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THE EFFECTS OF HARVESTING AND TIME DELAY ON PREDATOR-PREY SYSTEMS WITH HOLLING TYPE II FUNCTIONAL RESPONSE*

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Abstract. In this paper, the effects of harvesting and time delay on two different types of predator-prey systems with delayed predator specific growth and Holling type II functional response are studied by applying the normal form theory of retarded functional differential equations developed by Faria and Magalhães [*J. Differential Equations*, 122 (1995), pp. 181–200, *J. Differential Equations*, 122 (1995), pp. 201–224]. Hopf bifurcations are demonstrated in models with harvesting of the prey at a constant rate by taking the delay as a bifurcation parameter, and numerical examples supporting our theoretical prediction are also given. Furthermore, bifurcation analysis indicates that delayed predator-prey systems with predator harvesting exhibit Bogdanov–Takens bifurcation. The versal unfoldings of the models at the Bogdanov–Takens singularity are obtained, and numerical simulations and bifurcation diagrams are given to illustrate the obtained results.

Key words. predator-prey system, harvesting, time delay, Hopf bifurcation, Bogdanov–Takens bifurcation

AMS subject classifications. 34K18, 37G05, 37G10, 92D25

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1. Introduction.

The Canadian Grand Banks off the coast of Newfoundland were prime fishing grounds for several centuries. The population of Atlantic cod (*Godus morhua*) used to be so abundant that, in 1968, about 40,000 people were employed in the fishing industry and more than 800,000 tons of Atlantic cod were harvested (Schiermeier [34]). However, the Atlantic cod stocks collapsed in about 10 years and in 1992, the Department of Fisheries and Oceans of Canada belatedly closed the fisheries (Hutchings and Myers [21], Myers et al. [27, 28], Hutchings [20]). More than a decade later, there is still no sign of recovery of Atlantic cod stocks in the Grand Banks. Today, the fear of a rapid depletion of world fish stocks because of overexploitation is increasing (Pauly et al. [31], Myers and Worm [29]). Depletions of marine fish stocks not only endanger the future of marine fisheries but also may lead to species extinction and ecosystem regime shifts (Casey and Myers [6], Jackson et al. [22]). Thus, understanding the dynamics of fishing becomes very important.

There are several types of interactions within fisheries systems. Biological, harvest technical, and market interactions are the most important ones in bioeconomic modeling (Flaaten [16]). The biological interactions are predator-prey relations and competition between species. In the case of predator-prey interactions it is well known that the reduction of the predator stock level may increase the surplus production of the prey. However, harvesting becomes controversial when it comes to predators like whales and seals (May et al. [26], Flaaten [15], Yodzis [38]). The purpose of this paper is to study the importance of harvesting on multispecies management within

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the framework of predator-prey models (Clark [8], Flaaten [16]).

The study of predator-prey models with harvesting has attracted the attention of many researchers. Let $x(t)$ and $y(t)$ denote the density of the prey and predators at time t, respectively. Consider constant-yield harvesting of a predator-prey system with Holling type II functional response:

$$\begin{cases} \dot{x} = rx\left(1 - \dfrac{x}{K}\right) - \left(\dfrac{m}{\delta}\right)\dfrac{yx}{A+x} - H_1, \\ \dot{y} = y\left(-D + \dfrac{mx}{A+x}\right) - H_2, \end{cases} \tag{1.1}$$

where r is the intrinsic growth rate of the prey; K is the carrying capacity of the prey; m is the maximum growth rate of predators; δ is the yield conversion factor for predators feedings on the prey; A is the half saturation constant for the predators which is the prey density at which the functional response is half maximal; D is the death rate of predators; H_1 and H_2 are constant harvesting rates for the prey and predators, respectively. Note that we need to assume that $\dot{x} \geq 0$ and $\dot{y} \geq 0$ for all time. Model (1.1) and its variants have been studied extensively; see, for example, Brauer and Soudack [3, 4, 5], Beddington and Cooke [1], Dai and Tang [11], Hogarth et al. [19], Myerscough et al. [30], Xiao and Jennings [35], and Xiao and Ruan [36]. Very rich and interesting dynamical behaviors such as the existence of multiple equilibria, homoclinic loop, and Hopf bifurcation have been observed.

Let $\bar{y} = y/\delta, \bar{t} = mt, \bar{r} = r/m, \bar{H}_1 = H_1/m, \bar{D} = D/m$, and $\bar{H}_2 = H_2/m\delta$. Dropping the bars we obtain the dimensionless model

$$\begin{cases} \dot{x} = rx\left(1 - \dfrac{x}{K}\right) - \dfrac{yx}{A+x} - H_1, \\ \dot{y} = y\left(-D + \dfrac{x}{A+x}\right) - H_2. \end{cases} \tag{1.2}$$

When $H_1 = 0$, Xiao and Ruan [36] carried out a bifurcation analysis of model (1.2). In particular they showed that codimension 2 bifurcations occur in a two-dimensional parameter region. Under some conditions they proved that system (1.2) undergoes Bogdanov–Takens bifurcation; i.e., it can exhibit qualitatively different dynamical behavior, including Hopf bifurcation, saddle-node bifurcation, as well as homoclinic bifurcation.

On the other hand, population models with time delay are of current research interest in mathematical biology because of their realistic meaning; we refer to the monographs of Cushing [10], Gopalsamy [17], and Kuang [23] for general delayed biological systems and a survey paper of Ruan [33] and the references cited therein for studies on delayed predator-prey systems. Brauer [2] was the first to consider the combined effects of time delay and constant harvesting on predator-prey models. Further studies were performed by Martin and Ruan [25] who studied the combined effects of the prey harvesting and time delay on the dynamics of the generalized Gause-type predator-prey models and the Wangersky–Cunningham model. It is shown that in these models the time delay can cause a stable equilibrium to become unstable and even a switching of stabilities, while the prey harvesting rate has a stabilizing effect on the equilibrium if it is under the critical harvesting level. In particular, one of these models loses stability when the delay varies and then regains its stability when the prey harvesting rate is increased.

In this paper, following the work of Martin and Ruan [25] and Xiao and Ruan [36], we continue studying how time delay and harvesting affect the dynamics of the predator-prey systems. We assume that a time delay $\tau(>0)$ occurs in the predator response term. It represents a gestation time of the predators. The reproduction of predators after predating the prey is not instantaneous but will be mediated by some discrete time lag required for gestation of the predators. We consider two cases.

Model 1. The prey population is harvested at a constant rate:

$$\begin{cases} \dot{x}(t) = rx(t)\left(1 - \dfrac{x(t)}{K}\right) - \dfrac{x(t)y(t)}{A + x(t)} - H, \\ \dot{y}(t) = y(t)\left(-D + \dfrac{x(t-\tau)}{A + x(t-\tau)}\right). \end{cases} \tag{1.3}$$

Model 2. The predator population is harvested at a constant rate:

$$\begin{cases} \dot{x}(t) = rx(t)\left(1 - \dfrac{x(t)}{K}\right) - \dfrac{x(t)y(t)}{A + x(t)}, \\ \dot{y}(t) = y(t)\left(-D + \dfrac{x(t-\tau)}{A + x(t-\tau)}\right) - H. \end{cases} \tag{1.4}$$

The initial conditions for these systems are

$$x(\theta) = \phi_1(\theta), \quad y(\theta) = \phi_2(\theta) \tag{1.5}$$

for all $\theta \in [-\tau, 0]$, where $(\phi_1, \phi_2) \in C([-\tau, 0], \mathbb{R}_+^2)$, $x(0) = \phi_1(0) > 0$, and $y(0) = \phi_2(0) > 0$.

Following Xiao and Ruan [37], we first show that Hopf bifurcation occurs in Model 1 and determine the direction of the Hopf bifurcation. We then show that Model 2 exhibits Bogdanov–Takens bifurcation.

The paper is organized as follows. In section 2, we show that the system (1.3) with prey harvesting undergoes Hopf bifurcation as the delays cross some critical values and determine the stability of the bifurcating periodic solutions by using the method of Faria and Magalhães [13]. Numerical simulations are performed to illustrate the obtained results. In section 3, following the technique of Faria and Magalhães [13, 14], we show that the model (1.4) with predator harvesting exhibits Bogdanov–Takens bifurcation and obtain versal unfoldings at the Bogdanov–Takens singularity under some conditions. A brief discussion is given in section 4 to conclude the paper.

2. Prey harvesting.

In this section, we are concerned with Model 1, i.e., system (1.3). We first study the interior equilibrium of system (1.3). An easy computation shows that when

$$rAD(K - KD - AD) - HK(1-D)^2 > 0 \quad \text{and} \quad 0 < D < 1, \tag{2.1}$$

system (1.3) has a unique positive equilibrium $E = (x_0, y_0)$, where $x_0 = \frac{AD}{1-D}$, $y_0 = (r - \frac{rx_0}{K} - \frac{H}{x_0})(A + x_0)$. Through transformation $z_1(t) = x(t) - x_0, z_2(t) = y(t) - y_0$, the equilibrium $E = (x_0, y_0)$ is translated to the origin $O = (0, 0)$ and system (1.3) can be rewritten as the following equivalent system:

$$\begin{cases} \dot{z}_1(t) = \alpha_1 z_1(t) + \alpha_2 z_2(t) + \displaystyle\sum_{i+j\geq 2} \frac{1}{i!j!} f_{ij}^{(1)} z_1^i(t) z_2^j(t), \\ \dot{z}_2(t) = \beta_1 z_1(t-\tau) + \displaystyle\sum_{i+j\geq 2} \frac{1}{i!j!} f_{ij}^{(2)} z_1^i(t-\tau) z_2^j(t), \end{cases} \tag{2.2}$$

where

$$\alpha_1 = r - \frac{2x_0 r}{K} - \frac{Ay_0}{(A+x_0)^2}, \quad \alpha_2 = -\frac{x_0}{A+x_0}, \quad \beta_1 = \frac{Ay_0}{(A+x_0)^2},$$
$$f_{ij}^{(1)} = \left.\frac{\partial^{i+j} f^{(1)}}{\partial x^i \partial y^j}\right|_{(x_0,y_0)}, \quad f_{ij}^{(2)} = \left.\frac{\partial^{i+j} f^{(2)}}{\partial x^i \partial y^j}\right|_{(x_0,y_0)}, \quad i, j \geq 0,$$
$$f^{(1)} = rx\left(1 - \frac{x}{K}\right) - \frac{xy}{A+x} - H, \quad f^{(2)} = y\left(-D + \frac{x}{A+x}\right).$$

