligned}$$

for any $x \in \Omega$ and $t \geq 0$, we have

$$S_*(x,t) \leq \max_{\bar{\Omega}} S_0(x) + \frac{d_S N \|J\|_{L^\infty(\Omega)} + \max_{\bar{\Omega}}\{\beta(x) I_0(x)\}}{\alpha}.$$

Now, due to the fact (3.25), by the same method in [35], we can obtain that

$$I_*(x,t) \to 0 \ \text{ uniformly on } \bar{\Omega} \text{ as } t \to +\infty.$$

So, $\int_\Omega S_*(x,t)dx \to N$ as $t \to +\infty$. Let

$$S_*(x,t) = S_1(x,t) + \frac{1}{|\Omega|} \int_\Omega S_*(x,t)dx.$$

The direct computation yields that $S_1(x,t)$ satisfies

$$\frac{\partial S_1}{\partial t} = d_S \int_\Omega J(x-y)(S_1(y,t) - S_1(x,t))dy + f(x,t) \tag{3.26}$$

for $x \in \Omega$, $t > 0$ and $\int_\Omega S_1(x,t)dx = 0$, in which

$$f(x,t) = \frac{\beta(x) I_*^2(x,t)}{S_*(x,t) + I_*(x,t)} - \frac{1}{|\Omega|} \int_\Omega \frac{\beta(x) I_*^2(x,t)}{S_*(x,t) + I_*(x,t)} dx.$$

Obviously, we have $\lim\limits_{t \to +\infty} f(x,t) = 0$. Note that

$$\int_\Omega S_1(x,t) f(x,t) dx = \int_\Omega \frac{\beta(x) S_1(x,t) I_*^2(x,t)}{S_*(x,t) + I_*(x,t)} dx := g(t).$$

Hence, there is

$$|g(t)| \leq \int_\Omega \left| \frac{\beta(x) S_1(x,t) I_*^2(x,t)}{S_*(x,t) + I_*(x,t)} \right| dx \leq C \int_\Omega |S_1(x,t)| I_*(x,t) dx \leq \tilde{C} \int_\Omega I_*(x,t) dx$$

for some positive constant $\tilde{C}$. Thus, we have $\lim\limits_{t \to +\infty} g(t) = 0$. Let α be defined as (3.5) and $h(x) = d_S \int_\Omega J(x-y)dy$. Then, there is $0 < \alpha \leq \min\limits_{\bar{\Omega}} h(x)$. Define $U(t) = \int_\Omega S_1^2(x,t)dx$. By direct calculation, we get

$$\frac{dU(t)}{dt} = 2 \int_\Omega S_1(x,t) \frac{\partial S_1(x,t)}{\partial t} dx$$

$$
\begin{aligned}
&= 2\int_{\Omega} S_1(x,t)\left[d_S\int_{\Omega} J(x-y)(S_1(y,t)-S_1(x,t))dy + f(x,t)\right]dx \\
&= -d_S\int_{\Omega}\int_{\Omega} J(x-y)(S_1(y,t)-S_1(x,t))^2 dydx + 2g(t) \\
&\le -2\alpha U(t) + 2g(t).
\end{aligned}
$$

Thus, we have

$$
U(t) \le U(0)e^{-2\alpha t} + 2e^{-2\alpha t}\int_0^t e^{2\alpha s} g(s)ds.
$$

That is

$$
\|S_1(\cdot,t)\|_{L^2(\Omega)} \le \left(U(0)e^{-2\alpha t} + 2e^{-2\alpha t}\int_0^t e^{2\alpha s} g(s)ds\right)^{\frac{1}{2}}.
$$

This implies that $\lim_{t\to+\infty}\|S_1(\cdot,t)\|_{L^2(\Omega)} = 0$. On the other hand, following (3.26), there is

$$
S_1(x,t) = e^{-h(x)t}S_1(x,0) + e^{-h(x)t}\int_0^t e^{h(x)s}\left[d_S\int_{\Omega} J(x-y)S_1(y,s)dy + f(x,s)\right]ds.
$$

Note that

$$
\lim_{t\to+\infty} e^{-h(x)t}\int_0^t e^{h(x)s} f(x,s)ds = \lim_{t\to+\infty}\frac{f(x,t)}{h(x)} = 0
$$

and

$$
e^{-h(x)t}\int_0^t e^{h(x)s}\int_{\Omega} J(x-y)|S_1(y,s)|dyds \le Ce^{-h(x)t}\int_0^t e^{h(x)s}\|S_1(\cdot,s)\|_{L^2(\Omega)}ds
$$

for some positive constant C. Thus, we get $\lim_{t\to+\infty}|S_1(x,t)| = 0$ for all $x \in \bar{\Omega}$. This implies that $\lim_{t\to+\infty} S_*(x,t) = \frac{N}{|\Omega|}$ uniformly on $\bar{\Omega}$ as $t \to +\infty$. On the other hand, noticed that system (3.24) is quasi-monotone increasing, then the same arguments in [35] can get that $(\frac{N}{|\Omega|}, 0)$ is asymptotically stable. This ends the proof. □

