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Stability and bifurcation in a neural network model with two delays

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Abstract

A simple neural network model with two delays is considered. Linear stability of the model is investigated by analyzing the associated characteristic transcendental equation. For the case without self-connection, it is found that the Hopf bifurcation occurs when the sum of the two delays varies and passes a sequence of critical values. The stability and direction of the Hopf bifurcation are determined by applying the normal form theory and the center manifold theorem. An example is given and numerical simulations are performed to illustrate the obtained results. ©1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

There has been great interest recently in dynamical characteristics of neural networks or neural nets since Hopfield [1] constructed a simplified neural network model, in which each neuron is represented by a linear circuit consisting of a resistor and a capacitor, and is connected to other neurons via nonlinear sigmoidal activation functions, called transfer functions. Based on the Hopfield neural network model, Marcus and Westervelt [2] argued that the nonlinear sigmoidal activation functions which connected to the other neurons would include discrete delays and proposed the following differential equations with delays:

$$C_i\dot{u}_i(t') = -\frac{1}{R_i}u_i(t') + \sum_{j=1}^{n} T_{ij} f_j(u_j(t'-\tau_j')), \quad i = 1, 2, \dots, n. \tag{1}$$

The variable $u_i(t')$ represents the voltage on the input of the ith neuron. Each neuron is characterized by an input of capacitance C_i, a delay τ_i', and a transfer function f_i. The nonlinear transfer function $f_i(u)$ is sigmoidal. Assume

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that the neurons have the same capacitance, resistance, and transfer function, that is, $C_i = C$, $R_i = R$, and $f_i = f$, then system (1) becomes

$$\dot{u}_i(t) = -u_i(t) + \sum_{j=1}^{n} a_{ij} f(u_j(t-\tau_j)), \quad i = 1, 2, \dots, n \tag{2}$$

after recalling time, delay, and $T_{ij} : t = t'/RC, \tau_j = \tau_j'/RC, a_{ij} = RT_{ij}$. In the case of a single delay when $\tau_j \equiv \tau$, Marcus and Westervelt [2] studied the linear stability of system (2) when $(a_{ij})_{n\times n}$ is a symmetric matrix and discussed stability for three specific network topologies: symmetric rings of neurons, symmetric random networks, and associated memory networks. They found out that the delay can destabilize the network as a whole and create oscillatory behavior .

In the case of multiple delays, the dynamics of system (2) could be more complicated and interesting. For example, Campbell [3] has observed not only Hopf bifurcation but also codimension two and Hopf–Hopf bifurcations in neural networks with multiple delays. Since an exhaustive analysis of the dynamics of large systems such as (2) is difficult, Babcock and Westervelt [4] suggested examining carefully the dynamical behavior of some simple networks. One of the simple networks they studied is the following two-neuron network model with two delays:

$$\begin{cases} \dfrac{\mathrm{d}u_1(t)}{\mathrm{d}t} = -u_1(t) + a_1 \tanh[u_2(t-\tau_2)], \\ \dfrac{\mathrm{d}u_2(t)}{\mathrm{d}t} = -u_2(t) + a_2 \tanh[u_1(t-\tau_1)], \end{cases} \tag{3}$$

where a_1, a_2, τ_1, and τ_2 are positive constants. Babcock and Westervelt showed that system (3) exhibits very interesting and rich dynamics including underdamped ringing transients, stable and unstable limit cycles, etc. Equations similar to (3) have been used by an der Heiden [5] and Willson and Cowan [6] to model the neuron interactions where the delays reflect the finite signal propagation speeds along the dendrites and axons. Gopalsamy and Leung [7] considered system (3) with $\tau_1 = \tau_2$ and u_1 and u_2 denote the activating and inhibiting potentials, respectively. They showed that under certain conditions, the delay induces to a Hopf-type bifurcation. Furthermore, the supercritical property of the Hopf bifurcation and orbitally asymptotically stability of the bifurcation periodic solutions are studied.

Recently, Olien and Bélair [8] investigated system (2) with two delays for $n = 2$, that is,

$$\dot{u}_i(t) = -u_i(t) + \sum_{j=1}^{2} a_{ij} f(u_j(t-\tau_j)), \quad i = 1, 2. \tag{4}$$

They discussed several cases, such as $\tau_1 = \tau_2$, $a_{11} = a_{22} = 0$, etc. They obtained some sufficient conditions for the stability of the stationary point of (4) and showed that (4) may undergo some bifurcations at certain values of the parameters. A similar model representing a single pair of neurons with self-connections has been considered by Destexhe and Gaspard [9]. We refer to Campbell and Shayer [10], Majer and Roy [11], Ruan and Wei [12] and the references therein for related work on two-neuron networks with delays.

Usually it is difficult to analyze differential equations with multiple delays since the characteristic equation is transcendental (see Hale and Verduyn Lunel [13]). For instance, Olien and Bélair [8] derived the following characteristic equation:

$$(\lambda+1)^2 - (\lambda+1)(\alpha_{21}\mathrm{e}^{-\lambda\tau_1} + \alpha_{22}\mathrm{e}^{-\lambda\tau_2}) + (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})\mathrm{e}^{-\lambda(\tau_1+\tau_2)} = 0 \tag{5}$$

in studying system (4). They pointed out that “finding all the parameter values for all the roots of Eq. (5) to have negative real part is hopeless.” It indicates the difficulty in investigating the distribution of the zeros of Eq. (5).

One of the purposes of the present paper is to derive some sufficient conditions which guarantee that all roots of Eq. (5) have negative real parts. Hence, we obtain the stability property for the model (4) with two delays. Our method is based on Rouché's theorem (see Dieudonné [14]) and the technique in Cooke and Grossman [15]. These will appear in Section 2. In Section 3, by regarding $\tau = \tau_1 + \tau_2$ as a parameter, we study the stability of the zero solution and the Hopf bifurcation for the non self-connection networks with two delays. In Section 4, based on the normal form method and the center manifold theory introduced by Hassard, Kazarinoff and Wan [16], we derive the formula for determining the properties of Hopf bifurcation of the non self-connection networks with two delays, such as the direction of Hopf bifurcation, stability of the bifurcating periodic solutions and so on. Finally, in Section 5, we give an example to illustrate the results obtained in Section 4. In particular, we find out that under some conditions, Eq. (3) undergoes a Hopf bifurcation which is supercritical and the bifurcating periodic solutions are orbitally asymptotically stable. Numerical simulations support our observation.

2. Local analysis of a neural network with two delays

Consider the neural network model with two delays:

$$\begin{cases} \dot{u}_1(t) = -u_1(t) + a_{11} f(u_1(t-\tau_1)) + a_{12} f(u_2(t-\tau_2)), \\ \dot{u}_2(t) = -u_2(t) + a_{21} f(u_1(t-\tau_1)) + a_{22} f(u_2(t-\tau_2)). \end{cases} \tag{6}$$

Suppose that $f \in C'(R)$, $f(0) = 0$, and

$$uf(u) > 0 \quad \text{for}\, u \neq 0.$$

It is clear that the origin $(0, 0)$ is a stationary point of Eq. (6).

The linearization of Eq. (6) at the origin $(0, 0)$ is

$$\begin{cases} \dot{u}_1(t) = -u_1(t) + \alpha_{11} u_1(t-\tau_1) + \alpha_{12} u_2(t-\tau_2), \\ \dot{u}_2(t) = -u_2(t) + \alpha_{21} u_1(t-\tau_1) + \alpha_{22} u_2(t-\tau_2), \end{cases} \tag{7}$$

where $\alpha_{ij} = a_{ij} f'(0), \quad i, j = 1, 2$. The associated characteristic equation of (7) is

$$\det \begin{pmatrix} \lambda + 1 - \alpha_{11} e^{-\lambda\tau_1} & -\alpha_{12} e^{-\lambda\tau_2} \\ -\alpha_{21} e^{-\lambda\tau_1} & \lambda + 1 - \alpha_{22} e^{-\lambda\tau_2} \end{pmatrix} = 0.$$

This characteristic equation determines the local stability of the equilibrium solution: the latter is stable if and only if all the characteristic roots λ, the solutions of

$$(\lambda + 1)^2 - (\lambda + 1)(\alpha_{11} e^{-\lambda\tau_1} + \alpha_{22} e^{-\lambda\tau_2}) + (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}) e^{-\lambda(\tau_1+\tau_2)} = 0, \tag{8}$$

have negative real parts.