We obtain the linearized system

$$\begin{cases} \dot{z}_1(t) = \alpha_1 z_1(t) + \alpha_2 z_2(t), \\ \dot{z}_2(t) = \beta_1 z_1(t-\tau). \end{cases} \tag{2.3}$$

The characteristic equation of the linearized system (2.3) is a transcendental equation of the following form (see Cooke and Grossman [9] and Ruan [32]):

$$\lambda^2 - \lambda\alpha_1 - \alpha_2\beta_1 e^{-\lambda\tau} = 0. \tag{2.4}$$

In [25], Martin and Ruan considered the following system:

$$\begin{cases} \dot{x}(t) = x(t)[f(x(t)) - y(t)h(x(t))] - H, \\ \dot{y}(t) = y(t)[-d + cx(t-\tau)h(x(t-\tau))]. \end{cases} \tag{2.5}$$

We note that when $f(x)$ is the logistic growth function and $xh(x)$ is the Holling type II response function in (2.5), then system (2.5) becomes system (1.3). Martin and Ruan studied the stability of the equilibria and existence of Hopf bifurcation for model (2.5). However, the direction of the Hopf bifurcation and stability of the bifurcating periodic solutions were not considered in [25]. In this section, we shall determine the direction and stability of the bifurcating periodic solutions by employing the normal form theory due to Faria and Magalhães [13].

For convenience, some useful lemmas and theorems from [25] are rewritten as follows.

LEMMA 2.1. *Assume that* (2.1) *holds and*

$$rKAD^2 + (1-D)^3HK - rA^2D^2 - rKAD^3 - rA^2D^3 < 0. \tag{2.6}$$

Then at

$$\tau_k = \frac{1}{\sigma_+}\left(\arccos\frac{-\sigma_+^2}{\alpha_2\beta_1} + 2k\pi\right), \quad k = 0, 1, 2, \ldots, \tag{2.7}$$

(2.4) *has a simple pair of purely imaginary roots* $\pm i\sigma_+$, *where*

$$\sigma_+ = \sqrt{\frac{-\alpha_1^2 + \sqrt{\alpha_1^4 + 4\alpha_2^2\beta_1^2}}{2}}. \tag{2.8}$$

Furthermore,
(1) *if* $\tau \in [0, \tau_0)$, *all roots of* (2.4) *have strictly negative real parts;*
(2) *if* $\tau = \tau_0$, *all roots of* (2.4), *except* $\pm i\sigma_+$, *have strictly negative real parts.*

LEMMA 2.2. *The following transversality condition*

$$\operatorname{sign}\left\{ \operatorname{Re}\left(\frac{d\lambda}{d\tau}\right) \right\}_{\tau=\tau_k} > 0$$

is satisfied, where $\lambda(\tau) = \mu(\tau) \pm i\sigma(\tau)$ *are the roots of* (2.4) *near* $\tau = \tau_k (k = 0, 1, 2, \dots)$ *satisfying* $\mu(\tau_k) = 0$, $\sigma(\tau_k) = \sigma_+$, *and* $k = 0, 1, 2, \dots$.

LEMMA 2.3. *Suppose that* (2.1) *and* (2.6) *hold, then* (2.4) *has at least one eigenvalue with strictly positive real part for* $\tau > \tau_0$.

THEOREM 2.4. *Suppose that conditions* (2.1) *and* (2.6) *are satisfied. Then*
(1) *when* $\tau \in [0, \tau_0)$, *the zero solution of* (2.2) *is locally asymptotically stable;*
(2) *when* $\tau > \tau_0$, *the zero solution of* (2.2) *is unstable;*
(3) $\tau_k (k = 0, 1, 2, \dots)$ *are Hopf bifurcation values for system* (2.2).

From the above theorem, we know that system (1.3) undergoes Hopf bifurcations at the critical values $\tau_k (k = 0, 1, 2, \dots)$. In the following, we shall determine the direction and stability of the bifurcating periodic solutions.

Let $z_1(t) = x(\tau t) - x_0$, $z_2(t) = y(\tau t) - y_0$. Then system (1.3) is transformed into functional differential equations in $C([-1, 0], \mathbb{R}^2)$

$$\text{(2.9)} \qquad \begin{cases} \dot{z}_1(t) = \tau \left[\alpha_1 z_1(t) + \alpha_2 z_2(t) + \displaystyle\sum_{i+j \geq 2} \frac{1}{i!j!} f_{ij}^{(1)} z_1^i(t) z_2^j(t) \right], \\ \dot{z}_2(t) = \tau \left[\beta_1 z_1(t-1) + \displaystyle\sum_{i+j \geq 2} \frac{1}{i!j} f_{ij}^{(2)} z_1^i(t-1) z_2^j(t) \right], \end{cases}$$

and the linearized system is

$$\text{(2.10)} \qquad \begin{cases} \dot{z}_1(t) = \tau[\alpha_1 z_1(t) + \alpha_2 z_2(t)], \\ \dot{z}_2(t) = \tau \beta_1 z_1(t-1). \end{cases}$$

System (2.10) has the characteristic equation

$$\text{(2.11)} \qquad \lambda^2 - \lambda \tau \alpha_1 - \tau^2 \alpha_2 \beta_1 e^{-\lambda} = 0.$$

Letting $\lambda = \xi \tau$, then (2.11) becomes

$$\text{(2.12)} \qquad \xi^2 - \alpha_1 \xi - \alpha_2 \beta_1 e^{-\xi \tau} = 0.$$

Obviously, (2.12) is the same as (2.4). We know that for fixed $k \in N$, (2.12) has a simple pair of conjugate complex roots $\xi(\tau) = \mu(\tau) \pm i\sigma(\tau)$ which satisfy

$$\mu(\tau_k) = 0, \quad \sigma(\tau_k) = \sigma_+, \quad \mu'(\tau_k) \neq 0,$$

where τ_k is given in (2.7). Therefore, the characteristic equation (2.11) has two complex roots $\lambda(\tau) = \tau\mu(\tau) \pm i\tau\sigma(\tau)$, which satisfy

$$\frac{d\mathrm{Re}\lambda(\tau)}{d\tau}|_{\tau=\tau_k} = \tau \frac{d\mathrm{Re}\xi(\tau)}{d\tau}|_{\tau=\tau_k} + \mathrm{Re}\xi(\tau)|_{\tau=\tau_k} = \tau_k \mu^{'}(\tau_k) \neq 0.$$

Let $z = (z_1, z_2)^T$. System (2.9) can further be written as

$$\text{(2.13)} \qquad \dot{z}(t) = L(\tau)(z_t) + F(z_t, \tau),$$

where

$$L(\tau)(\varphi) = \tau \begin{pmatrix} \alpha_1\varphi_1(0) + \alpha_2\varphi_2(0) \\ \beta_1\varphi_1(-1) \end{pmatrix},$$

$$F(\varphi,\tau) = \tau \begin{pmatrix} \Sigma_{i+j\geq 2} \dfrac{1}{i!j!} f_{ij}^{(1)} \varphi_1^i(0)\varphi_2^j(0) \\ \\ \displaystyle\sum_{i+j\geq 2} \dfrac{1}{i!j} f_{ij}^{(2)} \varphi_1^i(-1)\varphi_2^j(0) \end{pmatrix},$$

where $\varphi = (\varphi_1, \varphi_2)^T \in C([-1,0], \mathbb{R}^2)$.

By the Riesz representation theorem, there exists an $n \times n$ matrix $\eta(\theta, \tau)$, whose elements are of bounded variation for $\theta \in [-1, 0]$, such that for $\varphi \in C([-1, 0], \mathbb{R}^2)$,

$$L(\tau)(\varphi) = \int_{-1}^{0} d\eta(\theta,\tau)\varphi(\theta).$$

Choose

$$\eta(\theta,\tau) = \tau \begin{pmatrix} \alpha_1 & \alpha_2 \\ 0 & 0 \end{pmatrix} H(\theta) + \tau \begin{pmatrix} 0 & 0 \\ -\beta_1 & 0 \end{pmatrix} H(\theta+1),$$

where $H(\theta)$ is the Heaviside function. We expand $F(\varphi, \tau)$ about φ as the Taylor expansion

$$F(\varphi,\tau) = \frac{1}{2!}F_2(\varphi,\tau) + \frac{1}{3!}F_3(\varphi,\tau) + O(|\varphi|^4), \tag{2.14}$$

where $F_l(\varphi, \tau)$ is given by

$$\frac{1}{l!}F_l(\varphi,\tau) = \tau \begin{pmatrix} \displaystyle\sum_{i+j=l} \dfrac{1}{i!j!} f_{ij}^{(1)} \varphi_1^i(0)\varphi_2^j(0) \\ \displaystyle\sum_{i+j=l} \dfrac{1}{i!j!} f_{ij}^{(2)} \varphi_1^i(-1)\varphi_2^j(0) \end{pmatrix}. \tag{2.15}$$

Setting a new parameter $\alpha = \tau - \tau_k$, system (2.13) is rewritten as

$$\dot{z}(t) = L(\tau_k)(z_t) + F_0(z_t, \alpha), \tag{2.16}$$

where $F_0(\varphi, \alpha) = L(\alpha)(\varphi) + F(\varphi, \tau_k + \alpha)$.