Remark 3.12. Note that when $d_S = d_I$, the epidemic disease will persist as $R_0 > 1$ and die out as $R_0 < 1$. When $d_S \ne d_I$, Theorem 3.4 gives that the epidemic disease will be extinct as $R_0 < 1$. The case $R_0 > 1$ ($d_S \ne d_I$) is very complicated, we will study it in a further work.

Remark 3.13. It is known from Theorem 3.4 that the epidemic disease always dies out as $R_0 < 1$. But for the case $R_0 = 1$, it is open. However, if the rate of the disease transmission is proportional to the rate of the disease recovery (i.e. $\beta(x) = r\gamma(x)$ for some positive constant r), then Theorem 3.11 implies that the epidemic disease will be completely extinct as $R_0 \le 1$ (i.e. $r \le 1$). For the case $R_0 > 1$ (i.e. $r > 1$), we conjecture that the epidemic disease will be persistence and leave it as an open problem.

4. The effect of the large diffusion rates

In this section, we discuss the effect of the large diffusion rate on the transmission of the disease. Throughout this section, we always assume that $\int_\Omega \beta(x)dx > \int_\Omega \gamma(x)dx$. Following Corollary 2.13, we know $R_0 > 1$ for all $d_I > 0$ in this condition. Then, the positive solution $(\tilde{S}, \tilde{I})$ of (3.3) exists.

Theorem 4.1. *If we let $d_S, d_I \to +\infty$, then*

$$(\tilde{S}, \tilde{I}) \to \left(\frac{N}{|\Omega|} \frac{\int_\Omega \gamma(x)dx}{\int_\Omega \beta(x)dx}, \frac{N}{|\Omega|} \left(1 - \frac{\int_\Omega \gamma(x)dx}{\int_\Omega \beta(x)dx} \right) \right).$$

Proof. Arguing as above, we know if $(\tilde{S}, \tilde{I})$ is the solution of (3.3), then $\tilde{S}, \tilde{I} \in C(\bar{\Omega})$. Since $\int_\Omega (\tilde{S}(x) + \tilde{I}(x))dx = N$, the continuity of $\tilde{S}, \tilde{I}$ gives that

$$\|\tilde{S}(\cdot)\|_{L^\infty(\Omega)} \le \tilde{M} \text{ and } \|\tilde{I}(\cdot)\|_{L^\infty(\Omega)} \le \tilde{M},$$

where $\tilde{M}$ is a positive constant independent of d_S and d_I.

Choosing sequences $\{d_{S,n}\}_{n=1}^\infty$ and $\{d_{I,n}\}_{n=1}^\infty$ with $d_{S,n} \to +\infty$ and $d_{I,n} \to +\infty$ as $n \to +\infty$. Meanwhile, the corresponding solution of (3.3) is $(\tilde{S}_n, \tilde{I}_n)$. Thus, there are subsequences still denoted by $\tilde{S}_n$ and $\tilde{I}_n$, and $\tilde{S}_*, \tilde{I}_*$ such that

$$\tilde{S}_n(x) \to \tilde{S}_*(x) \text{ and } \tilde{I}_n(x) \to \tilde{I}_*(x) \text{ weakly in } L^2(\Omega).$$

Note that

$$\left\| \frac{\beta(\cdot)\tilde{I}_n(\cdot)\tilde{S}_n(\cdot)}{\tilde{I}_n(\cdot) + \tilde{S}_n(\cdot)} - \gamma(\cdot)\tilde{I}_n(\cdot) \right\|_{L^\infty(\Omega)} \le C_*$$

for some positive constant C_* dependent only on β, γ and Ω. Let

$$g_n(x) = \frac{\beta(x)\tilde{I}_n(x)\tilde{S}_n(x)}{\tilde{I}_n(x) + \tilde{S}_n(x)} - \gamma(x)\tilde{I}_n(x).$$