In this section, we shall find some conditions which ensures that all roots of Eq. (8) have negative real parts.

For convenience, first of all, we introduce some notations (see Olien and Bélair [8]):

$$D = \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}, \quad T = \tfrac{1}{2}(\alpha_{11} + \alpha_{22}). \tag{9}$$

Then Eq. (8) becomes

$$(\lambda + 1)^2 - (\lambda + 1)(\alpha_{11} e^{-\lambda\tau_1} + \alpha_{22} e^{-\lambda\tau_2}) + D e^{-\lambda(\tau_1+\tau_2)} = 0. \tag{10}$$

Denote

$$a = 2 - \alpha_{11}, \quad b = -\alpha_{22}, \quad c = 1 - \alpha_{11}, \quad d = D - \alpha_{22}.$$

When $\tau_1 = 0$, Eq. (10) becomes

$$\lambda^2 + a\lambda + b\lambda e^{-\lambda\tau_2} + c + d e^{-\lambda\tau_2} = 0. \tag{11}$$

For convenience, we make some hypotheses as follows:
(H_1) $T^2 - D \le 0$.
(H_2) $T^2 - D > 0$.
(H_3) $c^2 < d^2$.
(H_4) $c^2 > d^2$, $b^2 - a^2 + 2c > 0$, and $(b^2 - a^2 + 2c)^2 > 4(c - d^2)$.
(H_5) Neither (H_3) nor (H_4).

Clearly, when $\tau_1 = \tau_2 = 0$, (6) becomes a system of ODEs. If (H_1) holds, then all roots of Eq. (10) have negative real parts if and only if $T < 1$; if (H_2) holds, then all roots of Eq. (10) have negative real parts if and only if $T < 1$ and $D > 2T - 1$.

In order to study the characteristic Eq. (10) with two delays, we first consider Eq. (11) with one delay τ_2. By using τ_2 as a parameter and employ Rouché's theorem (in the form of the Lemma in Cooke and Grossman [15]), we shall find stable intervals for τ_2 in which all roots of Eq. (11) have negative real parts. Then we consider (10) with τ_2 in its stable intervals. Use Rouché's theorem a second time and this time regard τ_1 as a parameter, we shall find a stable interval (depending on τ_2) for τ_1. This will be the stable interval for the characteristic Eq. (10).

Now we consider the case when $\tau_1 = 0$, that is, consider Eq. (11). Applying the Lemma in Cooke and Grossman [15], we obtain the following results.

Lemma 1. *For Eq.* (11), *we have*
1. *if* (H_3) *holds and* $\tau_2 = \tau_{2,n}^{(1)}$, *then Eq.* (11) *has a pair of purely imaginary roots* $\pm \mathrm{i} w_+$;
2. *if* (H_4) *holds and* $\tau_2 = \tau_{2,n}^{(1)} (res. \tau_2 = \tau_{2,n}^{(2)})$, *then Eq.* (11) *has a pair of imaginary roots* $\pm \mathrm{i} w_+ (res. \pm \mathrm{i} w_-)$;
3. *if* (H_5) *holds and* $\tau_2 > 0$, *then Eq.* (11) *has no purely imaginary root*,

where

$$w_\pm^2 = \tfrac{1}{2}(b^2 - a^2 + 2c) \pm \left[\tfrac{1}{4}(b^2 - a^2 + 2c) - (c^2 - d^2)\right]^{1/2}, \tag{12}$$

$$\tau_{2,n}^{(1)} = \frac{1}{w_+} \cos^{-1}\left\{\frac{d(w_+^2 - c) - w_+^2 ab}{b^2 w_+^2 + d^2}\right\} + \frac{2n\pi}{w_+}, \tag{13}$$

$$\tau_{2,n}^{(2)} = \frac{1}{w_-} \cos^{-1}\left\{\frac{d(w_-^2 - c) - w_-^2 ab}{b^2 w_-^2 + d^2}\right\} + \frac{2n\pi}{w_-} \quad (n = 0, 1, \ldots). \tag{14}$$

Denote

$$\lambda_{k,n} = \alpha_{k,n}(\tau_2) + \mathrm{i} w_{k,n}(\tau_2), \quad k = 1, 2; \quad n = 0, 1, 2, \ldots$$

the root of Eq. (11) satisfying

$$\alpha_{1,n}(\tau_{2,n}^{(1)}) = 0, \quad w_{1,n}(\tau_{2,n}^{(1)}) = w_+$$

and

$$\alpha_{2,n}(\tau_{2,n}^{(2)}) = 0, \quad w_{2,n}(\tau_{2,n}^{(2)}) = w_-.$$

To see if $\tau_{2,n}^{(1)}$ and $\tau_{2,n}^{(2)}$ are bifurcation values, we need to verify if the transversality conditions hold. In fact, we have the following

Lemma 2. *The following transversality conditions*:

$$\frac{\mathrm{dRe}\lambda_{1,n}(\tau_{2,n}^{(1)})}{\mathrm{d}\tau_2} > 0, \quad \frac{\mathrm{dRe}\lambda_{2,n}(\tau_{2,n}^{(2)})}{\mathrm{d}\tau_2} < 0$$

are satisfied.

Thus, we know the distribution of the characteristic roots of Eq. (11).

Lemma 3. *For Eq.* (11), *we have the following*

1. *If* (H_3) *and either (1)* (H_1) *and* $T < 1$ *or (2)* (H_2), $T < 1$ *and* $D > 2T - 1$ *hold, then when* $\tau_2 \in [0, \tau_{2,0}^{(1)})$ *all roots of Eq.* (11) *have negative real parts, and when* $\tau_2 > \tau_{2,0}^{(1)}$ *Eq.* (11) *has at least one root with positive real part.*
2. *If* (H_4) *and either* (H_1) *or* (H_2) *hold, then there are* k *switches from stability to instability, that is, when* $\tau_2 \in (\tau_{2,n}^{(2)}, \tau_{2,n+1}^{(1)})$, $n = -1, 0, 1, \dots, k-1$, *all roots of Eq.* (11) *have negative real parts, where* $\tau_{2,-1}^{(2)} = 0$, *and when* $\tau_2 \in [\tau_{2,n}^{(1)}, \tau_{2,n}^{(2)})$ *and* $\tau_2 > \tau_{2,k}^{(1)}$, $n = 0, 1, \dots, k-1$, *Eq.* (11) *has at least one root with positive real part.*

Next, we consider Eq. (10) with τ_2 in its stable intervals. Regard τ_1 as a parameter, we have

Lemma 4. *If all roots of Eq.* (11) *have negative real parts, then there exists a* $\tau_1(\tau_2) > 0$, *such that when* $\tau_1 \in [0, \tau_1(\tau_2))$ *all roots of Eq.* (8) *have negative real parts.*

Proof. Notice that Eq. (11) has no root with nonnegative real part, that is, Eq. (8) with $\tau_1 = 0$ has no root with nonnegative real part. We regard τ_1 as a parameter. It is clear that the left side of Eq. (8) is analytic in λ and τ_1.