Let A_0 be the infinitesimal generator corresponding to $\dot{z}(t) = L(\tau_k)z_t$. Then A_0 has a pair of purely imaginary characteristic roots $\pm i\sigma_k(\sigma_k = \tau_k\sigma_+)$, which are simple, and no other characteristic roots with zero real part. Consider $\Lambda = \{-i\sigma_k, i\sigma_k\}$ and denote by P the invariant space of A_0 corresponding to Λ, where dim $P = 2$. Let the phase space $C = C([-1, 0], \mathbb{R}^2)$ be decomposed by Λ as $C = P \oplus Q$ by applying the formal adjoint theory for functional differential equations in Hale [18]. Consider complex coordinates and still write as $C = C([-1, 0], \mathbb{C}^2)$. Suppose that $\Phi = (\Phi_1, \Phi_2)$ is the basis for P and $\Phi_1(\theta) = e^{i\sigma_k\theta}v^T = e^{i\sigma_k\theta}(1, v_2)^T$, $\Phi_2(\theta) = \overline{\Phi_1(\theta)}$, $-1 \leq \theta \leq 0$, where $v = (1, v_2)^T$ is a vector in $\mathbb{C}^2$ and $v_2 = \frac{\tau_k\beta_1 e^{-i\sigma_k}}{i\sigma_k}$, which satisfies

$$L(\tau_k)(\Phi_1) = i\sigma_k v^T.$$

Also, the two eigenvectors Ψ_1 and Ψ_2 of the formal adjoint operator A_0^*, corresponding to $i\sigma_k$ and $-i\sigma_k$, respectively, form a basis $\Psi(s) = \mathrm{col}(\Psi_1(s), \Psi_2(s))$ of the adjoint space P^* of P, where $\Psi_1(s) = e^{-i\sigma_k s}(u_1, u_2)$, $\Psi_2(s) = \overline{\Psi_1(s)}$, $0 \le s \le 1$. We know $-\dot{\Psi}_1(0) = \int_{-1}^{0} \Psi_1(-\theta)d\eta(\theta)$; therefore, we can choose $u_2 = \frac{\tau_k \alpha_2}{i\sigma_k} u_1$. In order to have $(\Psi, \Phi) = ((\Psi_j, \Phi_i), i, j = 1, 2) = I_2$, where I_2 is the second-order identical matrix, $(\cdot, \cdot)$ is a bilinear inner product form

$$(\psi, \varphi) = \psi(0)\varphi(0) - \int_{-1}^{0} \int_{0}^{\theta} \psi(\xi - \theta)d\eta(\theta)\varphi(\xi)d\xi,$$

we have $u_1 = \frac{\tau_k^2 \beta_1 \alpha_2}{\tau_k^2 \beta_1 \alpha_2 + (i\sigma_k - \tau_k \alpha_1)^2 e^{i\sigma_k} + (i\sigma_k - \tau_k \alpha_1)\tau_k^2 \alpha_2 \beta_1}$. Note that $\dot{\Phi} = \Phi B$, where B is a diagonal matrix

$$B = \begin{pmatrix} i\sigma_k & 0 \\ 0 & i\sigma_k \end{pmatrix}.$$

Take the enlarged phase space $BC := \{\varphi : [-1, 0] \to \mathbb{C}^2 | \varphi \text{ is continuous on } [-1, 0), \exists \lim_{\theta \to 0^-} \varphi(\theta)\}$. The elements of BC have the form $\psi = \varphi + X_0 \alpha$, where $\varphi \in C$, $\alpha \in \mathbb{C}^2$, $X_0 = X_0(\theta)$ is given by

$$X_0(\theta) = \begin{cases} I, & \theta = 0, \\ 0, & -1 \le \theta < 0. \end{cases}$$

The projection of C upon P, $\varphi \mapsto \Phi(\Psi, \varphi)$, associated with the decomposition $C = P \oplus Q$ is replaced by $\pi : BC \mapsto P$ such that

$$\pi(\varphi + X_0 \alpha) = \Phi[(\Psi, \varphi) + \Psi(0)\alpha].$$

Thus, we have the decomposition $BC = P \oplus \ker \pi$. Use the decomposition $z_t = \Phi x(t) + y_t$, $x(t) \in \mathbb{C}^2$, $y_t \in \ker \pi \cap C^1 \triangleq Q^1$, where $C^1 = C^1([-1, 0]; \mathbb{R}^2)$ denotes the space of continuously differentiable functions from $[-1, 0]$ to $\mathbb{R}^2$. The equation (2.16) is equivalent to the system

$$\begin{cases} \dot{x} = Bx + \Psi(0)F_0(\Phi x + y, \alpha), \\ \dfrac{d}{dt} y = A_{Q^1} y + (I - \pi)X_0 F_0(\Phi x + y, \alpha). \end{cases} \tag{2.17}$$

We have the Taylor expansions,

$$\Psi(0)F_0(\Phi x + y, \alpha) = \frac{1}{2} f_2^1(x, y, \alpha) + \frac{1}{3!} f_3^1(x, y, \alpha) + h.o.t., \tag{2.18}$$
$$(I - \pi)X_0 F_0(\Phi x + y, \alpha) = \frac{1}{2} f_2^2(x, y, \alpha) + \frac{1}{3!} f_3^2(x, y, \alpha) + h.o.t.,$$

where $f_j^1(x, y, \alpha)$ and $f_j^2(x, y, \alpha)$ are homogeneous polynomials in (x, y, α) of degree j, $j = 2, 3$, with coefficients in $\mathbb{C}^2$ and $\ker \pi$, respectively, and h.o.t. stands for higher order terms. The normal form method implies a normal form on the center manifold of the origin for (2.17) as

$$\dot{x} = Bx + \frac{1}{2} g_2^1(x, 0, \alpha) + \frac{1}{3!} g_3^1(x, 0, \alpha) + h.o.t, \tag{2.19}$$

where g_2^1, g_3^1 are the second- and third-order terms in (x,α), respectively. Following [13], $V_j^{m+p}(X)$ denotes the linear space of homogeneous polynomials of degree j in $m+p$ real variables, $x=(x_1,\ldots,x_m),\alpha=(\alpha_1,\ldots,\alpha_p)$ with coefficients in X; that is,

$$V_j^{m+p}(X)=\left\{\sum_{|(q,l)|=j} c_{(q,l)}x^q\alpha^l : (q,l)\in N_0^{m+p}, c_{(q,l)}\in X\right\}.$$

The operators M_j^1 are defined by

$$(M_j^1 p)(x,\alpha)=D_x p(x,\alpha)Bx-Bp(x,\alpha),\quad j\geq 2,$$

and M_j^1 act in $V_j^3(\mathbb{C}^2)$, $V_j^3(\mathbb{C}^2)=\mathrm{Im}(M_j^1)\oplus\ker(M_j^1)$ and

$$\ker(M_j^1)=\mathrm{span}\{x^q\alpha^l e_k : (q,\bar\lambda)=\lambda_k, k=1,2, q\in\mathbb{N}_0^2, l\in\mathbb{N}_0, |(q,l)|=j\},$$

where $\{e_1,e_2\}$ is the canonical basis of $\mathbb{C}^2$. Hence,

$$\begin{aligned}\ker(M_2^1)&=\mathrm{span}\left\{\begin{pmatrix}x_1\alpha\\0\end{pmatrix},\begin{pmatrix}0\\x_2\alpha\end{pmatrix}\right\},\\ \ker(M_3^1)&=\mathrm{span}\left\{\begin{pmatrix}x_1^2x_2\\0\end{pmatrix},\begin{pmatrix}x_1\alpha^2\\0\end{pmatrix},\begin{pmatrix}0\\x_1x_2^2\end{pmatrix},\begin{pmatrix}0\\x_2\alpha^2\end{pmatrix}\right\}.\end{aligned}$$

For (2.18), it follows that

$$f_2^1(x,y,\alpha)=\Psi(0)[2L(\alpha)(\Phi x+y)+F_2(\Phi x+y,\tau_k)], \tag{2.20}$$

where F_2 is given in (2.15). By computing, we obtain

$$f_2^1(x,0,\alpha)=\begin{pmatrix}2A_1x_1\alpha+2A_2x_2\alpha+a_{20}x_1^2+2a_{11}x_1x_2+a_{02}x_2^2\\ 2\bar A_2x_1\alpha+2\bar A_1x_2\alpha+\bar a_{02}x_1^2+2\bar a_{11}x_1x_2+\bar a_{20}x_2^2\end{pmatrix}, \tag{2.21}$$

where

$$\begin{aligned}A_1&=\frac{i\sigma_k}{\tau_k}(u_1+u_2v_2),\\ A_2&=\frac{-i\sigma_k}{\tau_k}(u_1+u_2\bar v_2),\\ a_{20}&=\tau_k[u_1f_{20}^{(1)}+2u_1v_2f_{11}^{(1)}+u_2f_{20}^{(2)}e^{-2i\sigma_k}+2u_2v_2f_{11}^{(2)}e^{-i\sigma_k}],\\ a_{11}&=\tau_k[u_1f_{20}^{(1)}+u_1f_{11}^{(1)}(v_2+\bar v_2)+u_2f_{20}^{(2)}+u_2f_{11}^{(2)}(e^{-i\sigma_k}\bar v_2+e^{i\sigma_k}v_2)],\\ a_{02}&=\tau_k[u_1f_{20}^{(1)}+2u_1\bar v_2f_{11}^{(1)}+u_2f_{20}^{(2)}e^{2i\sigma_k}+2u_2\bar v_2f_{11}^{(2)}e^{i\sigma_k}].\end{aligned} \tag{2.22}$$

Since the second-order term in (α,x) of the normal form on the center manifold is given by

$$\frac{1}{2}g_2^1(x,0,\alpha)=\frac{1}{2}\mathrm{Proj}_{\ker(M_2^1)}f_2^1(x,0,\alpha),$$

it implies that

$$\frac{1}{2}g_2^1(x,0,\alpha)=\begin{pmatrix}A_1x_1\alpha\\ \bar A_1x_2\alpha\end{pmatrix}, \tag{2.23}$$

where $A_1 = \frac{i\sigma_k}{\tau_k}(u_1 + u_2 v_2)$. Note that

$$g_3^1(x,0,\alpha) \in \ker(M_3^1) = \operatorname{span}\left\{ \begin{pmatrix} x_1^2 x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1\alpha^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1 x_2^2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2\alpha^2 \end{pmatrix} \right\}.$$

However, the terms $O(|x|\alpha^2)$ are irrelevant to determine the generic Hopf bifurcation. Hence, we need only to compute the coefficients of $\binom{x_1^2x_2}{0}$ and $\binom{0}{x_1x_2^2}$. Notice that

$$\frac{1}{3!} g_3^1(x,0,\alpha) = \frac{1}{3!} \operatorname{Proj}_{\ker(M_3^1)} \tilde{f}_3^1(x,0,\alpha) = \frac{1}{3!} \operatorname{Proj}_s \tilde{f}_3^1(x,0,0) + O(|x|\alpha^2),$$

where

$$s := \operatorname{span}\left\{ \begin{pmatrix} x_1^2 x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1 x_2^2 \end{pmatrix} \right\}$$

and the term $\tilde{f}_3^1(x,0,0)$ is given by

$$\tilde{f}_3^1(x,0,0) = f_3^1(x,0,0) + \frac{3}{2}[(D_x f_2^1)U_2^1 - (D_x U_2^1)g_2^1]_{(x,0,0)} + \frac{3}{2}[(D_y f_2^1)h]_{(x,0,0)}.$$

Now we compute $\frac{1}{3!} g_3^1(x,0,\alpha)$ step by step.