Then, $\tilde{S}_n(x)$ and $\tilde{I}_n(x)$ satisfy

$$\tilde{S}_n(x) = \left[\int_\Omega J(x-y)dy \right]^{-1} \left[\int_\Omega J(x-y)\tilde{S}_n(y)dy - \frac{g_n(x)}{d_{S,n}} \right] \tag{4.1}$$

and

$$\tilde{I}_n(x) = \left[\int_\Omega J(x-y)dy \right]^{-1} \left[\int_\Omega J(x-y)\tilde{I}_n(y)dy + \frac{g_n(x)}{d_{I,n}} \right], \tag{4.2}$$

respectively. It is well-known that

$$\int_\Omega J(x-y)\tilde{S}_n(y)dy \to \int_\Omega J(x-y)\tilde{S}_*(y)dy, \quad \int_\Omega J(x-y)\tilde{I}_n(y)dy \to \int_\Omega J(x-y)\tilde{I}_*(y)dy$$

for all $x \in \Omega$ as $n \to +\infty$. Thus, following from (4.1) and (4.2), we have

$$\tilde{S}_n(x) \to \tilde{S}_*(x) \text{ and } \tilde{I}_n(x) \to \tilde{I}_*(x) \quad \text{in } C(\bar{\Omega}) \text{ as } n \to +\infty.$$

On the other hand, $\tilde{S}_n(x)$ and $\tilde{I}_n(x)$ satisfy

$$\int_\Omega J(x-y)(\tilde{S}_n(y) - \tilde{S}_n(x))dy = \frac{g_n(x)}{d_{S,n}}$$

and

$$\int_\Omega J(x-y)(\tilde{I}_n(y) - \tilde{I}_n(x))dy = -\frac{g_n(x)}{d_{I,n}}.$$

Thus, $\tilde{S}_*(x)$ and $\tilde{I}_*(x)$ satisfy

$$\int_\Omega J(x-y)(\tilde{S}_*(y) - \tilde{S}_*(x))dy = 0 \text{ and } \int_\Omega J(x-y)(\tilde{I}_*(y) - \tilde{I}_*(x))dy = 0$$

for $x \in \Omega$, respectively. This implies that $\tilde{S}_*(x)$ and $\tilde{I}_*(x)$ are all constants, still denoted by $\tilde{S}_*$ and $\tilde{I}_*$ for the convenience.

Below, we need to show that $\tilde{S}_*$ and $\tilde{I}_*$ are all positive.

Case I: Assume $\tilde{I}_* = 0$, $\tilde{S}_* > 0$. Let $\hat{I}_n(x) = \frac{\tilde{I}_n(x)}{\|\tilde{I}_n(\cdot)\|_{L^\infty(\Omega)}}$. Thus, $\hat{I}_n(x)$ satisfies

$$d_{I,n} \int_\Omega J(x-y)(\hat{I}_n(y) - \hat{I}_n(x))dy + \frac{\beta(x)\tilde{S}_n(x)\hat{I}_n(x)}{\tilde{S}_n(x) + \tilde{I}_n(x)} - \gamma(x)\hat{I}_n(x) = 0 \text{ in } \Omega. \tag{4.3}$$

The same arguments as above yield that $\hat{I}_n(x) \to 1$ as $n \to +\infty$ for all $x \in \Omega$. Integrating both sides of (4.3) over Ω and letting $n \to +\infty$, we have $\int_\Omega \beta(x)dx = \int_\Omega \gamma(x)dx$, which is a contradiction.

Case II: Assume $\tilde{I}_* > 0$, $\tilde{S}_* = 0$. Integrating (4.3) on Ω and letting $n \to +\infty$, we have a contradiction with $-\int_\Omega \gamma(x)dx = 0$.

Case III: Assume $\tilde{I}_* = 0$, $\tilde{S}_* = 0$. This is impossible because of $\int_\Omega (\tilde{S}_n(x) + \tilde{I}_n(x))dx = N$. Thus, we get $\tilde{S}_* > 0$ and $\tilde{I}_* > 0$. Meanwhile, we know

$$\int_\Omega \frac{\beta(x)\tilde{S}_*\tilde{I}_*}{\tilde{S}_* + \tilde{I}_*} dx = \int_\Omega \gamma(x)\tilde{I}_* dx \text{ and } \tilde{S}_* + \tilde{I}_* = \frac{N}{|\Omega|}.$$

Hence, the direct computation gives that

$$\tilde{S}_* = \frac{N\int_\Omega \gamma(x)dx}{|\Omega|\int_\Omega \beta(x)dx}, \quad \tilde{I}_* = \frac{N}{|\Omega|}\left(1 - \frac{\int_\Omega \gamma(x)dx}{\int_\Omega \beta(x)dx}\right).$$

This completes the proof. □

Theorem 4.2. *If $d_S \to +\infty$, then*

$$(\tilde{S}(x), \tilde{I}(x)) \to \left(\frac{d_I N}{\int_\Omega (d_I + \theta_*(x))dx}, \frac{N\theta_*(x)}{\int_\Omega (d_I + \theta_*(x))dx}\right),$$

where $\theta_*(x)$ is the unique positive solution of the following problem

$$d_I \int_\Omega J(x-y)(u(y) - u(x))dy + (\beta(x) - \gamma(x))u - \frac{\beta(x)u^2}{d_I + u} = 0 \ \text{ in } \Omega. \tag{4.4}$$