Similar to the proof of the Lemma of Cooke and Grossman [15], we can prove that:

Let $f(\lambda, \tau_1, \tau_2) = \lambda^2 + a_1\lambda + a_2\lambda e^{-\lambda\tau_2} + a_3 e^{-\lambda\tau_2} + b_1\lambda e^{-\lambda\tau_1} + b_2 e^{-\lambda\tau_1} + c$, where $a_1, a_2, a_3, b_1, b_2, c, \tau_1, \tau_2$ are all real numbers, $\tau_1 \ge 0$, and $\tau_2 \ge 0$. Then, as τ_1 varies, the sum of the multiplicities of zeros of f in the open right half-plane can change only if a zero appears on or crosses the imaginary axis.

Applying this conclusion and noticing that Eq. (8) with $\tau_1 = 0$ has no root with nonnegative real part, we obtain that there is $\tau_1^0 > 0$ such that all roots of Eq. (8) with $\tau_1 \in [0, \tau_1^0)$ have negative real parts. The proof is complete. □

Summarizing the above lemmas, we obtain the following sufficient conditions for all characteristic roots of Eq. (10) to have negative real parts.

Theorem 1. *Suppose either* (H_1) or (H_2) *is satisfied.*

1. *If* (H_3) *holds, then for any* $\tau_2 \in [0, \tau_{2,0}^{(1)})$, *there exists a* $\tau_1(\tau_2) > 0$ *such that when* $\tau_1 \in [0, \tau_1(\tau_2))$, *all roots of Eq.* (8) *have negative real parts.*
2. *If* (H_4) *holds, then for any* $\tau_2 \in \bigcup_{n=-1}^{k-1}(\tau_{2,n}^{(2)}, \tau_{2,n+1}^{(1)})$ *there exists a* $\tau_1(\tau_2) > 0$, *such that when* $\tau_1 \in [0, \tau_1(\tau_2))$, *all roots of Eq.* (8) *have negative real parts, where* $\tau_{2,j}^{(1)}$ *and* $\tau_{2,j}^{(2)}$ *are defined by Eq.* (13) *and* (14), *respectively, and* $\tau_{2,-1}^{(2)} = 0$.
3. *If* (H_5) *holds, then for any* $\tau_2 \ge 0$, *there exists a* $\tau_1(\tau_2) > 0$, *such that when* $\tau_1 \in [0, \tau_1(\tau_2))$ *all roots of Eq.* (3) *have negative real parts.*

Applying Theorem 1, we then obtain sufficient conditions for local stability of system (6).

3. The non self-connection neural network parameterized by delays

When there is no self-connection in the network, that is, $a_{11} = a_{22} = 0$, system (6) becomes

$$\begin{cases} \dot{u}_1(t) = -u_1(t) + a_{12} f(u_2(t - \tau_2)), \\ \dot{u}_2(t) = -u_2(t) + a_{21} f(u_1(t - \tau_1)) \end{cases} \tag{15}$$

and the characteristic Eq. (8) reduces to

$$\lambda^2 + 2\lambda - \alpha_{21}\alpha_{12}\mathrm{e}^{-\lambda\tau} + 1 = 0, \tag{16}$$

where $\tau = \tau_1 + \tau_2$.

In this section, we regard the sum of delays, $\tau = \tau_1 + \tau_2$, as the parameter to give some conditions that separate the first quadrant of the (τ_1, τ_2) plane into two parts, one is the stable region, another is the unstable region, and the boundary is the Hopf bifurcation curve.

We first consider the characteristic Eq. (16).

Lemma 5. *Suppose*

$$\alpha_{12}\alpha_{21} < -1. \tag{17}$$

Then we have the following:

1. *When*

$$\tau = \tau^j \stackrel{\text{def}}{=} \frac{1}{w_0}\left[\sin^{-1}\left(-\frac{2w_0}{\alpha_{12}\alpha_{21}}\right) + 2j\pi\right], \quad j = 0, 1, 2, \ldots, \tag{18}$$

   Eq. (16) *has a simple pair of purely imaginary roots* $\pm \mathrm{i}w_0$, *where*

$$w_0 = \sqrt{|\alpha_{12}\alpha_{21}| - 1}. \tag{19}$$

2. *For* $\tau \in [0, \tau^0)$, *all roots of Eq.* (16) *have strictly negative real parts.*
3. *When* $\tau = \tau^0$, *Eq.* (16) *has a pair of imaginary roots* $\pm \mathrm{i}w_0$ *and all other roots have strictly negative real parts.*

Proof. $\pm \mathrm{i}w (w > 0)$ is a pair of purely imaginary roots of (16) if and only if w satisfies

$$-w^2 + \mathrm{i}2w - \alpha_{12}\alpha_{21}\cos w\tau + \mathrm{i}\alpha_{12}\alpha_{21}\sin w\tau + 1 = 0.$$

Separate the real and imaginary parts, we have

$$\begin{cases} w^2 - 1 = -\alpha_{12}\alpha_{21}\cos w\tau, \\ 2w = -\alpha_{12}\alpha_{21}\sin w\tau. \end{cases} \tag{20}$$

It follows from (20) that

$$w^4 + 2w^2 + 1 = \alpha_{12}^2\alpha_{21}^2,$$

hence, $w^2 = -1 \pm |\alpha_{12}\alpha_2|$, i.e., $w = \sqrt{|\alpha_{12}\alpha_{21}| - 1}$. It is clear that w is well-defined if $|\alpha_{12}\alpha_2| > 1$.

Denote

$$w_0 = \sqrt{|\alpha_{12}\alpha_{21}| - 1}.$$

Let

$$\tau^j = \frac{1}{w_0}\left[\sin^{-1}\left(-\frac{2w_0}{\sqrt{|\alpha_{12}\alpha_{21}| - 1}}\right) + 2j\pi\right], \quad j = 0, 1, 2, \ldots.$$

From (20) we know that Eq. (16) with $\tau = \tau^j (j = 0, 1, \dots)$ has a pair of imaginary roots $\pm w_0$, which are simple.

Consider the Eq. (16) with $\tau = 0$, that is,

$$\lambda^2 + 2\lambda + (1 - \alpha_{12}\alpha_{21}) = 0. \tag{21}$$

It is obvious that all roots of Eq. (21) have negative real parts. Applying the Lemma in Cooke and Grossman [15], we obtain the conclusion (2) and (3). This completes the proof. □

From Lemma 5 it seems that $\tau^j (j = 0, 1, 2, \dots)$ are bifurcation values. In fact, we have

Lemma 6. *Denote* $\lambda_j(\tau) = \alpha_j(\tau) + \mathrm{i}w_j(\tau)$ *as the root of Eq.* (16) *satisfying* $\alpha_j(\tau^j) = 0$, $w_j(\tau^j) = w_0$, $j = 0, 1, \dots$. *We have the following transversality condition:*

$$\frac{\mathrm{dRe}\lambda_j(\tau^j)}{\mathrm{d}\tau} > 0. \tag{22}$$

Indeed, we can directly compute that

$$\frac{\mathrm{dRe}\lambda_j(\tau^j)}{\mathrm{d}\tau} = \frac{2w_0^2(w_0^2 + 1)}{\Delta},$$

where

$$\Delta = (2 + \alpha_{12}\alpha_{21}\tau^j \cos\omega_0\tau^j)^2 + (2\omega_0 - \alpha_{12}\alpha_{21}\tau^j \sin\omega_0\tau^j)^2.$$

From Lemma 6, we obtain the following lemma.

Lemma 7. *Suppose Eq.* (17) *is satisfied. If* $\tau > \tau^0$, *then Eq.* (16) *has at least one root with strictly positive real part.*

In fact, by the Lemma in Cooke and Grossman [15] and Lemma 6, we can see that when $\tau \in (\tau^j, \tau^{j+1})$, Eq. (16) has $2(j + 1)$ $(j = 0, 1, \dots)$ roots with positive real parts.

By Lemmas 5–7, we have the following result on stability and bifurcation in system (15).

Theorem 2. *For system* (15), *we have*
1. *If* $\tau \in [0, \tau^0)$, *then the zero solution of* (15) *is asymptotically stable.*
2. *If* $\tau > \tau^0$, *then the zero solution of Eq.* (15) *is unstable.*
3. τ^j $(j = 0, 1, 2, \dots)$ *are Hopf bifurcation values for Eq.* (15).