Step 1. We compute $\operatorname{Proj}_s[(D_x f_2^1)U_2^1]_{(x,0,0)}$. Following [13], we take

$$U_2^1(x,0) = (M_2^1)^{-1} P_{I,2}^1 f_2^1(x,0,0).$$

We have from (2.21) that

$$f_2^1(x,0,0) = \begin{pmatrix} a_{20}x_1^2 + 2a_{11}x_1x_2 + a_{02}x_2^2 \\ \bar{a}_{02}x_1^2 + 2\bar{a}_{11}x_1x_2 + \bar{a}_{20}x_2^2 \end{pmatrix}.$$

According to the definition of M_2^1, the equation $M_2^1 U_2^1(x,0) = f_2^1(x,0,0)$ can be written as the following differential equations:

$$\begin{cases} x_1 \dfrac{\partial u_1}{\partial x_1} - x_2 \dfrac{\partial u_1}{\partial x_2} - u_1 = \dfrac{1}{i\sigma_k}(a_{20}x_1^2 + 2a_{11}x_1x_2 + a_{02}x_2^2), \\ x_1 \dfrac{\partial u_2}{\partial x_1} - x_2 \dfrac{\partial u_2}{\partial x_2} + u_2 = \dfrac{1}{i\sigma_k}(\bar{a}_{02}x_1^2 + 2\bar{a}_{11}x_1x_2 + \bar{a}_{20}x_2^2). \end{cases}$$

A straightforward calculation shows that

$$U_2^1(x,0) = \begin{pmatrix} \dfrac{1}{i\sigma_k}\left(a_{20}x_1^2 - 2a_{11}x_1x_2 - \dfrac{1}{3}a_{02}x_2^2\right) \\ \dfrac{1}{i\sigma_k}\left(\dfrac{1}{3}\bar{a}_{02}x_1^2 + 2\bar{a}_{11}x_1x_2 - \bar{a}_{20}x_2^2\right) \end{pmatrix}.$$

Consequently,

$$\operatorname{Proj}_s[(D_x f_2^1)U_2^1]_{(x,0,0)} = \begin{pmatrix} \dfrac{2}{i\sigma_k}\left(\dfrac{1}{3}|a_{02}|^2 + 2|a_{11}|^2 - a_{11}a_{20}\right) x_1^2 x_2 \\ \dfrac{-2}{i\sigma_k}\left(\dfrac{1}{3}|a_{02}|^2 + 2|a_{11}|^2 - \bar{a}_{11}\bar{a}_{20}\right) x_1 x_2^2 \end{pmatrix}.$$

Step 2. We compute $\mathrm{Proj}_s[(D_xU_2^1)g_2^1]_{(x,0,0)}$. From (2.23), we know that $g_2^1(x,0,0) = 0$. Thus, $[(D_xU_2^1)g_2^1]_{(x,0,0)} = 0$.

Step 3. We compute $\mathrm{Proj}_s[(D_yf_2^1)h]_{(x,0,0)}$, where $h = (h^1,h^2)^T$ is a second degree homogeneous polynomial in (x_1,x_2,α) with coefficients in Q^1, which has the form

$$h = h(x_1,x_2,\alpha) = h_{110}x_1x_2 + h_{101}x_1\alpha + h_{011}x_2\alpha + h_{200}x_1^2 + h_{020}x_2^2 + h_{002}\alpha^2,$$

and $h = h(x_1,x_2,\alpha)$ is the unique solution in $V_2^3(Q^1)$ of the equation

$$(M_2^2h)(x,\alpha) = (I-\pi)X_0[2L(\alpha)(\Phi x) + F_2(\Phi x,\tau_k)].$$

Since

$$\begin{aligned}(M_2^2h)(x,\alpha) &= D_xh(x,\alpha)Bx - A_{Q^1}(h(x,\alpha))\\ &= D_xh(x,\alpha)Bx - \dot{h}(x,\alpha) - X_0[L(\tau_k)(h(x,\alpha)) - \dot{h}(x,\alpha)(0)]\\ &= (I-\pi)X_0[2L(\alpha)(\Phi x) + F_2(\Phi x,\tau_k)];\end{aligned}$$

thus, $h = h(x,0)(\theta)$ can be evaluated by the system

$$\dot{h}(x) - D_xh(x)Bx = \Phi\Psi(0)F_2(\Phi x,\tau_k), \tag{2.24}$$

$$\dot{h}(x)(0) - L(\tau_k)(h(x)) = F_2(\Phi x,\tau_k), \tag{2.25}$$

where $\dot{h}$ denotes the derivative of $h(x)(\theta)$ with respect to θ. From (2.20), we obtain

$$\begin{aligned}f_2^1(x,y,0) &= \Psi(0)F_2(\Phi x + y,\tau_k)\\ &= \begin{pmatrix} u_1 & u_2\\ \bar{u}_1 & \bar{u}_2\end{pmatrix}\tau_k\begin{pmatrix} f_{20}^{(1)}p_1^2 + 2f_{11}^{(1)}p_1p_2\\ f_{20}^{(2)}l_1^2 + 2f_{11}^{(2)}l_1p_2\end{pmatrix},\end{aligned}$$

where

$$p_1 = x_1 + x_2 + y_1(0), \quad p_2 = v_2x_1 + \bar{v}_2x_2 + y_2(0), \quad l_1 = e^{-i\sigma_k}x_1 + e^{i\sigma_k}x_2 + y_1(-1).$$

Hence, we have

$$[(D_yf_2^1)h]_{(x,0,0)} = \tau_k\begin{pmatrix} u^T\begin{pmatrix} 2f_{20}^{(1)}p_1'h^1(0) + 2f_{11}^{(1)}p_2'h^1(0) + 2f_{11}^{(1)}p_1'h^2(0)\\ 2f_{20}^{(2)}l_1'h^1(-1) + 2f_{11}^{(2)}p_2'h^1(0) + 2f_{11}^{(2)}l_1'h^2(-1)\end{pmatrix}\\ \bar{u}^T\begin{pmatrix} 2f_{20}^{(1)}p_1'h^1(0) + 2f_{11}^{(1)}p_2'h^1(0) + 2f_{11}^{(1)}p_1'h^2(0)\\ 2f_{20}^{(2)}l_1'h^1(-1) + 2f_{11}^{(2)}p_2'h^1(0) + 2f_{11}^{(2)}l_1'h^2(-1)\end{pmatrix}\end{pmatrix},$$

where

$$p_1' = x_1 + x_2, \quad p_2' = v_2x_1 + \bar{v}_2x_2, \quad l_1' = e^{-i\sigma_k}x_1 + e^{i\sigma_k}x_2. \tag{2.26}$$

Thus,

$$\mathrm{Proj}_s[(D_yf_2^1)h]_{(x,0,0)} = \begin{pmatrix} 2c_3x_1^2x_2\\ 2\bar{c}_3x_1x_2^2\end{pmatrix},$$

where

$$c_3 = u^T\tau_k\begin{pmatrix} ah_{110}^1(0) + \bar{a}h_{200}^1(0) + bh_{110}^2(0) + \bar{b}h_{200}^2(0)\\ ch_{110}^1(-1) + \bar{c}h_{200}^1(-1) + dh_{110}^2(-1) + \bar{d}h_{200}^2(-1) + eh_{110}^1(0) + \bar{e}h_{200}^1(0)\end{pmatrix}$$

and

$$a = f_{20}^{(1)} + f_{11}^{(1)} v_2, \quad b = f_{11}^{(1)}, \quad c = f_{20}^{(2)} e^{-i\sigma_k}, \quad d = f_{11}^{(2)} e^{-i\sigma_k}, \quad e = f_{11}^{(2)} v_2.$$

In the following, we compute $h_{110}(\theta)$ and $h_{200}(\theta)$. Combining (2.24) with (2.25), we know that $h_{110} = (h_{110}^1, h_{110}^2)^T$ and $h_{200} = (h_{200}^1, h_{200}^2)^T$ are, respectively, the solution of the following two systems:

$$\begin{cases} \dot{h}_{110} = (\Phi_1 \quad \Phi_2) \begin{pmatrix} 2a_{11} \\ 2\bar{a}_{11} \end{pmatrix}, \\ \dot{h}_{110}(0) - L(\tau_k)(h_{110}) = \tau_k \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \end{cases} \tag{2.27}$$

and

$$\begin{cases} \dot{h}_{200} - 2i\sigma_k h_{200} = (\Phi_1 \quad \Phi_2) \begin{pmatrix} a_{20} \\ \bar{a}_{02} \end{pmatrix}, \\ \dot{h}_{200}(0) - L(\tau_k)(h_{200}) = \tau_k \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}, \end{cases} \tag{2.28}$$

where

$$\begin{aligned} a_1 &= 2f_{20}^{(1)} + 2(v_2 + \bar{v}_2) f_{11}^{(1)}, \quad b_1 = 2f_{20}^{(2)} + 2f_{11}^{(2)} (e^{-i\sigma_k} \bar{v}_2 + e^{i\sigma_k} v_2), \\ a_2 &= f_{20}^{(1)} + 2f_{11}^{(1)} v_2, \quad b_2 = f_{20}^{(2)} e^{-2i\sigma_k} + 2f_{11}^{(2)} v_2 e^{-i\sigma_k}. \end{aligned}$$