Proof. Inspired by the method in [34], let $\theta(x) = d_S I(x)$. Then, according to (3.10), we have

$$d_I \int_\Omega J(x-y)(\theta(y) - \theta(x))dy + (\beta(x) - \gamma(x))\theta - \frac{\beta(x)\theta^2}{\theta + d_I(1 - d_S^{-1}\theta)} = 0 \ \text{ in } \Omega. \tag{4.5}$$

Note that the positive solution $\theta(x)$ of (4.5) is monotone increasing on d_S. Indeed, for any $d_{S_1} < d_{S_2}$, letting $\theta_1(x)$ and $\theta_2(x)$ be solutions of (4.5) corresponding to $d_S = d_{S_1}$ and $d_S = d_{S_2}$ respectively, then there is

$$\begin{aligned} & d_I \int_\Omega J(x-y)(\theta_1(y) - \theta_1(x))dy + (\beta(x) - \gamma(x))\theta_1 - \frac{\beta(x)\theta_1^2}{\theta_1 + d_I(1 - d_{S_2}^{-1}\theta_1)} \\ = & \frac{\beta(x)\theta_1^2}{\theta_1 + d_I(1 - d_{S_1}^{-1}\theta_1)} - \frac{\beta(x)\theta_1^2}{\theta_1 + d_I(1 - d_{S_2}^{-1}\theta_1)} > 0. \end{aligned}$$

This gives that $\theta_1(x)$ is a subsolution of (4.5) with $d_S = d_{S_2}$. Thus, according to the super-sub solutions method ([21]) and the uniqueness of solution of (4.5), we have $\theta_1(x) < \theta_2(x)$ for all $x \in \Omega$. Since $\theta(\cdot) \in C(\bar{\Omega})$, there exists some $x_0 \in \bar{\Omega}$ such that $\theta(x_0) = \max_{\bar{\Omega}} \theta(x)$. Then, it follows from (4.5) that

$$(\beta(x_0)-\gamma(x_0))\theta(x_0)-\frac{\beta(x_0)\theta^2(x_0)}{\theta(x_0)+d_I(1-d_S^{-1}\theta(x_0))}\geq 0.$$

That is

$$\begin{aligned}\theta(x_0)&\leq\frac{d_I(\beta(x_0)-\gamma(x_0))}{\gamma(x_0)}(1-d_S^{-1}\theta(x_0))\\&\leq\frac{d_I|\beta(x_0)-\gamma(x_0)|}{\gamma(x_0)}.\end{aligned}$$

Thus, we have

$$\theta(x)\leq d_I\max_{\bar\Omega}\left(\frac{|\beta(x)-\gamma(x)|}{\gamma(x)}\right).$$

Since $\theta(x)$ is monotone increasing and uniformly bounded on d_S, there exists some sequences $\{d_{S,n}\}_{n=1}^{\infty}$ satisfying $d_{S,n}\to+\infty$ as $n\to+\infty$ such that $\theta_n(x)=d_{S,n}I_n(x)\to\theta_*(x)$ in $C(\bar\Omega)$ for some nonnegative function $\theta_*(x)$ as $n\to+\infty$, where $\theta_n(x)$ is the solution of (4.5) with $d_S=d_{S,n}$. Thus, $\theta_*(x)$ is the unique positive solution of (4.4). We claim that $\theta_*(x)\neq 0$. On the contrary, assume that $\theta_*(x)=0$. Let

$$\hat\theta_n(x)=\frac{\theta_n(x)}{\|\theta_n\|_{L^\infty(\Omega)}}.$$

Then, $\hat\theta_n(x)$ satisfies

$$d_I\int_\Omega J(x-y)(\hat\theta_n(y)-\hat\theta_n(x))dy+(\beta(x)-\gamma(x))\hat\theta_n-\frac{\beta(x)\hat\theta_n\theta_n}{\theta_n+d_I(1-d_{S,n}^{-1}\theta_n)}=0\ \text{ in }\Omega.$$

Note that there is some $\hat\theta(x)>0$ such that $\hat\theta_n(x)\to\hat\theta(x)$ as $n\to+\infty$ and $\hat\theta$ satisfies

$$d_I\int_\Omega J(x-y)(\hat\theta(y)-\hat\theta(x))dy+(\beta(x)-\gamma(x))\hat\theta(x)=0\quad\text{in }\Omega.$$

It follows from Lemma 2.5 that $\lambda_p(d_I)=0$. This is a contradiction according to the discussion in Section 3.