Theorem 2 can be illustrated by Fig. 1, where the shadowed region is the stable domain, Hopf bifurcations occur at the lines $\tau_1 + \tau_2 = \tau^j$ $(j = 0, 1, 2, \dots)$.

4. Direction and stability of the Hopf bifurcation

In Section 3 we obtained some conditions which guarantee that the non self-connection neural network model with two delays undergoes the Hopf bifurcation at some value of $\tau = \tau_1 + \tau_2$. In this section, we will study the direction, stability, and the period of the bifurcating periodic solutions. The method we used is based on the normal form method and the center manifold theory introduced by Hassard, Kazarinoff and Wan [16].

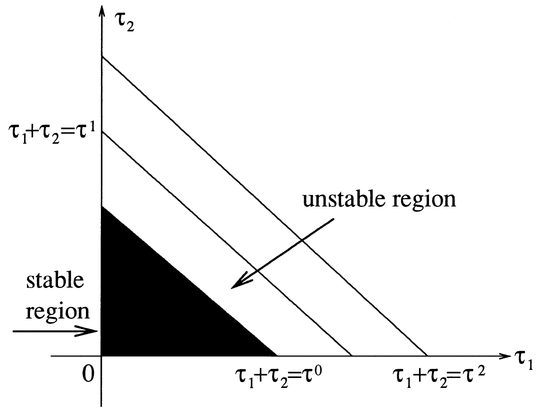

Fig. 1. The stable diagram in the (τ_1, τ_2) plane.

From conclusion (3) of Lemma 5 and Lemma 6, we know that if $\alpha_{12}\alpha_{21} = a_{12}a_{21}(f'(0))^2 < -1$, then all roots of Eq. (15) other than $\pm \mathrm{i}w_0$ have negative real parts, and

$$\lambda(\tau) = \alpha(\tau) + \mathrm{i}w(\tau),$$

the root of Eq. (15) satisfying $\alpha(\tau^0) = 0$, $w(\tau^0) = w_0$, has the property

$$\frac{\mathrm{d}\alpha(\tau^0)}{\mathrm{d}\tau} > 0.$$

For convenience, let $\tau = \tau^0 + \mu$, $\mu \in R$. Then $\mu = 0$ is the Hopf bifurcation value for Eq. (15). Without loss generality, we assume that $\tau_1^0 > \tau_2^0$ and let $|\mu| \leq \tau_1^0 - \tau_2^0$ since the analysis is local, where $\tau^0 = \tau_1^0 + \tau_2^0$ and $\tau = \tau_1^0 + (\tau_2^0 + \mu)$. Choosing the phase space as

$$C = C([-\tau_1^0, 0], R^2). \tag{23}$$

We assume that the function f satisfies
(P_1) $f \in C^3(R)$, $uf(u) \neq 0$ when $u \neq 0$.
Then Eq. (15) can be rewritten as

$$\begin{cases} \dot{u}_1(t) = -u_1(t) + a_{12}\left(f'(0)u_2(t-\tau_2) + \dfrac{f''(0)}{2}u_2^2(t-\tau_2) + \dfrac{f'''(0)}{6}u_2^3(t-\tau_2)\right) + \mathrm{O}(u_2^4), \\ \dot{u}_2(t) = -u_2(t) + a_{21}\left(f'(0)u_1(t-\tau_1) + \dfrac{f''(0)}{2}u_1^2(t-\tau_1) + \dfrac{f'''(0)}{6}u_1^3(t-\tau_1)\right) + \mathrm{O}(u_1^4). \end{cases} \tag{24}$$

For $\phi \in C$, let

$$L_\mu\phi = -I\phi(0) + B_1\phi(-\tau_2) + B_2\phi(-\tau_1), \tag{25}$$

where I is the identical matrix,

$$B_1 = \begin{pmatrix} 0 & \alpha_{12} \\ 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 \\ \alpha_{21} & 0 \end{pmatrix},$$

$$\alpha_{12} = a_{12}f'(0), \quad \alpha_{21} = a_{21}f'(0),$$

and

$$F(\mu,\phi)=\frac{f''(0)}{2}\begin{pmatrix}a_{12}\phi_2^2(-\tau_2)\\ a_{21}\phi_1^2(-\tau_1)\end{pmatrix}+\frac{f'''(0)}{6}\begin{pmatrix}a_{12}\phi_2^3(-\tau_2)\\ a_{21}\phi_1^3(-\tau_1)\end{pmatrix}+\mathrm{O}(|u|^4). \tag{26}$$

By the Riesz representation theorem, there exists a matrix whose components are bounded variation functions $\eta(\theta,\mu)$ in $\theta\in[-\tau_1^0,0]$ such that

$$L_\mu\phi=\int_{-\tau_1^0}^{0}\mathrm{d}\eta(\theta,\mu)\phi(\theta)\quad\text{for}\ \phi\in C. \tag{27}$$

In fact, we choose

$$\eta(\theta,\mu)=\begin{cases}-I, & \theta=0,\\ B_1\delta(\theta+\tau_2), & \theta\in[-\tau_2,0),\\ -B_2\delta(\theta+\tau_1^0), & \theta\in[-\tau_1^0,-\tau_2).\end{cases}$$

Then (27) is satisfied.

For $\phi\in C^1([-\tau_1^0,0],R^2)$, define

$$A(\mu)\phi=\begin{cases}\mathrm{d}\phi(\theta)/\mathrm{d}\theta, & \theta\in[-\tau_1^0,0),\\ \displaystyle\int_{-\tau_1^0}^{0}\mathrm{d}\eta(t,\mu)\phi(t), & \theta=0\end{cases} \tag{28}$$

and

$$R(\mu)\phi=\begin{cases}0, & \theta\in[-\tau_1^0,0),\\ F(\mu,\phi), & \theta=0.\end{cases} \tag{29}$$

Hence, we can rewrite (24) as the following form:

$$\dot{u}_t=A(\mu)u_t+R(\mu)u_t, \tag{30}$$

where $u=(u_1,u_2)^{\mathrm{T}}$, $u_t=u(t+\theta)$ for $\theta\in[-\tau_1^0,0]$.

For $\psi\in C^1[0,\tau_1^0]$, define

$$A^*\phi(s)=\begin{cases}-\mathrm{d}\psi(s)/\mathrm{d}s, & s\in(0,\tau_1^0),\\ \displaystyle\int_{-\tau_1^0}^{0}\mathrm{d}\eta(t,0)\psi(-t), & s=0.\end{cases}$$

For $\phi\in C[-\tau_1^0,0]$ and $\psi\in C[0,\tau_1^0]$, define the bilinear form

$$\langle\psi,\phi\rangle=\bar{\psi}(0)\phi(0)-\int_{-\tau_1^0}^{0}\int_{\xi=0}^{\theta}\bar{\psi}(\xi-\theta)\mathrm{d}\eta(\theta)\phi(\xi)\mathrm{d}\xi, \tag{31}$$

where $\eta(\theta)=\eta(\theta,0)$. Then A^* and $A(0)$ are adjoint operators.

By the results in Section 3, we assume that $\pm\mathrm{i}w_0$ are eigenvalues of $A(0)$. Thus, they are also eigenvalues of A^*.