Solving (2.27) and (2.28), we obtain

$$h_{110}(\theta) = \frac{2}{i\sigma_k} (a_{11} e^{i\sigma_k \theta} v - \bar{a}_{11} e^{-i\sigma_k \theta} \bar{v}) + c_1 \tag{2.29}$$

and

$$h_{200}(\theta) = -\frac{1}{i\sigma_k} \left(a_{20} e^{i\sigma_k \theta} v + \frac{1}{3} \bar{a}_{02} e^{-i\sigma_k \theta} \bar{v} \right) + e^{2i\sigma_k \theta} c_2, \tag{2.30}$$

where $c_1 = (c_1^{(1)}, c_1^{(2)})^T$, $c_2 = (c_2^{(1)}, c_2^{(2)})^T$, and

$$\begin{aligned} c_1^{(1)} &= -\frac{b_1}{\beta_1}, \quad c_1^{(2)} = \frac{\alpha_1 b_1 - \beta_1 a_1}{\alpha_2 \beta_1}, \\ c_2^{(1)} &= \frac{2i\sigma_k \tau_k a_2 + \tau_k^2 \alpha_2 b_2}{2i\sigma_k (2i\sigma_k - \tau_k \alpha_1) - \tau_k^2 \alpha_2 \beta_1 e^{-2i\sigma_k}}, \\ c_2^{(2)} &= \frac{\tau_k^2 a_2 \beta_1 e^{-2i\sigma_k} + 2i\sigma_k \tau_k b_2 - \tau_k^2 \alpha_1 b_2}{2i\sigma_k (2i\sigma_k - \tau_k \alpha_1) - \tau_k^2 \alpha_2 \beta_1 e^{-2i\sigma_k}}. \end{aligned}$$

Step 4. We compute $\mathrm{Proj}_s f_3^1(x, 0, 0)$. From (2.18), $f_3^1(x, 0, 0)$ is given by

$$f_3^1(x, 0, 0) = \Psi(0) F_3(\Phi x, \tau_k),$$

where $F_3(\Phi x, \tau_k)$ is defined in (2.15). Thus, F_3 can be computed as follows:

$$F_3(\Phi x, \tau_k) = \tau_k \begin{pmatrix} f_{30}^{(1)} p_1'^3 + 3f_{21}^{(1)} p_1'^2 p_2' \\ f_{30}^{(2)} l_1'^3 + 3f_{21}^{(2)} l_1'^2 p_2' \end{pmatrix},$$

where $p_1^{'}$, $p_2^{'}$, $l_1^{'}$ are given in (2.26). Therefore, we obtain

$$\operatorname{Proj}_s f_3^1(x,0,0) = \begin{pmatrix} 3a_{21}x_1^2x_2 \\ 3\bar{a}_{21}x_1x_2^2 \end{pmatrix},$$

where

$$a_{21} = \tau_k u_1(f_{30}^{(1)} + f_{21}^{(1)}\bar{v}_2 + 2f_{21}^{(1)}v_2) + \tau_k u_2(f_{30}^{(2)}e^{-i\sigma_k} + f_{21}^{(2)}e^{-2i\sigma_k}\bar{v}_2 + 2f_{21}^{(2)}v_2).$$

Summarizing Steps 1–4, we obtain

$$\frac{1}{3!}g_3^1(x,0,0) = \begin{pmatrix} A_3x_1^2x_2 \\ \bar{A}_3x_1x_2^2 \end{pmatrix},$$

where

$$A_3 = \frac{1}{2i\sigma_k}\left(\frac{1}{3}|a_{02}|^2 + 2|a_{11}|^2 - a_{11}a_{20}\right) + \frac{1}{2}(a_{21} + c_3). \tag{2.31}$$

In consequence, the normal form of (2.19) has the form

$$\begin{aligned} \dot{x} &= Bx + \frac{1}{2}g_2^1(x,0,\alpha) + \frac{1}{3!}g_3^1(x,0,\alpha) + h.o.t. \\ &= Bx + \begin{pmatrix} A_1x_1\alpha \\ \bar{A}_1x_2\alpha \end{pmatrix} + \begin{pmatrix} A_3x_1^2x_2 \\ \bar{A}_3x_1x_2^2 \end{pmatrix} + O(|x|\alpha^2 + |x|^4). \end{aligned}$$

The normal form (2.19) relative to P can be written in real coordinates (w_1, w_2) through the change of variables $x_1 = w_1 - iw_2$, $x_2 = w_1 + iw_2$. Followed by the use of polar coordinates (ρ, ξ), $w_1 = \rho\cos\xi$, $w_2 = \rho\sin\xi$, this formal form becomes

$$\begin{cases} \dot{\rho} = k_1\alpha\rho + k_2\rho^3 + O(\alpha^2\rho + |(\rho,\alpha)|^4), \\ \dot{\xi} = -\sigma_k + O(|(\rho,\alpha)|), \end{cases} \tag{2.32}$$

where $k_1 = \operatorname{Re} A_1$, $\quad k_2 = \operatorname{Re} A_3$. Following [7], we know that the sign of k_1k_2 determines the direction of the bifurcation and the sign of k_2 determines the stability of the nontrivial periodic solution bifurcating from Hopf bifurcation. Therefore, summarizing the above discussion, we have the following theorem.

THEOREM 2.5. *The flow of* (2.16) *on the center manifold of the origin at* $\tau = \tau_k$ *is given by* (2.32). *Hopf bifurcation is supercritical if* $k_1k_2 < 0$ *and subcritical if* $k_1k_2 > 0$. *Moreover, the nontrivial periodic solution is stable if* $k_2 < 0$ *and unstable if* $k_2 > 0$.

As an example, we consider system (1.3) with $A = 1$, $D = \frac{2}{3}$, $H = \frac{1}{20}$, $K = 4$, and $r = \frac{1}{2}$; that is,

$$\begin{cases} \dot{x} = \frac{1}{2}x\left(1 - \frac{x}{4}\right) - \frac{xy}{1+x} - \frac{1}{20}, \\ \dot{y} = y\left(-\frac{2}{3} + \frac{x(t-\tau)}{1+x(t-\tau)}\right). \end{cases} \tag{2.33}$$

In this case, system (2.33) has a unique positive equilibrium $(x_0, y_0) = (2, \frac{27}{40})$ and conditions (2.1) and (2.6) are satisfied. Thus, the equilibrium (x_0, y_0) is asymptotically stable when $\tau \in [0, \tau_0)$, where $\tau_0 = 1.5279$. By means of the software Maple, we

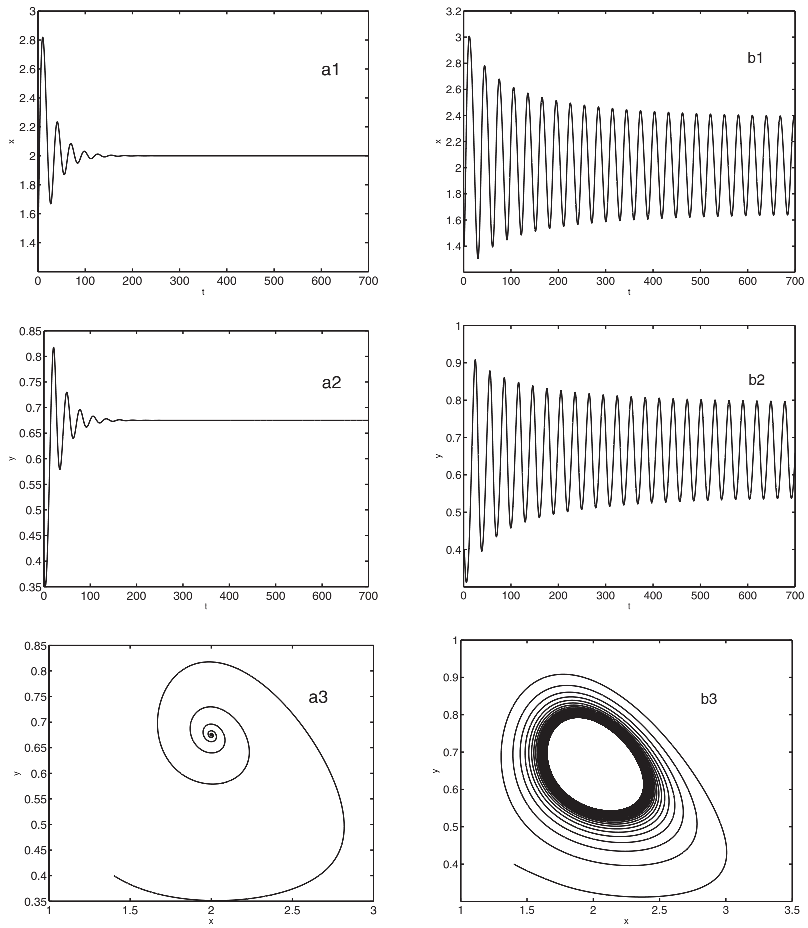

FIG. 1. *The solution trajectories and phase portraits of the system* (2.33) *before* (a1–a3) *and after* (b1–b3) *Hopf bifurcation. In* (a1)–(a3), $\tau = 1.45$*; in* (b1)–(b3), $\tau = 1.6$.

can compute the following values:

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -0.1125 - 0.3261i \end{pmatrix}, u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0.4729 + 0.0029i \\ -0.0088 + 1.4501i \end{pmatrix}$$

and

$$\begin{aligned} a_{20} &= -0.0602 - 0.1399i, \quad a_{11} = -0.1258 - 0.1115i, \\ a_{02} &= -0.1904 - 0.2455i, \quad a_{21} = -0.0722 + 0.1053i, \\ c_3 &= 0.1306 + 0.1172i. \end{aligned}$$

Therefore, from (2.22) and (2.31), we can compute

$$A_1 = 0.0342 + 0.2058i, \quad A_3 = -0.0074 - 0.0343i.$$

Thus, $k_1 = 0.0342$, $k_2 = -0.0074$. By Theorem 2.5, we know that there is a supercritical Hopf bifurcation for (2.33) at $\tau = \tau_0$ and the nontrivial periodic solution associated with Hopf bifurcation is stable. The numerical simulations are depicted in Figure 1.

Remark 2.6. Martin and Ruan [25] studied the Hopf bifurcation in the following predator-prey system with prey harvesting and delayed prey specific growth:

$$\begin{cases} \dot{x}(t) = rx(t)(1 - \dfrac{x(t-\tau)}{K}) - \dfrac{x(t)y(t)}{A + x(t)} - H, \\ \dot{y}(t) = y(t)(-D + \dfrac{x(t)}{A + x(t)}). \end{cases} \tag{2.34}$$

Similarly, as in Theorem 2.5 we can determine the direction of the Hopf bifurcation in system (2.34).