On the other hand, we know $\theta_n(x)=d_{S,n}I_n(x)$. Thus, there holds $I_n(x)=\frac{\theta_n(x)}{d_{S,n}}\to 0$ as $n\to+\infty$. Due to $d_{S,n}S_n(x)=1-I_n(x)$, we have $d_{S,n}S_n(x)\to 1$ as $n\to+\infty$. Hence, applying (3.10) yields that

$$\tilde S_n(x)=kS_n(x)=\frac{d_INS_n(x)}{\int_\Omega(d_IS_n(x)+I_n(x))dx}=\frac{d_INd_{S,n}S_n(x)}{\int_\Omega(d_Id_{S,n}S_n(x)+d_{S,n}I_n(x))dx}$$

and

$$\tilde I_n(x)=\frac{k}{d_I}I_n(x)=\frac{NI_n(x)}{\int_\Omega(d_IS_n(x)+I_n(x))dx}=\frac{Nd_{S,n}(x)I_n(x)}{\int_\Omega(d_Id_{S,n}S_n(x)+d_{S,n}I_n(x))}.$$

Consequently, we obtain that

$$\tilde{S}_n(x) \to \frac{d_I N}{\int_\Omega (d_I + \theta_*(x))dx} \text{ as } n \to +\infty$$

and

$$\tilde{I}_n(x) \to \frac{N\theta^*(x)}{\int_\Omega (d_I + \theta^*(x))dx} \text{ as } n \to +\infty.$$

This ends the proof. $\square$

Theorem 4.3. *If $d_I \to +\infty$, then*

$$(\tilde{S}(x), \tilde{I}(x)) \to (S^*(x), I^*) \quad in\ C(\bar{\Omega}),$$

where $S^*(x)$ is a positive function and I^* is a positive constant. Moreover, $(S^*(x), I^*)$ satisfies

$$\begin{cases} d_S \int_\Omega J(x-y)(S^*(y) - S^*(x))dy + \gamma(x)I^* - \frac{\beta(x)S^*(x)I^*}{S^*(x)+I^*} = 0, & x \in \Omega, \\ \int_\Omega (S^*(x) + I^*)dx = N. \end{cases} \tag{4.6}$$

Proof. Choose some sequence $\{d_{I,n}\}_{n=1}^\infty$ satisfying $d_{I,n} \to +\infty$ as $n \to +\infty$ and let $(\tilde{S}_n(x), \tilde{I}_n(x))$ be the solutions corresponding to system (3.3). By the same discussion as in Theorem 4.1, we have that there is some constant I^* such that $\tilde{I}_n(x) \to I^*$ as $n \to +\infty$. On the other hand, since $\tilde{S}_n(x)$ is bounded, we can find some subsequence still denoted by $\{\tilde{S}_n\}_{n=1}^\infty$, weakly converges to some nonnegative function $S^*(x)$ in $L^2(\Omega)$. Now, denote

$$a(x) = d_S \int_\Omega J(x-y)dy, \quad h_n(x) = d_S \int_\Omega J(x-y)\tilde{S}_n(y)dy,$$
$$G_n(x) = (a(x) - \gamma(x) + \beta(x))\tilde{I}_n(x) - h_n(x), \quad H_n(x) = \gamma \tilde{I}_n^2(x) + h_n(x)\tilde{I}_n(x).$$

Thus, we have

$$h_n(x) \to d_S \int_\Omega J(x-y)S^*(y)dy \text{ as } n \to +\infty$$

and

$$G_n(x) \to (a(x) - \gamma(x) + \beta(x))I^* - d_S \int_\Omega J(x-y)S^*(y)dy \text{ as } n \to +\infty.$$

Meanwhile,

$$H_n(x) \to \gamma {I^*}^2 + d_S I^* \int_\Omega J(x-y)S^*(y)dy \text{ as } n \to +\infty.$$

Seen from the first equation of (3.3), $S_n(x)$ satisfies

$$a(x)S_n^2(x) + G_n(x)S_n(x) - H_n(x) = 0.$$

Consequently, we have

$$S_n(x) = \frac{-G_n(x) + \sqrt{G_n^2(x) + 4a(x)H_n(x)}}{2a(x)}.$$

This implies that

$$S_n(x) \to S^*(x) \quad \text{in } C(\bar{\Omega}) \text{ as } n \to +\infty.$$

Additionally, the same arguments as in Theorem 4.1 yield that $S^*(x) > 0$ and $I^* > 0$. Obviously, $(S^*(x), I^*)$ satisfies (4.6). The proof is complete. $\square$

5. Discussion

In the current paper, we firstly give the basic reproduction number R_0 of system (1.1), which is an important threshold value to discuss the dynamic behavior of (1.1). We prove that the disease persists when $R_0 > 1$, but when $R_0 < 1$, the disease dies out. Moreover, we also consider the effect of the large diffusion rates for the susceptible individuals or the infected individuals on the disease transmission and find that the nonlocal movement of the susceptible individuals or infected individuals will enhance the persistence of the disease.