By direct computation, we obtain that

$$q(\theta)=\begin{pmatrix}1\\ \dfrac{1+\mathrm{i}w_0}{\alpha_{12}}\mathrm{e}^{\mathrm{i}w_0\tau_2^0}\end{pmatrix}\mathrm{e}^{\mathrm{i}w_0\theta}$$

is the eigenvector of $A(0)$ corresponding to $\mathrm{i}w_0$; and

$$q^*(s) = D\begin{pmatrix} \dfrac{1-\mathrm{i}w_0}{\alpha_{12}}\mathrm{e}^{\mathrm{i}w_0\tau_2^0} \\ 1 \end{pmatrix}^{\mathrm{T}} \mathrm{e}^{\mathrm{i}w_0 s}$$

is the eigenvector of A^* corresponding to $-\mathrm{i}w_0$. Moreover,

$$\langle q^*, q\rangle = 1, \quad \langle q^*, \bar{q}\rangle = 0,$$

where

$$\begin{aligned} D &= \tfrac{1}{2}\left\{\frac{[1+\tau_2^0(1-\mathrm{i}w_0)](1-\mathrm{i}w_0)}{\alpha_{12}}\mathrm{e}^{-\mathrm{i}w_0\tau_2^0} + \alpha_{21}\tau_1^0\mathrm{e}^{\mathrm{i}w_0\tau_1^0}\right\}^{-1} \\ &= \tfrac{1}{2}\left\{\frac{(1-\mathrm{i}w_0)[1+\tau^0(1-\mathrm{i}w_0)]}{\alpha_{12}}\mathrm{e}^{-\mathrm{i}w_0\tau_2^0}\right\}^{-1}. \end{aligned}$$

Now, we verify that $\langle q^*, q\rangle = 1$. In fact, from (31), we have

$$\begin{aligned} \langle q^*, q\rangle &= \bar{D}\left\{\left(\frac{1+\mathrm{i}w_0}{\mathrm{d}_{12}}\mathrm{e}^{\mathrm{i}w_0\tau_2^0}, 1\right)\begin{pmatrix} 1 \\ \dfrac{1+\mathrm{i}w_0}{\alpha_{12}}\mathrm{e}^{\mathrm{i}w_0\tau_2^0} \end{pmatrix}\right. \\ &\qquad \left. - \int_{-\tau_1^0}^{0}\int_{\xi=0}^{\theta}\begin{pmatrix} \dfrac{1+\mathrm{i}w_0}{\alpha_{12}}\mathrm{e}^{\mathrm{i}w_0\tau_2^0} \\ 1 \end{pmatrix}^{\mathrm{T}} \mathrm{e}^{-\mathrm{i}w_0(\xi-\theta)}\mathrm{d}\eta(\theta)\begin{pmatrix} 1 \\ \dfrac{1+\mathrm{i}w_0}{\alpha_{12}}\mathrm{e}^{\mathrm{i}w_0\tau_2^0} \end{pmatrix}\mathrm{e}^{\mathrm{i}w_0\xi}\mathrm{d}\xi\right\} \\ &= 2\bar{D}\left\{\frac{1+\mathrm{i}w_0}{\alpha_{12}}\mathrm{e}^{\mathrm{i}w_0\tau_2^0} - \int_{-\tau_1^0}^{0}\begin{pmatrix} \dfrac{1+\mathrm{i}w_0}{\alpha_{12}}\mathrm{e}^{\mathrm{i}w_0\tau_2^0} \\ 1 \end{pmatrix}^{\mathrm{T}} \theta\mathrm{e}^{-\mathrm{i}w_0\theta}\mathrm{d}\eta(\theta)\begin{pmatrix} 1 \\ \dfrac{1+\mathrm{i}w_0}{\alpha_{12}}\mathrm{e}^{\mathrm{i}w_0\tau_2^0} \end{pmatrix}\right\} \\ &= 2\bar{D}\left\{\frac{1+\mathrm{i}w_0}{\alpha_{12}}\mathrm{e}^{\mathrm{i}w_0\tau_2^0} + \frac{\tau_2^0(1+\mathrm{i}w_0)^2}{\alpha_{12}}\mathrm{e}^{\mathrm{i}w_0\tau_2^0} + \alpha_{21}\tau_1^0\mathrm{e}^{-\mathrm{i}w_0\tau_1^0}\right\} \\ &= 2\bar{D}\left\{\frac{[1+\tau_2^0(1+\mathrm{i}w_0)](1+\mathrm{i}w_0)}{\alpha_{12}}\mathrm{e}^{\mathrm{i}w_0\tau_2^0} + \alpha_{21}\tau_1^0\mathrm{e}^{-\mathrm{i}w_0\tau_1^0}\right\} = 1. \end{aligned}$$

Using the same notations as in Hassard, Kazarinoff and Wan [16], we first compute the coordinates to describe the center manifold $\mathcal{C}_0$ at $\mu = 0$. Let u_t be the solution of Eq. (15) when $\mu = 0$.

Define

$$z(t) = \langle q^*, u_t\rangle, \qquad W(t,\theta) = u_t(\theta) - 2\mathrm{Re}\{z(t)q(\theta)\}.$$

On the center manifold $\mathcal{C}_0$ we have

$$W(t,\theta) = W(z(t), \bar{z}(t), \theta),$$

where

$$W(z,\bar{z},\theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + W_{30}\frac{z^3}{6} + \cdots,$$

z and $\bar{z}$ are local coordinates for center manifold $\mathcal{C}_0$ in the direction of q^* and $\bar{q}^*$. Note that W is real if u_t is real. We consider only real solutions.

For solution $u_t \in \mathcal{C}_0$ of (15), since $\mu = 0$,

$$\begin{aligned}\dot{z}(t) &= \mathrm{i}w_0 z + \langle q^*(\theta), F(W + 2\mathrm{Re}\{z(t)q(\theta)\})\rangle \\ &= \mathrm{i}w_0 z + \bar{q}^*(0)F(W(z,\bar{z},0) + 2\mathrm{Re}\{z(t)q(0)\}) \stackrel{\text{def}}{=} \mathrm{i}w_0 z + \bar{q}^*(0)F_0(z,\bar{z}).\end{aligned}$$

We rewrite this as

$$\dot{z}(t) = \mathrm{i}w_0 z(t) + g(z,\bar{z}), \tag{32}$$

where

$$g(z,\bar{z}) = \bar{q}^*(0)F(W(z,\bar{z},0) + 2\mathrm{Re}\{z(t)q(0)\}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots. \tag{33}$$

By (30) and (32), we have

$$\dot{w} = \dot{u}_t - \dot{z}q - \dot{\bar{z}}\bar{q} = \begin{cases} AW - 2\mathrm{Re}\{\bar{q}^*(0)F_0 q(\theta)\}, & \theta \in [-r_0, 0) \\ AW - 2\mathrm{Re}\{\bar{q}^*(0)F_0 q(\theta)\} + F_0, & \theta = 0, \end{cases}$$

$$\stackrel{\text{def}}{=} AW + H(z,\bar{z},\theta),$$

where

$$H(z,\bar{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots. \tag{34}$$

Expanding the above series and comparing the coefficients, we obtain

$$(A - 2\mathrm{i}w_0)W_{20}(\theta) = -H_{20}(\theta), \quad AW_{11}(\theta) = -H_{11}(\theta), (A + 2\mathrm{i}w_0)W_{02}(\theta) = -H_{02}(\theta), \quad \cdots. \tag{35}$$

Notice that

$$q^*(0) = D\left(\frac{1 - \mathrm{i}w_0}{\alpha_{12}}\mathrm{e}^{-\mathrm{i}w_0\tau_2^0}, 1\right),$$

$$u_2(t - \tau_2^0) = \frac{1 + \mathrm{i}w_0}{\alpha_{12}}z + \frac{1 - \mathrm{i}w_0}{\alpha_{12}}\bar{z} + W^{(2)}(t, -\tau_2^0),$$

and

$$u_1(t - \tau_1^0) = \mathrm{e}^{-\mathrm{i}w_0\tau_1^0}z + \mathrm{e}^{\mathrm{i}w_0\tau_1^0}\bar{z} + W^{(1)}(t, -\tau_1^0),$$

where

$$W^{(2)}(t, -\tau_2^0) = W_{20}^{(2)}(-\tau_2^0)\frac{z^2}{2} + W_{11}^{(2)}(-\tau_2^0)z\bar{z} + W_{02}^{(2)}(-\tau_2^0)\frac{\bar{z}^2}{2} + \cdots,$$

$$W^{(1)}(t, -\tau_1^0) = W_{20}^{(1)}(-\tau_1^0)\frac{z^2}{2} + W_{11}^{(1)}(-\tau_1^0)z\bar{z} + W_{02}^{(1)}(-\tau_1^0)\frac{\bar{z}^2}{2} + \cdots,$$

and

$$F_0 = \frac{f''(0)}{2}\begin{pmatrix} a_{12}u_2^2(t - \tau_2^0) \\ a_{21}u_1^2(t - \tau_1^0)\end{pmatrix} + \frac{f'''(0)}{6}\begin{pmatrix} a_{12}u_2^3(t - \tau_2^0) \\ a_{21}u_1^3(t - \tau_1^0)\end{pmatrix} + \cdots.$$