3. Predator harvesting. Following Xiao and Ruan [36], we know that system (1.4) has a unique interior equilibrium $E = (x_0, y_0)$ provided that

$$\left(1 - D + \frac{AD}{K}\right)^2 - 4\frac{(1-D)(ADr + H)}{Kr} = 0 \quad \text{and} \quad K > \frac{AD}{1-D} \tag{3.1}$$

hold and (x_0, y_0) is given by $x_0 = \frac{K(1-D)+AD}{2(1-D)}$ and $y_0 = r(1-\frac{x_0}{K})(A+x_0)$. We assume throughout this section that $0 < D < 1$.

We translate the equilibrium (x_0, y_0) of system (1.4) to the origin. Setting $z_1(t) = x(t) - x_0, z_2(t) = y(t) - y_0$, system (1.4) can be written as the following system:

$$\begin{cases} \dot{z}_1(t) = \alpha_1 z_1(t) + \alpha_2 z_2(t) + \displaystyle\sum_{i+j\geq 2} \frac{1}{i!j!} f_{ij}^{(1)} z_1^i(t) z_2^j(t), \\ \dot{z}_2(t) = \beta_1 z_1(t-\tau) + \beta_2 z_2(t) + \displaystyle\sum_{i+j\geq 2} \frac{1}{i!j!} f_{ij}^{(2)} z_1^i(t-\tau) z_2^j(t), \end{cases} \tag{3.2}$$

where

$$\begin{aligned} &\alpha_1 = r - \frac{2x_0 r}{K} - \frac{Ay_0}{(A+x_0)^2}, \quad \alpha_2 = -\frac{x_0}{A+x_0}, \\ &\beta_1 = \frac{Ay_0}{(A+x_0)^2}, \quad \beta_2 = -D + \frac{x_0}{A+x_0}, \\ &f_{ij}^{(1)} = \left.\frac{\partial^{i+j} f^{(1)}}{\partial x^i \partial y^j}\right|_{(x_0,y_0)}, \quad f_{ij}^{(2)} = \left.\frac{\partial^{i+j} f^{(2)}}{\partial x^i \partial y^j}\right|_{(x_0,y_0)}, \quad i, j \geq 0, \\ &f^{(1)} = rx\left(1 - \frac{x}{K}\right) - \frac{xy}{A+x}, \quad f^{(2)} = y\left(-D + \frac{x}{A+x}\right) - H. \end{aligned} \tag{3.3}$$

Consider the linearized system of (3.2) at the zero equilibrium

$$\begin{cases} \dot{z}_1(t) = \alpha_1 z_1(t) + \alpha_2 z_2(t), \\ \dot{z}_2(t) = \beta_1 z_1(t-\tau) + \beta_2 z_2(t). \end{cases} \tag{3.4}$$

The characteristic equation for system (3.4) takes the form

$$\lambda^2 - \lambda(\alpha_1 + \beta_2) + \alpha_1\beta_2 - \alpha_2\beta_1 e^{-\lambda\tau} = 0. \tag{3.5}$$

In fact, we have $\alpha_1\beta_2 - \alpha_2\beta_1 = 0$ (the proof can be found in the Appendix). We can easily see that (3.5) has two zero eigenvalues and no other eigenvalues on the imaginary axis if and only if $\tau = \tau_0 = \frac{\alpha_1+\beta_2}{\alpha_2\beta_1}$ and $\tau_0 \neq \sqrt{\frac{2}{\alpha_2\beta_1}}$. Because $\tau_0 > 0$ and $\alpha_2\beta_1 < 0$, we know that $\tau_0 \neq \sqrt{\frac{2}{\alpha_2\beta_1}}$.

Normalizing the delay τ in system (1.4) by the time-scaling $t \to t/\tau$, system (1.4) is transformed into

$$\begin{cases} \dot{x}(t) = \tau \left[rx(t)\left(1 - \dfrac{x(t)}{K}\right) - \dfrac{x(t)y(t)}{A + x(t)} \right], \\ \dot{y}(t) = \tau y(t) \left[-D + \dfrac{x(t-1)}{A + x(t-1)} \right] - \tau H. \end{cases} \tag{3.6}$$

Setting $z_1(t) = x(t) - x_0, z_2(t) = y(t) - y_0$, system (3.6) can be rewritten as functional differential equations in $C := C([-1, 0], \mathbb{R}^2)$

$$\begin{cases} \dot{z}_1(t) = \tau \left[\alpha_1 z_1(t) + \alpha_2 z_2(t) + \displaystyle\sum_{i+j\geq 2} \frac{1}{i!j!} f_{ij}^{(1)} z_1^i(t) z_2^j(t) \right], \\ \dot{z}_2(t) = \tau \left[\beta_1 z_1(t-1) + \beta_2 z_2(t) + \displaystyle\sum_{i+j\geq 2} \frac{1}{i!j!} f_{ij}^{(2)} z_1^i(t-1) z_2^j(t) \right]. \end{cases} \tag{3.7}$$

Let $z = (z_1, z_2)^T$. System (3.7) is written as

$$\dot{z}(t) = \tau L(z_t) + \tau F(z_t), \tag{3.8}$$

where $L : C \to \mathbb{R}^2$, $F : C \to \mathbb{R}^2$ are given by

$$L(\varphi) = \begin{pmatrix} \alpha_1 \varphi_1(0) + \alpha_2 \varphi_2(0) \\ \beta_1 \varphi_1(-1) + \beta_2 \varphi_2(0) \end{pmatrix}, \quad F(\varphi) = \begin{pmatrix} \displaystyle\sum_{i+j\geq 2} \frac{1}{i!j!} f_{ij}^{(1)} \varphi_1^i(0) \varphi_2^j(0) \\ \displaystyle\sum_{i+j\geq 2} \frac{1}{i!j!} f_{ij}^{(2)} \varphi_1^i(-1) \varphi_2^j(0) \end{pmatrix} \tag{3.9}$$

and $\varphi = \mathrm{col}(\varphi_1, \varphi_2)$. We consider the formal Taylor expansion of F

$$F(\varphi) = \sum_{j\geq 2} \frac{1}{j!} F_j(\varphi), \quad \varphi \in C \tag{3.10}$$

and define $L_0 = \tau_0 L$. $L_0(\varphi)$ is a continuous linear function. Therefore, there exists a 2×2 matrix function $\eta(\theta), -1 \leq \theta \leq 0$, whose elements are of bounded variation such that

$$L_0(\varphi) = \int_{-1}^{0} d\eta(\theta)\varphi(\theta).$$

Choose

$$\eta(\theta) = \tau_0 \begin{pmatrix} \alpha_1 & \alpha_2 \\ 0 & \beta_2 \end{pmatrix} H(\theta) + \tau_0 \begin{pmatrix} 0 & 0 \\ -\beta_1 & 0 \end{pmatrix} H(\theta + 1).$$

Let A_0 be the infinitesimal generator corresponding to $\dot{z}(t) = L_0(z_t)$. Then A_0 has two zero characteristic roots. Define $\Lambda = \{0\}$ and denote by P the invariant space of A_0 corresponding to Λ, where the dimension of P equals to 2. Let $\Phi = (\Phi_1, \Phi_2)$ and $\Psi = \mathrm{col}(\Psi_1, \Psi_2)$ be the bases for P and P^*, the adjoint space of P, respectively. We note that $\dot{\Phi} = \Phi B$, thus, for $\Phi = (\Phi_1, \Phi_2)$, and the matrix B is given by

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We know that $\Phi_i(0)$, $\Psi_i(0)(i=1,2)$ satisfy $\dot{\Phi}_i(0)=\int_{-1}^{0}d\eta(\theta)\Phi_i(\theta)$, $-\dot{\Psi}_i(0)=\int_{-1}^{0}\Psi_i(-\theta)d\eta(\theta)$, and $(\Psi,\Phi)=((\Psi_j,\Phi_i),i,j=1,2)=I_2$. Accordingly, we can obtain

$$\Phi(\theta)=\begin{pmatrix} -\alpha_2 & -\alpha_2\theta-\alpha_2 \\ \alpha_1 & \alpha_1\theta+\alpha_1-\tau_0^{-1}\end{pmatrix},$$
$$\Psi(s)=\begin{pmatrix} -es+g & -fs+h \\ e & f\end{pmatrix},\quad -1\le\theta\le 0,\quad 0\le s\le 1,$$

where e, f, g, h satisfy the following equations:

$$\begin{cases} \alpha_2 e+\beta_2 f=0,\\ \tau_0\alpha_2 g+\tau_0\beta_2 h=f,\\ h(\alpha_1-\tau_0\alpha_2\beta_1)-\alpha_2 g+\dfrac{1}{2}\tau_0\alpha_2\beta_1 f=1,\\ h\left(\alpha_1-\tau_0^{-1}-\dfrac{1}{2}\tau_0\beta_1\alpha_2\right)-\alpha_2 g+\dfrac{1}{3}\tau_0\beta_1\alpha_2 f=0,\\ -e\alpha_2+f\left(\alpha_1-\tau_0^{-1}-\dfrac{1}{2}\tau_0\beta_1\alpha_2\right)=1.\end{cases}\tag{3.11}$$

According to [14], we obtain that the normal form for (3.8) is as follows

$$\begin{cases}\dot{x}_1=x_2+O(|(x_1,x_2)|^3),\\ \dot{x}_2=B_1x_1^2+B_2x_1x_2+O(|(x_1,x_2)|^3),\end{cases}$$

where

$$B_1=\frac{1}{2}\tau\alpha_2^2(ef_{20}^{(1)}+ff_{20}^{(2)})-\tau\alpha_1\alpha_2(ef_{11}^{(1)}+ff_{11}^{(2)}),$$
$$\begin{aligned}B_2=\tau\alpha_2^2(gf_{20}^{(1)}+hf_{20}^{(2)}+ef_{20}^{(1)})-\tau\alpha_1\alpha_2(2gf_{11}^{(1)}+2ef_{11}^{(1)}\\ +2hf_{11}^{(2)}+ff_{11}^{(2)})+\tau\tau_0^{-1}\alpha_2(ef_{11}^{(1)}+ff_{11}^{(2)}),\end{aligned}$$

and α_i, $f_{ij}^{(1)}$, $f_{ij}^{(2)}(i,j=1,2)$ are given in (3.3). In fact, the Bogdanov–Takens bifurcation for a planar system with two discrete delays have been studied in Faria [12]. Applying the formula in [12], we can also derive the above normal form. Therefore, we have the following result.