In Section 2, we have proved the main result Theorem 2.10, and established the relations between R_0 and $\lambda_p(d_I)$ even if $\lambda_p(d_I)$ is not always a principal eigenvalue of the operator $\mathcal{M}$ defined by (2.1). Note that if $\beta(x) = \beta$ and $\gamma(x) = \gamma$ are all positive constants, then the linear problem

$$-d_I \int_{\Omega} J(x-y)(u(y) - u(x))dy + \gamma u(x) = \mu\beta u(x) \text{ in } \Omega$$

admits a principal eigenpair $(\mu_p, \varphi(x))$, where $\mu_p = \frac{\gamma}{\beta}$. Thus, it follows from Lemma 2.16 that $R_0 = \frac{1}{\mu_p} = \frac{\beta}{\gamma}$ in this case. By the same discussion as Subsections 3.1 and 3.2, we have that the disease persists if $\beta > \gamma$ and the disease dies out if $\beta < \gamma$. But when the spatial heterogeneity is concerned, we know from Corollaries 2.12 and 2.13 that the disease may persist even though there are some sites such that $\beta(x) < \gamma(x)$. That is, the spatial heterogeneity can enhance the spread of the disease. In fact, from [2], x is a low-risk site if the local disease transmission rate $\beta(x)$ is lower than the local disease recovery rate $\gamma(x)$, and the high-risk site is defined in reverse. Meanwhile, Ω is a low-risk domain if $\int_\Omega \beta(x)dx \le \int_\Omega \gamma(x)dx$ and a high-risk domain if $\int_\Omega \beta(x)dx > \int_\Omega \gamma(x)dx$. In the view of the biological point, Corollary 2.12 implies that the disease may spread even if the habitat of the species is low-risk as long as there is some high-risk site and the movement of the infected individuals is slow. But the quick movement of the infected individuals may suppress the spread of the disease. Following from Corollary 2.13, we know that the disease will always persist if the species live in a high-risk domain and be extinct if the habitat of the species is filled with the low-risk sites. We hope these results will be useful for the disease control.

Additionally, due to the effect of the nonlocal dispersal for system (1.1), we only discuss the effect of the large diffusion rates of the susceptible individuals or the infected individuals on the disease transmission. Other cases are left for future work. Also, we know that the diffusive ability of the species is different, here the diffusive ability represents the diffusive rates and the dispersal distance. Thus, it is more realistic to discuss that the susceptible individuals and the infected individuals have different dispersal strategy, that is the dispersal kernel functions are distinct from each other. This problem is of interest and it may have more complex dynamic results.

Acknowledgments

The authors are grateful to the referee for the careful reading and valuable comments which led to improvements of our original manuscript. The authors would like to thank Dr. Liang Zhang (Lanzhou University) for his helpful suggestions. F.-Y. Yang was partially supported by NSF of China (11601205), W.-T. Li was partially supported by NSF of China (11671180, 11731005) and S. Ruan was partially supported by National Science Foundation (DMS-1412454).