We have

$$
\begin{aligned}
g(z,\bar z) &= \bar q^*(0)F_0\\
&= \frac{\bar D}{2}\left[a_{12}M\mathrm{e}^{\mathrm{i}w_i\tau_2^0}\left(f''(0)u_2^2(t-\tau_2^0)+\frac{f'''(0)}{3}u_2^3(t-\tau_2^0)\right)\right.\\
&\quad\left.+a_{21}\left(f''(0)u_1^2(t-\tau_1^0)+\frac{f'''(0)}{3}u_1^3(t-\tau_1^0)\right)\right]+\mathrm{O}(u^4)\\
&= \frac{\bar D}{2}\{a_{12}M\mathrm{e}^{\mathrm{i}w_i\tau_0}[f''(0)(M^2z^2+\bar M^2\bar z^2+2|M|z\bar z)+(f''(0)(\bar M W_{20}^{(2)}(-\tau_2^0)\\
&\quad+2MW_{11}^{(2)}(-\tau_2^0))+f'''(0)|M|^2M)z^2\bar z]+a_{21}[f''(0)(\mathrm{e}^{-2\mathrm{i}w_i\tau_1^0}z^2+\mathrm{e}^{2\mathrm{i}w_0\tau_1^0}\bar z^2+2z\bar z)\\
&\quad+(f''(0)(\mathrm{e}^{\mathrm{i}w_0\tau_1^0}W_{20}^{(1)}(-\tau_1^0)+2\mathrm{e}^{-\mathrm{i}w_0\tau_1^0}W_{11}^{(1)}(-\tau_1^0))+f'''(0)\mathrm{e}^{-\mathrm{i}w_0\tau_1^0})z^2\bar z]\},
\end{aligned}
$$

where $M=1+\mathrm{i}w_0/\alpha_{12}$.

Comparing the coefficients with (33), we have

$$
\begin{aligned}
g_{20} &= \bar D f''(0)[a_{12}M^3\mathrm{e}^{\mathrm{i}w_0\tau_2^0}+a_{21}\mathrm{e}^{-2\mathrm{i}w_0\tau_1^0}],\\
g_{11} &= \bar D f''(0)[a_{12}|M|M\mathrm{e}^{\mathrm{i}w_0\tau_2^0}+a_{21}],\\
g_{02} &= \bar D f''(0)[a_{12}|M|^2\bar M\mathrm{e}^{\mathrm{i}w_0\tau_2^0}+a_{21}\mathrm{e}^{-2\mathrm{i}w_0\tau_1^0}],\\
g_{21} &= \bar D\{a_{21}\mathrm{e}^{\mathrm{i}w_0\tau_2^0}[f''(0)(|M|^2W_{20}^{(2)}(-\tau_2^0)+2M^2W_{11}^{(2)}(-\tau_2^0))+f'''(0)|M|^2M^2]\\
&\quad+a_{21}[f''(0)(\mathrm{e}^{\mathrm{i}w_0\tau_2^0}W_{20}^{(1)}(-\tau_1^0)+2\mathrm{e}^{-\mathrm{i}w_0\tau_2^0}W_{11}^{(1)}(-\tau_1^0))+f'''(0)\mathrm{e}^{-\mathrm{i}w_0\tau_2^0}]\}. \qquad (36)
\end{aligned}
$$

We still need to compute $W_{20}(\theta)$ and $W_{11}(\theta)$ for $\theta\in[-\tau_1^0,0)$, we have

$$
\begin{aligned}
H(z,\bar z,\theta) &= 2\mathrm{Re}\{\bar z^*(0)F_0q(\theta)\}=-gq(\theta)-\bar g\bar q(\theta)\\
&= -\left(g_{20}\frac{z^2}{2}+g_{11}z\bar z+g_{02}\frac{\bar z^2}{2}+\cdots\right)q(\theta)-\left(\bar g_{20}\frac{\bar z^2}{2}+\bar g_{11}z\bar z+\bar g_{02}\frac{z^2}{2}+\cdots\right)\bar q(\theta).
\end{aligned}
$$

Comparing the coefficients with (34) gives that

$$H_{20}(\theta)=-g_{20}q(\theta)-\bar g_{02}\bar q(\theta)$$

and

$$H_{11}(\theta)=-g_{11}q(\theta)-\bar g_{11}\bar q(\theta).$$

It follows from (35) that

$$\dot W_{20}(\theta)=2\mathrm{i}w_0W_{20}(\theta)-g_{20}q(0)\mathrm{e}^{\mathrm{i}w_0\theta}-\bar g_{02}\bar q(0)\mathrm{e}^{-\mathrm{i}w_0\theta}.$$

Solving for $W_{20}(\theta)$, we obtain

$$W_{20}(\theta)=\frac{g_{20}}{\mathrm{i}w_0}q(0)\mathrm{e}^{\mathrm{i}w_0\theta}-\frac{\bar g_{20}}{3\mathrm{i}w_0}\bar q(0)\mathrm{e}^{-\mathrm{i}w_0\theta}+E_1\mathrm{e}^{2\mathrm{i}w_i\theta}, \qquad (37)$$

and similarly

$$W_{11}(\theta)=\frac{g_{11}}{\mathrm{i}w_0}q(0)\mathrm{e}^{\mathrm{i}w_0\theta}-\frac{\bar g_{11}}{\mathrm{i}w_0}\bar q(0)\mathrm{e}^{-\mathrm{i}w_0\theta}+E_2, \qquad (38)$$

where E_1 and E_2 are both two-dimensional vectors, and can be determined by setting $\theta=0$ in H. In fact, since

$$H(z,\bar z,0)=-2\mathrm{Re}\{\bar q^*(0)F_0q(0)\}+F_0,$$

we have

$$H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + f''(0)\begin{pmatrix} a_{12}M^2 \\ a_{21}\mathrm{e}^{-2\mathrm{i}w_0 r_0} \end{pmatrix} \tag{39}$$

and

$$H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + f''(0)\begin{pmatrix} a_{21}|M|^2 \\ a_{21} \end{pmatrix}. \tag{40}$$

From (35) and the definition of A, we have

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} W_{20}(0) + \begin{pmatrix} 0 & \alpha_{12} \\ 0 & 0 \end{pmatrix} W_{20}(-\tau_2^0) + \begin{pmatrix} 0 & 0 \\ \alpha_{21} & 0 \end{pmatrix} W_{20}(-r_0) = 2\mathrm{i}w_0 W_{20}(0) - H_{20}(0) \tag{41}$$

and

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} W_{11}(0) + \begin{pmatrix} 0 & \alpha_{12} \\ 0 & 0 \end{pmatrix} W_{11}(-\tau_2^0) + \begin{pmatrix} 0 & 0 \\ \alpha_{21} & 0 \end{pmatrix} W_{11}(-\tau_1^0) = -H_{11}(0). \tag{42}$$