THEOREM 3.1. *Suppose that* (3.1) *holds. Then the equilibrium* E *of* (1.4) *is a Bogdanov–Takens singularity for* $\tau=\tau_0=\frac{\alpha_1+\beta_2}{\alpha_2\beta_1}$.

We know that system (1.4) undergoes Bogdanov–Takens bifurcation as τ crosses the critical value τ_0. Next, we are interested in determining a versal unfolding for system (1.4) with a Bogdanov–Takens singularity. Note that $\alpha_1\beta_2-\alpha_2\beta_1=0$ (see the Appendix). We introduce two bifurcation parameters $\mu=(\mu_1,\mu_2)$ by setting $\tau=\tau_0+\mu_1$, $\beta_1=\frac{\alpha_1\beta_2}{\alpha_2}+\mu_2$. System (3.8) is rewritten as

$$\dot{z}(t)=L_0(z_t)+L_1(\mu)z_t+\tilde{F}(z_t,\mu),\tag{3.12}$$

where

$$L_1(\mu)\varphi=\mu_1L(\varphi)+\tau_0\mu_2\begin{pmatrix}0\\ \varphi_1(-1)\end{pmatrix},\tag{3.13}$$

$$\tilde{F}(\varphi,\mu) = \mu_1\mu_2 \begin{pmatrix} 0 \\ \varphi_1(-1) \end{pmatrix} + (\tau_0+\mu_1)F(\varphi) \quad \text{for} \quad \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}. \tag{3.14}$$

In the phase space $BC = P \oplus \ker\pi$, we decompose z_t in the form $z_t = \Phi x + y$. Thus, the system (3.12) is equivalent to

$$\begin{cases} \dot{x} = Bx + \Psi(0)[L_1(\mu)(\Phi x + y) + \tilde{F}(\Phi x + y, \mu)], \\ \dfrac{d}{dt}y = A_{Q^1}y + (I-\pi)X_0[L_1(\mu)(\Phi x + y) + \tilde{F}(\Phi x + y,\mu)], \end{cases} \tag{3.15}$$

where $L_1(\mu)\varphi$, $\tilde{F}(\varphi,\mu)$ are given by (3.13) and (3.14) and $\varphi =$col(φ_1,φ_2). Define (3.16)

$$\Psi(0)[L_1(\mu)(\Phi x + y) + \tilde{F}(\Phi x + y,\mu)] = \frac{1}{2}f_2^1(x,y,\mu) + \frac{1}{3!}f_3^1(x,y,\mu) + h.o.t,$$
$$(I-\pi)X_0[L_1(\mu)(\Phi x + y) + \tilde{F}(\Phi x + y,\mu)] = \frac{1}{2}f_2^2(x,y,\mu) + \frac{1}{3!}f_3^2(x,y,\mu) + h.o.t..$$

Therefore, system (3.15) becomes

$$\begin{cases} \dot{x} = Bx + \dfrac{1}{2}f_2^1(x,y,\mu) + h.o.t., \\ \dfrac{d}{dt}y = A_{Q^1}y + \dfrac{1}{2}f_2^2(x,y,\mu) + h.o.t., \end{cases} \tag{3.17}$$

where $f_2^1(x,y,\mu)$ and $f_2^2(x,y,\mu)$ are homogeneous polynomials in (x,y,μ) of degree 2, with coefficients in $\mathbb{R}^2$ and $\ker\pi$, respectively. Thus, the normal form for (3.12) on the center manifold of the origin has the form

$$\dot{x} = Bx + \frac{1}{2}g_2^1(x,0,\mu) + h.o.t., \tag{3.18}$$

where g_2^1 is the second-order term in x,μ and determined by

$$\frac{1}{2}g_2^1(x,0,\mu) = \frac{1}{2}\operatorname{Proj}_{\operatorname{Im}(M_2^1)^c} f_2^1(x,0,\mu),$$

where $\operatorname{Im}(M_2^1)^c$ is a complementary space of $\operatorname{Im}(M_2^1)$ in $V_2^4(\mathbb{R}^2)$ and $V_2^4(\mathbb{R}^2)$ is the space of homogeneous polynomials of degree 2 in the variables (x_1,x_2,μ_1,μ_2). According to (3.13), (3.14), and (3.16), we know that

$$\begin{aligned} f_2^1(x,0,\mu) &= \Psi(0)[2L_1(\mu)(\Phi x) + \tau_0 F_2(\Phi x)] \\ &= \Psi(0)\left[\begin{pmatrix} -2\alpha_2\tau_0^{-1}\mu_1 x_2 \\ 2\beta_2(\alpha_1 - \tau_0^{-1})\mu_1 x_2 - 2\tau_0\alpha_2\mu_2 x_1 \end{pmatrix} + \tau_0 F_2(\Phi x)\right], \end{aligned}$$

where F_2 is given in (3.9) and (3.10). We consider the canonical basis of $V_2^4(\mathbb{R}^2)$

$$\begin{pmatrix} x_1^2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_2^2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1\mu_i \\ 0 \end{pmatrix}, \begin{pmatrix} x_2\mu_i \\ 0 \end{pmatrix}, \begin{pmatrix} \mu_1^2 \\ 0 \end{pmatrix}, \begin{pmatrix} \mu_1\mu_2 \\ 0 \end{pmatrix}, \begin{pmatrix} \mu_2^2 \\ 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 \\ x_1^2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1x_2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2^2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1\mu_i \end{pmatrix}, \begin{pmatrix} 0 \\ x_2\mu_i \end{pmatrix}, \begin{pmatrix} 0 \\ \mu_1^2 \end{pmatrix}, \begin{pmatrix} 0 \\ \mu_1\mu_2 \end{pmatrix},$$
$$\begin{pmatrix} 0 \\ \mu_2^2 \end{pmatrix}, (i=1,2).$$

The operators M_j^1 are defined by

$$(M_j^1 p)(x) = D_x p(x) Bx - Bp(x), \quad j \geq 2;$$

thus, M_2^1 acting in $V_2^4(\mathbb{R}^2)$ has the following form

$$M_2^1 p(x_1, x_2) = \begin{pmatrix} x_2 \dfrac{\partial p_1}{\partial x_1} - p_2 \\ x_2 \dfrac{\partial p_2}{\partial x_1} \end{pmatrix},$$

where $p \in V_2^4(\mathbb{R}^2)$. The images of the basis under M_2^1 are, respectively,

$$\begin{pmatrix} 2x_1x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_2^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} x_2\mu_i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -x_1^2 \\ 2x_1x_2 \end{pmatrix}, \begin{pmatrix} -x_1x_2 \\ x_2^2 \end{pmatrix}, \begin{pmatrix} -x_2^2 \\ 0 \end{pmatrix}, \begin{pmatrix} -x_1\mu_i \\ x_2\mu_i \end{pmatrix}, \begin{pmatrix} -x_2\mu_i \\ 0 \end{pmatrix},$$
$$\begin{pmatrix} -\mu_1^2 \\ 0 \end{pmatrix}, \begin{pmatrix} -\mu_1\mu_2 \\ 0 \end{pmatrix}, \begin{pmatrix} -\mu_2^2 \\ 0 \end{pmatrix}, \quad (i = 1, 2).$$

A complementary space of $\mathrm{Im}(M_2^1)$ in $V_2^4(\mathbb{R}^2)$ is

$$\mathrm{Im}(M_2^1)^c = \mathrm{span}\left\{ \begin{pmatrix} 0 \\ x_1^2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1x_2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1\mu_1 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1\mu_2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2\mu_1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 \\ x_2\mu_2 \end{pmatrix}, \begin{pmatrix} 0 \\ \mu_1^2 \end{pmatrix}, \begin{pmatrix} 0 \\ \mu_1\mu_2 \end{pmatrix}, \begin{pmatrix} 0 \\ \mu_2^2 \end{pmatrix} \right\}.$$

Following [12], we know that the normal form of (3.12) on the center manifold is given by

$$\begin{cases} \dot{x}_1 = x_2 + h.o.t., \\ \dot{x}_2 = \lambda_1 x_1 + \lambda_2 x_2 + B_1 x_1^2 + B_2 x_1 x_2 + h.o.t., \end{cases} \tag{3.19}$$

where

$$\lambda_1 = -\tau_0 \alpha_2 f \mu_2, \qquad \lambda_2 = \beta_2 \alpha_1 f \mu_1 - \tau_0 \alpha_2 h \mu_2 \tag{3.20}$$

and

$$\begin{aligned} B_1 &= \frac{\tau_0}{2}\left[\alpha_2^2(ef_{20}^{(1)} + ff_{20}^{(2)}) - 2\alpha_1\alpha_2(ef_{11}^{(1)} + ff_{11}^{(2)})\right], \\ B_2 &= \tau_0\left\{\alpha_2^2(gf_{20}^{(1)} + hf_{20}^{(2)}) - 2\alpha_1\alpha_2(gf_{11}^{(1)} + hf_{11}^{(2)}) + \alpha_2^2 e f_{20}^{(1)} \right. \\ &\quad \left. - \alpha_2\left[(2\alpha_1 - \tau_0^{-1}) f_{11}^{(1)} e + (\alpha_1 - \tau_0^{-1}) f_{11}^{(2)} f\right]\right\}. \end{aligned} \tag{3.21}$$

The above arguments imply the following theorem.

THEOREM 3.2. *Let μ_1, μ_2 be defined by $\tau = \tau_0 + \mu_1$, $\beta_1 = \frac{\alpha_1\beta_2}{\alpha_2} + \mu_2$, where $\tau_0 = \frac{\alpha_1+\beta_2}{\alpha_2\beta_1}$. For $E = (x_0, y_0)$ and $\mu_1 = 0$, $\mu_2 = 0$, system* (1.4) *exhibits a Bogdanov–Takens bifurcation. The normal form on the center manifold for* (1.4) *of E is given by* (3.19), *with λ_1, λ_2, B_1, B_2 defined by* (3.20) *and* (3.21).