References

[1] L.J.S. Allen, B.M. Bolker, Y. Lou, A.L. Nevai, Asymptotic profiles of the steady states for an SIS epidemic patch model, SIAM J. Appl. Math. 67 (2007) 1283–1309.
[2] L.J.S. Allen, B.M. Bolker, Y. Lou, A.L. Nevai, Asymptotic profiles of the steady states for an SIS epidemic reaction-diffusion model, Discrete Contin. Dyn. Syst. 21 (2008) 1–20.
[3] L.J.S. Allen, Y. Lou, A.L. Nevai, Spatial patterns in a discrete-time SIS patch model, J. Math. Biol. 58 (2009) 339–375.
[4] F. Andreu-Vaillo, J.M. Mazón, J.D. Rossi, J.J. Toledo-Melero, Nonlocal Diffusion Problems, Mathematical Surveys and Monographs, vol. 165, Amer. Math. Soc., Providence, Rhode Island, 2010.
[5] D.G. Aronson, The asymptotic speed of propagation of a simple epidemic, in: W.E. Fitzgibbon, H.F. Walker (Eds.), Nonlinear Diffusion, in: Res. Notes Math., vol. 14, Pitman, London, 1977, pp. 1–23.
[6] H. Berestycki, F. Hamel, L. Roques, Analysis of the periodically fragmented environment model. I. Species persistence, J. Math. Biol. 51 (2005) 75–113.
[7] P. Bates, On some nonlocal evolution equations arising in materials science, in: H. Brunner, X.Q. Zhao, X. Zou (Eds.), Nonlinear Dynamics and Evolution Equations, in: Fields Inst. Commun., vol. 48, Amer. Math. Soc., Providence, RI, 2006, pp. 13–52.
[8] P. Bates, P.C. Fife, X. Ren, X. Wang, Traveling waves in a convolution model for phase transition, Arch. Ration. Mech. Anal. 138 (1997) 105–136.
[9] P. Bates, G. Zhao, Existence, uniqueness and stability of the stationary solution to a nonlocal evolution equation arising in population dispersal, J. Math. Anal. Appl. 332 (2007) 428–440.
[10] E. Chasseigne, M. Chaves, J.D. Rossi, Asymptotic behavior for nonlocal diffusion equation, J. Math. Pures Appl. 86 (2006) 271–291.
[11] C. Cortázar, J. Coville, M. Elgueta, S. Martínez, A nonlocal inhomogeneous dispersal process, J. Differential Equations 241 (2007) 332–358.
[12] J. Coville, On a simple criterion for the existence of a principal eigenfunction of some nonlocal operators, J. Differential Equations 249 (2010) 2921–2953.
[13] J. Coville, J. Dávila, S. Martínez, Existence and uniqueness of solutions to a nonlocal equation with monostable nonlinearity, SIAM J. Math. Anal. 39 (2008) 1693–1709.
[14] J. Coville, J. Dávila, S. Martínez, Pulsating fronts for nonlocal dispersion and KPP nonlinearity, Ann. Inst. H. Poincaré Anal. Non Linéaire 30 (2013) 179–223.
[15] R. Cui, K.Y. Lam, Y. Lou, Dynamics and asymptotic profiles of steady states of an epidemic model in advective environments, J. Differential Equations 263 (2017) 2343–2373.