Substituting (37) into (41) and noting that

$$\begin{pmatrix} -1-\mathrm{i}w_0 & \alpha_{12}\mathrm{e}^{-\mathrm{i}w_0\tau_2^0} \\ \alpha_{21}\mathrm{e}^{-\mathrm{i}w_0\tau_1^0} & -1-\mathrm{i}w_0 \end{pmatrix} q(0) = 0,$$

we have

$$\begin{pmatrix} -1-2\mathrm{i}w_0 & \alpha_{12}\mathrm{e}^{-2\mathrm{i}w_0\tau_2^0} \\ \alpha_{21}\mathrm{e}^{-2\mathrm{i}w_0\tau_1^0} & -1-2\mathrm{i}w_0 \end{pmatrix} E_1 = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) - H_{20}(0).$$

Substituting (39) into this, we get

$$\begin{pmatrix} -(1+2\mathrm{i}w_0) & \alpha_{12}\mathrm{e}^{-2\mathrm{i}w_0\tau_2^0} \\ \alpha_{21}\mathrm{e}^{-2\mathrm{i}w_0\tau_1^0} & -(1+2\mathrm{i}w_0) \end{pmatrix} E_1 = -f''(0)\begin{pmatrix} a_{12}M^2 \\ a_{21}\mathrm{e}^{-2\mathrm{i}w_0\tau_1^0} \end{pmatrix}.$$

Solving this equations for $(E_1^{(1)}, E_1^{(2)})^{\mathrm{T}} = E_1$, we obtain

$$E_1^{(1)} = \frac{f''(0)[a_{12}M^2(1+2\mathrm{i}w_0) + a_{21}\alpha_{12}\mathrm{e}^{-2\mathrm{i}w_0(\tau_2^0+\tau_1^0)}]}{(1+2\mathrm{i}w_i)^2 - \alpha_{12}\alpha_{21}\mathrm{e}^{-2\mathrm{i}w_0(\tau_2^0+\tau_1^0)}}$$

and

$$E_1^{(2)} = \frac{f''(0)\mathrm{e}^{-2\mathrm{i}w_0\tau_1^0}[a_{21}(1+2\mathrm{i}w_0) + a_{12}\alpha_{21}M^2]}{(1+2\mathrm{i}w_0)^2 - \alpha_{12}\alpha_{21}\mathrm{e}^{-2\mathrm{i}w_0(\tau_2^0+\tau_1^0)}}.$$

Similarly, we can get

$$\begin{pmatrix} -1 & \alpha_{12} \\ \alpha_{21} & -1 \end{pmatrix} E_2 = -f''(0)\begin{pmatrix} a_{12}|M|^2 \\ a_{21} \end{pmatrix},$$

and hence,

$$E_2^{(1)} = \frac{f''(0)[a_{12}|M|^2 + \alpha_{12}a_{21}]}{1-\alpha_{12}\alpha_{21}}, \quad E_2^{(2)} = \frac{f''(0)[a_{21} + a_{12}\alpha_{21}|M|^2]}{1-\alpha_{12}\alpha_{21}}.$$

Based on the above analysis, we can see that each g_{ij} in (36) is determined by the parameters and delays in (15). Thus, we can compute the following quantities:

$$\begin{aligned} C_1(0) &= \frac{\mathrm{i}}{2w_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \tfrac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{\mathrm{Re}\{C_1(0)\}}{\mathrm{Re}\lambda'(0)}, \\ T_2 &= -\frac{\mathrm{Im}\{C_1(0)\} + \mu_2 \mathrm{Im}\lambda'(0)}{w_0}, \\ \beta_2 &= 2\mathrm{Re}\{C_1(0)\}. \end{aligned} \tag{43}$$

We know that (see Hassard, Kazarinoff and Wan [16]) μ_2 determines the directions of the Hopf bifurcation: if $\mu_2 > 0 (< 0)$, then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for $\tau = \tau_1 + \tau_2 > \tau^0\ (< \tau^0)$; β_2 determines the stability of the bifurcating periodic solutions: the bifurcating periodic solutions are orbitally stable (unstable) if $\beta_2 < 0\ (> 0)$; and T_2 determines the period of the bifurcating periodic solutions: the period increase (decrease) if $T_2 > 0\ (< 0)$. In (43),

$$\lambda(\mu) = \alpha(\mu) + \mathrm{i}w(\mu)$$

is a solution of (15), where $\tau = \tau^0 + \mu$, satisfying $\alpha(0) = 0$, $w(0) = w_0$.

For system Eq. (15), according to the properties of the function $\tanh(u)$, we make the following assumption on function f:

(P_2) $\quad f'(0) \neq 0, \quad f''(0) = 0 \quad$ and $\quad f'''(0) \neq 0.$

Then we have the main result in this section.

Theorem 3. *If* (P_1)–(P_2) *are satisfied and* $a_{12}a_{21}[f'(0)]^2 < -1$, *then there exists a* $\tau^0 > 0$ *such that when* $\tau = \tau_1 + \tau_2 \in [0, \tau_0)$, *the zero solution of Eq.* (15) *is asymptotically stable. When* $\tau = \tau^0$, *Eq.* (15) *undergoes a Hopf bifurcation. The direction of the Hopf bifurcation and stability of bifurcating periodic solutions are determined by* $\mathrm{sign}\, f'''(0)/f'(0)$. *In fact, if* $f'''(0)/f'(0) < 0 (> 0)$, *then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions are orbitally asymptotically stable (unstable).*

Proof. We only need to prove the last part of the theorem. Since $f''(0) = 0$, from (36) we have $g_{11} = g_{20} = g_{02} = 0$, and

$$g_{21} = \bar{D}[a_{12} f'''(0)\mathrm{e}^{\mathrm{i}w_0\tau_2^0}|M|^2M^2 + a_{21} f'''(0)\mathrm{e}^{-\mathrm{i}w_0\tau_1^0}] = f'''(0)\bar{D}[a_{12}|M|^2M^2 + a_{21}\mathrm{e}^{-\mathrm{i}w_0\tau_1^0}]. \tag{44}$$

Note that

$$D = \tfrac{1}{2}\left[\frac{(1-\mathrm{i}w_0)[1+\tau^0(1-\mathrm{i}w_0)]}{\alpha_{12}}\mathrm{e}^{-\mathrm{i}w_0\tau^0}\right]^{-1},$$

$$\tau_1^0 = \tau^0 - \tau_2^0, \quad M = \frac{1+\mathrm{i}w_0}{\alpha_{12}}, \quad \mathrm{e}^{-\mathrm{i}w_0\tau^0} = \frac{(1+\mathrm{i}w_0)^2}{\alpha_{12}\alpha_{21}}.$$

From (44), we obtain

$$\begin{aligned} g_{21} &= \frac{\alpha_{12}}{2} f'''(0)\frac{[a_{12}|M|^2M^2 + a_{21}\mathrm{e}^{-\mathrm{i}w_0\tau^0}]}{(1+\mathrm{i}w_0)[1+\tau^0(1+\mathrm{i}w_0)]} = \frac{\alpha_{12}}{2} f'''(0)\frac{(a_{12}|M|^2(1+\mathrm{i}w_0)^2/\alpha_{12}^2)+(a_{21}(1+\mathrm{i}w_0)^2/\alpha_{12}\alpha_{21})}{(1+\mathrm{i}w_0)[(1+\tau^0)+i\tau^0w_0]} \\ &= \frac{f'''(0)}{2}\frac{(a_{12}|M|^2(1+\mathrm{i}w_0)/\alpha_{12})+(a_{21}(1+\mathrm{i}w_0)/\alpha_{21})}{(1+\tau^0)+i\tau^0w_0}. \end{aligned}$$