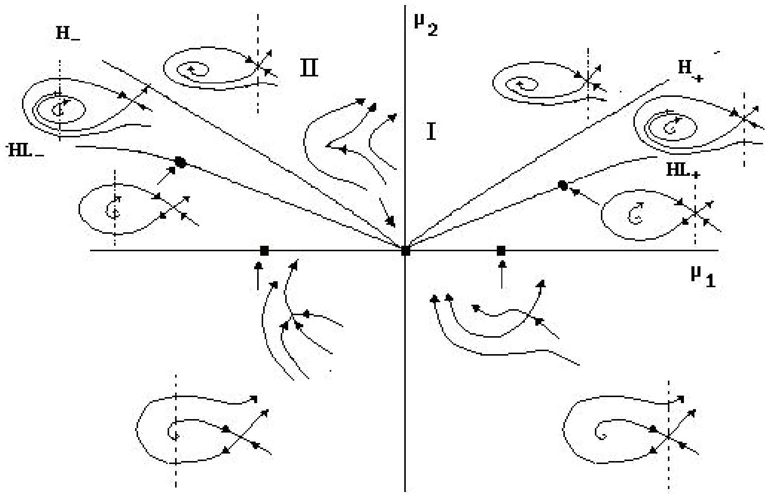

FIG. 2. *The Bogdanov–Takens bifurcation diagram and phase portraits for system* (3.22).

As an example, we consider system (1.4) with $r = 1$, $K = 2$, $A = 1$, $D = \frac{1}{3}$; that is,

$$
\begin{cases} \dot{x}(t) = x(t)\left(1 - \dfrac{x(t)}{2}\right) - \dfrac{x(t)y(t)}{1+x(t)}, \\ \dot{y}(t) = y(t)\left(-\dfrac{1}{3} + \dfrac{x(t-\tau)}{1+x(t-\tau)}\right) - H. \end{cases} \tag{3.22}
$$

According to Theorem 3.1, we know that system (3.22) has a Bogdanov–Takens singularity point $(x_0, y_0) = (\frac{5}{4}, \frac{27}{32})$ when $(H_0, \tau_0) = (\frac{3}{16}, \frac{21}{10})$. We will show that system (3.22) undergoes Bogdanov–Takens bifurcation when τ and H vary in a small neighborhood of τ_0 and H_0. Now we introduce two bifurcation parameters μ_1, μ_2 by setting $\tau = \frac{21}{10} + \mu_1$, $\beta_1 = \frac{\alpha_1\beta_2}{\alpha_2} + \mu_2$; i.e., $H = H_0 + \mu_2$. Following the analysis in this section, we obtain the versal unfolding for system (3.22) as follows:

$$
\begin{cases} \dot{x}_1 = x_2 + h.o.t., \\ \dot{x}_2 = \lambda_1 x_1 + \lambda_2 x_2 + B_1 x_1^2 + B_2 x_1 x_2 + h.o.t., \end{cases} \tag{3.23}
$$

where

$$
\begin{aligned} &\lambda_1 = -2.034602076\mu_2, \quad \lambda_2 = 0.1614763552\mu_1 + 1.919613031\mu_2, \\ &B_1 = 0.3767781622, \quad B_2 = -0.715752274. \end{aligned}
$$

Therefore, we know that system (3.22) exhibits Bogdanov–Takens bifurcation when the parameters μ_1, μ_2 vary in a small neighborhood of the origin. On the lines $H_\pm$, there exists stable Hopf bifurcation, while there exists curves $HL_\pm$ corresponding to homoclinic bifurcation. The bifurcation diagram is depicted in Figure 2.

Remark 3.3. Similarly, we can study Bogdanov–Takens bifurcation in the following predator-prey model with predator harvesting and delayed prey specific growth

$$
\begin{cases} \dot{x}(t) = rx(t)\left(1 - \dfrac{x(t-\tau)}{K}\right) - \dfrac{x(t)y(t)}{A+x(t)}, \\ \dot{y}(t) = y(t)\left(-D + \dfrac{x(t)}{A+x(t)}\right) - H \end{cases} \tag{3.24}
$$

and obtain results similar to Theorems 3.1 and 3.2.

4. Discussion. Predator-prey models play a crucial role in studying the management of renewable resources (Clark [8]). The effect of constant-rate harvesting on the dynamics of predator-prey systems has been investigated by many authors; see, for example, Brauer and Soudack [3, 4, 5], Beddington and Cooke [1], Dai and Tang [11], Hogarth et al. [19], Myerscough et al. [30], Xiao and Jennings [35], and Xiao and Ruan [36]. Very rich and interesting dynamical behaviors, such as the existence of multiple equilibria, existence of Hopf bifurcation, limit cycles, homoclinic loops, and Bogdanov–Takens bifurcations, have been observed. It is also observed that in some cases, before a catastrophic harvest rate is reached the effect of harvesting is to stabilize the equilibrium of the population system.

Martin and Ruan [25] studied the combined effects of prey harvesting and delay on the dynamics of predator-prey systems and focused on three very well-studied delayed predator-prey models. Namely, they considered a generalized Gause-type predator-prey model with prey harvesting and a time delay in the prey specific growth term; a generalized Gause-type predator-prey model with prey harvesting and a time delay in the predator response function is analyzed; and the Wangersky–Cunningham predator-prey model with prey harvesting. It was shown that in the first and third models the time delay could cause not only instability and oscillations but also the switching of stabilities, while the prey harvesting changes only the equilibrium values but not the properties of solutions. In the second model, the time delay induces instability and bifurcation but there is no switching of stabilities; however, increasing the prey harvesting level will help the system to regain its stability. This indicates that the prey harvesting has a stabilizing effect on the dynamics of the model.

In this paper, following the work of Martin and Ruan [25] and Xiao and Ruan [36], we continued studying the combined effects of time delay and constant harvesting on the dynamics of predator-prey systems with Holling type II functional response. Two different types of models have been analyzed; namely, predator-prey systems with delayed predator response and (i) prey harvesting or (ii) predator harvesting. There are two types of bifurcation phenomena: Hopf bifurcation and Bogdanov–Takens bifurcation. More precisely, in the model with prey harvesting there is no bifurcation on the number of positive equilibria; time delay can induce oscillations of both species via Hopf bifurcation. While in the model with predator harvesting, multiple positive equilibria and degenerate equilibria can exist, Bogdanov–Takens bifurcation can occur.

In predator-prey interactions within fisheries systems, it is well known that the reduction of the predator stock level may increase the surplus production of the prey. Harvesting predators becomes controversial (May et al. [26], Flaaten [15], Yodzis [38]). The Bogdanov–Takens bifurcation diagram in the predator-prey models with predator harvesting in section 3 indicates that there are some parameter regions in which both predator and prey species can be driven to extinction. This may provide some explanations for the collapse of the Atlantic cod stocks in the Canadian Grand Banks (Hutchings and Myers [21], Myers et al. [27, 28], Hutchings [20]). Our study demonstrates that appropriate harvesting of predator population is crucial in the long-term survival of both predator and prey species, and in turn the fisheries systems. This is significant and useful in designing fishing policies for the fishery industry (Pauly et al. [31], Myers and Worm [29]).

Fishing is a seasonal activity. It will be very interesting to study how seasonal harvesting affects the existing Hopf and Bogdanov–Takens bifurcations in predator-prey systems with Holling type II functional response. We leave this for future consideration.

Appendix A. In this section, we verify that $\alpha_1\beta_2 - \alpha_2\beta_1 = 0$. Since

$$\begin{aligned}
\alpha_1\beta_2 - \alpha_2\beta_1 &= \left[r - \frac{2x_0 r}{K} - \frac{Ay_0}{(A+x_0)^2}\right]\left(-D + \frac{x_0}{A+x_0}\right) + \frac{Ax_0y_0}{(A+x_0)^3} \\
&= r\left(1 - \frac{2x_0}{K}\right)\left(-D + \frac{x_0}{A+x_0}\right) + \frac{DAy_0}{(A+x_0)^2} \\
&= r\left(1 - \frac{2x_0}{K}\right)\frac{H}{y_0} + \frac{DAr(1-\frac{x_0}{K})}{A+x_0} \\
&= \left(1 - \frac{2x_0}{K}\right)\frac{H}{(1-\frac{x_0}{K})(A+x_0)} + \frac{DAr(1-\frac{x_0}{K})}{A+x_0} \\
&= \frac{1}{A+x_0}\left[\frac{(1-\frac{2x_0}{K})H}{1-\frac{x_0}{K}} + DAr\left(1 - \frac{x_0}{K}\right)\right] \\
&= \frac{1}{A+x_0}\left[\frac{K-2x_0}{K-x_0}H + DAr\left(\frac{K-x_0}{K}\right)\right] \\
&= \frac{1}{A+x_0}\frac{K(K-2x_0)H + DAr(K-x_0)^2}{K(K-x_0)},
\end{aligned}$$

in order to prove $\alpha_1\beta_2 - \alpha_2\beta_1 = 0$, we need only to prove $K(K-2x_0)H + DAr(K-x_0)^2 = 0$. In fact,

$$\begin{aligned}
&K(K-2x_0)H + DAr(K-x_0)^2 \\
&\quad = K\left[K - 2\frac{K(1-D)+AD}{2(1-D)}\right]H + DAr\left[K - \frac{K(1-D)+AD}{2(1-D)}\right]^2 \\
&\quad = -\frac{ADKH}{1-D} + DAr\left[\frac{K}{2} - \frac{AD}{2(1-D)}\right]^2 \\
&\quad = \frac{-4KADH(1-D) + DAr\big[K(1-D) - AD\big]^2}{4(1-D)^2} \\
&\quad = \frac{DArK^2\big[(1-D-\frac{AD}{K})^2 - \frac{4H(1-D)}{Kr}\big]}{4(1-D)^2} \\
&\quad = \frac{DArK^2}{4(1-D)^2}\left[\left(1-D+\frac{AD}{K}\right)^2 - \frac{4ADr(1-D)+4H(1-D)}{Kr}\right] \\
&\quad = \frac{DArK^2}{4(1-D)^2}\left[\left(1-D+\frac{AD}{K}\right)^2 - \frac{4(1-D)(ADr+H)}{Kr}\right] = 0.
\end{aligned}$$

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