[16] R. Cui, Y. Lou, A spatial SIS model in advective heterogeneous environments, J. Differential Equations 261 (2016) 3305–3343.
[17] O. Diekmann, J.A.P. Heesterbeek, J.A.J. Metz, On the definition and the computation of the basic reproduction ratio R_0 in models for infectious diseases in heterogeneous populations, J. Math. Biol. 28 (1990) 365–382.
[18] P. Fife, Some nonclassical trends in parabolic and parabolic-like evolutions, in: M. Kirkilionis, S. Krömker, R. Rannacher, F. Tomi (Eds.), Trends in Nonlinear Analysis, Springer, Berlin, 2003, pp. 153–191.
[19] J. García-Melián, J.D. Rossi, On the principal eigenvalue of some nonlocal diffusion problems, J. Differential Equations 246 (2009) 21–38.
[20] J. García-Melián, J.D. Rossi, Maximum and antimaximum principles for some nonlocal diffusion operators, Nonlinear Anal. 71 (2009) 6116–6121.
[21] J. García-Melián, J.D. Rossi, A logistic equation with refuge and nonlocal diffusion, Commun. Pure Appl. Anal. 8 (2009) 2037–2053.
[22] J.M. Heffernan, R.J. Smith, L.M. Wahl, Perspectives on the basic reproductive ratio, J. R. Soc. Interface 2 (2005) 281–293.
[23] W. Huang, M. Han, K. Liu, Dynamics of an SIS reaction-diffusion epidemic model for disease transmission, Math. Biosci. Eng. 7 (2010) 51–66.
[24] V. Hutson, S. Martinez, K. Mischaikow, G.T. Vickers, The evolution of dispersal, J. Math. Biol. 47 (2003) 483–517.
[25] C.Y. Kao, Y. Lou, W. Shen, Random dispersal vs non-local dispersal, Discrete Contin. Dyn. Syst. 26 (2010) 551–596.
[26] D.G. Kendall, Discussion of 'Measles periodicity and community size' by M.S. Bartlett, J. R. Stat. Soc., A 120 (1977) 64–76.
[27] D.G. Kendall, Mathematical models of the spread of infection, in: Mathematics and Computer Science in Biology and Medicine, 1965, pp. 213–225.
[28] H. Li, R. Peng, F.B. Wang, Varying total population enhances disease persistence: qualitative analysis on a diffusive SIS epidemic model, J. Differential Equations 262 (2017) 885–913.
[29] W.T. Li, Y.J. Sun, Z.C. Wang, Entire solutions in the Fisher-KPP equation with nonlocal dispersal, Nonlinear Anal. Real World Appl. 11 (2010) 2302–2313.
[30] W.T. Li, Li Zhang, G.B. Zhang, Invasion entire solutions in a competition system with nonlocal dispersal, Discrete Contin. Dyn. Syst. 35 (2015) 1531–1560.
[31] D. Mollison, Possible velocities for a simple epidemic, Adv. in Appl. Probab. 4 (1972) 233–257.
[32] S. Pan, W.T. Li, G. Lin, Travelling wave fronts in nonlocal reaction-diffusion systems and applications, Z. Angew. Math. Phys. 60 (2009) 377–392.
[33] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, New York, 1983.
[34] R. Peng, Asymptotic profiles of the positive steady state for an SIS epidemic reaction-diffusion model, I, J. Differential Equations 247 (2009) 1096–1119.
[35] R. Peng, S. Liu, Global stability of the steady states of an SIS epidemic reaction-diffusion model, Nonlinear Anal. 71 (2009) 239–247.
[36] R. Peng, X.Q. Zhao, A reaction-diffusion SIS epidemic model in a time-periodic environment, Nonlinearity 25 (2012) 1451–1471.
[37] R. Peng, F. Yi, Asymptotic profile of the positive steady state for an SIS epidemic reaction-diffusion model: effects of epidemic risk and population movement, Phys. D 259 (2013) 8–25.
[38] L. Rass, J. Radcliffe, Spatial Deterministic Epidemics, Math. Surveys Monogr., vol. 102, Amer. Math. Soc., Providence, PI, 2003.
[39] S. Ruan, Spatial-temporal dynamics in nonlocal epidemiological models, in: Y. Takeuchi, K. Sato, Y. Iwasa (Eds.), Mathematics for Life Science and Medicine, Springer-Verlag, Berlin, 2007, pp. 99–122.
[40] W. Shen, A. Zhang, Spreading speeds for monostable equations with nonlocal dispersal in space periodic habitats, J. Differential Equations 15 (2010) 747–795.
[41] W. Shen, A. Zhang, Stationary solutions and spreading speeds of nonlocal monostable equations in space periodic habitats, Proc. Amer. Math. Soc. 140 (2012) 1681–1696.
[42] J.W. Sun, W.T. Li, F.Y. Yang, Approximate the Fokker–Planck equation by a class of nonlocal dispersal problems, Nonlinear Anal. 74 (2011) 3501–3509.
[43] J.W. Sun, F.Y. Yang, W.T. Li, A nonlocal dispersal equation arising from a selection-migration model in genetics, J. Differential Equations 257 (2014) 1372–1402.
[44] J.W. Sun, W.T. Li, Z.C. Wang, A nonlocal dispersal logistic model with spatial degeneracy, Discrete Contin. Dyn. Syst. 35 (2015) 3217–3238.

[45] Y.J. Sun, W.T. Li, Z.C. Wang, Entire solutions in nonlocal dispersal equations with bistable nonlinearity, J. Differential Equations 251 (2011) 551–581.
[46] H.R. Thieme, Spectral bound and reproduction number for infinite-dimensional population structure and time heterogeneity, SIAM J. Appl. Math. 70 (2009) 188–211.
[47] P. van den Driessche, J. Watmough, Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission, John A. Jacquez memorial volume, Math. Biosci. 180 (2002) 29–48.
[48] W. Wang, X. Zhao, A nonlocal and time-delayed reaction-diffusion model of dengue transmission, SIAM J. Appl. Math. 71 (2011) 147–168.
[49] W. Wang, X. Zhao, Basic reproduction numbers for reaction-diffusion epidemic models, SIAM J. Appl. Dyn. Syst. 11 (2012) 1652–1673.
[50] X. Wang, Metastability and stability of patterns in a convolution model for phase transitions, J. Differential Equations 183 (2002) 434–461.
[51] Y. Wu, X. Zou, Asymptotic profiles of steady states for a diffusive SIS epidemic model with mass action infection mechanism, J. Differential Equations 261 (2016) 4424–4447.
[52] F.Y. Yang, W.T. Li, Dynamics of a nonlocal dispersal SIS epidemic model, Commun. Pure Appl. Anal. 16 (2017) 781–797.
[53] F.Y. Yang, W.T. Li, J.W. Sun, Principal eigenvalues for some nonlocal eigenvalue problems and applications, Discrete Contin. Dyn. Syst. 36 (2016) 4027–4049.
[54] G.B. Zhang, W.T. Li, Y.J. Sun, Asymptotic behavior for nonlocal dispersal equations, Nonlinear Anal. 72 (2010) 4466–4474.
[55] G. Zhao, S. Ruan, Spatial and temporal dynamics of a nonlocal viral infection model, SIAM J. Appl. Math. 78 (2018) 1954–1980.