Notice that $\alpha_{12} = a_{12} f'(0), \alpha_{21} = a_{21} f'(0)$. Thus, we have

$$g_{21} = \frac{f'''(0)(|M|^2+1)}{2f'(0)} \frac{1+\mathrm{i}w_0}{(1+\tau^0)+\mathrm{i}\tau^0 w_0}$$
$$= \frac{f'''(0)}{f'(0)} \cdot \frac{(|M|^2+1)}{2\Delta} \{[(1+\tau^0)+\tau^0 w_0^2] + \mathrm{i}[(1+\tau^0)w_0 - \tau^0 w_0]\},$$

where

$$\Delta = (1+\tau^0)^2 + (\tau^0)^2 w_0^2, \qquad \mathrm{Re} g_{21} = \frac{f'''(0)}{f'(0)} \cdot \frac{1}{2\Delta} (|M|^2+1)(1+\tau^0+\tau^0 w_0^2).$$

From (43) and $g_{11} = g_{20} = g_{02} = 0$, we have

$$\mathrm{Re} c_1(0) = \tfrac{1}{2} \mathrm{Re} g_{21} = \frac{f'''(0)}{f'(0)} \cdot \frac{1}{4\Delta} (1+|M|^2)(1+\tau^0+\tau^0 w_0^2).$$

From Lemma 6 we know that

$$\mathrm{Re}\lambda'(0) > 0,$$

and hence, we have

$$\mu_2 = -\frac{\mathrm{Re}\{c_1(0)\}}{\mathrm{Re}\lambda'(0)} > 0\ (< 0) \text{ when } \frac{f'''(0)}{f'(0)} < 0\ (> 0)$$

and

$$\beta_2 = -2\mathrm{Re}\{c_1(0)\} < 0\ (> 0) \text{ when } \frac{f'''(0)}{f'(0)} < 0\ (> 0).$$

By the general results of Hassard, Kazarinoff and Wan [16], the conclusion of the theorem follows. □

5. An example

Babcock and Westervelt [4] considered the following simple two-neuron network with two delays:

$$\begin{cases} \dfrac{\mathrm{d}u_1(t)}{\mathrm{d}t} = -u_1(t) + a_1 \tanh[u_2(t-\tau_2)], \\ \dfrac{\mathrm{d}u_2(t)}{\mathrm{d}t} = -u_2(t) + a_2 \tanh[u_1(t-\tau_1)], \end{cases} \tag{45}$$

where a_1, a_2, τ_1 and τ_2 are positive constants.

Notice that $f'''(0)/f'(0) = \tanh'''(0)/\tanh'(0) = -2 < 0$, by Theorem 3, we obtain the following result on stability and bifurcation in system (45).

Corollary 1. *If $a_1 a_2 < -1$, then there exists a $\tau^0 > 0$ such that when $\tau_1 + \tau_2 \in [0, \tau^0)$, the zero solution of (45) is asymptotically stable and $\tau^0 = \tau_1^0 + \tau_2^0$ is a Hopf bifurcation value. A family of periodic solutions bifurcates from the equilibrium exist when $\tau = \tau_1 + \tau_2 > \tau^0$ and is orbitally asymptotically stable.*

Remark 1. Bakcock and Westervelt [4] showed that when $a_1 a_2 < -1$ and $\tau_1 + \tau_2$ is sufficiently small, the origin of system (45) is stable and the transients spiral toward it. When the total delay increases through a critical value, the origin becomes unstable and the input and output voltages oscillate in a limit cycle. In Corollary 1, we confirm

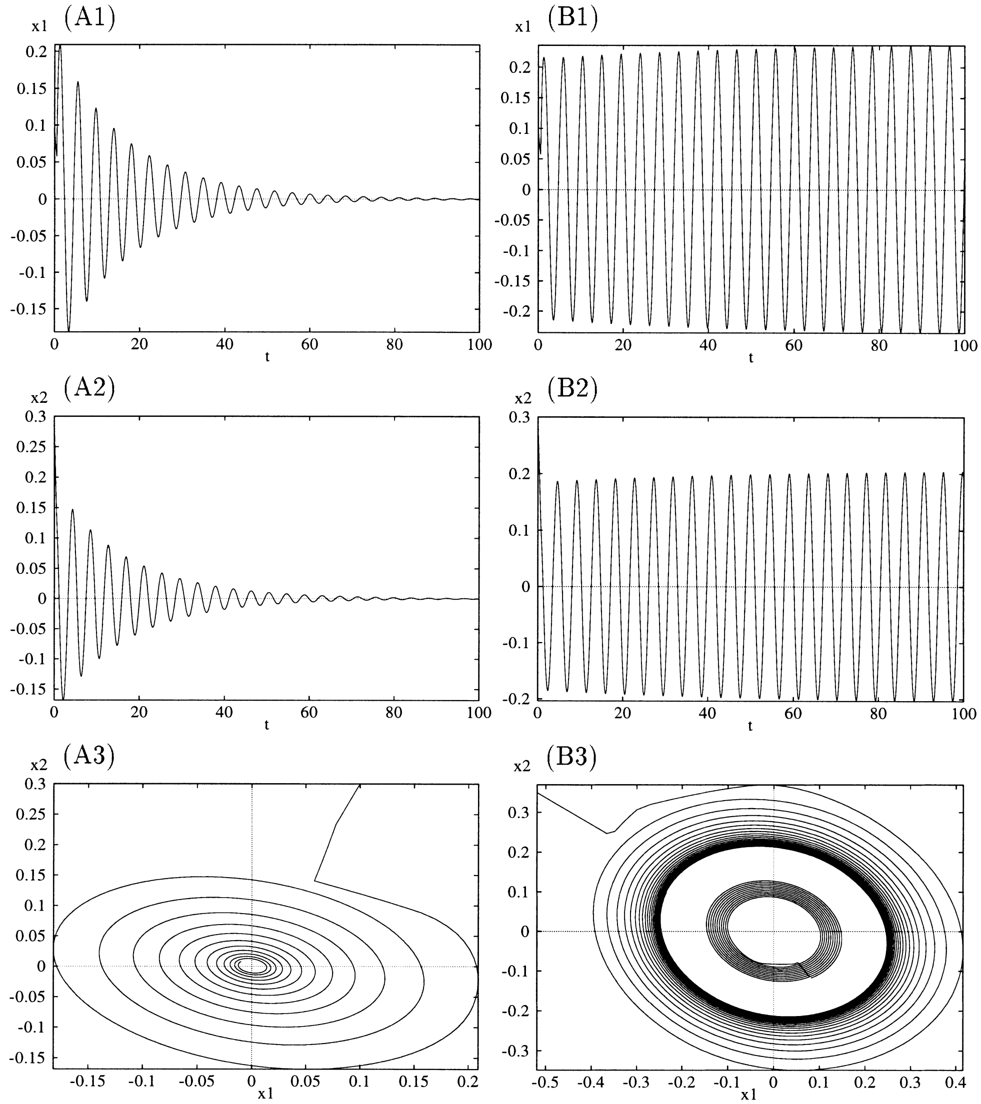

Fig. 2. (A1)–(A3): $\tau_1 + \tau_2 < 0.8$; (B1)–(B3): $\tau_1 + \tau_2 > 0.8$.

their analysis and mathematically prove that the bifurcating periodic solutions are orbitally asymptotically stable, that is, the limit cycle they obtained is indeed stable. Hence, Corollary 1 complements their analysis.

Remark 2. Notice that when $\tau_1 = \tau_2$, system (45) reduces to the model considered by Gopalsamy and Leung [7]. Thus, Corollary 1 also complements their results.

As an example, consider system (45) with $a_1 = 2$ and $a_2 = -1.5$. Then $\tau_0 = 0.8$. Choose $\tau_1 = 0.2$, $\tau_2 = 0.5$, then $\tau_1 + \tau_2 < 0.8$. Fig. 2 (A1)–(A3) show that the origin is asymptotically stable. By Corollary 1, a Hopf bifurcation occurs when $\tau_1 + \tau_2 = 0.8$, the origin loses its stability and a periodic solution bifurcates from the origin exists for $\tau_1 + \tau_2 > 0.8$. The bifurcation is supercritical and the bifurcating periodic solution is orbitally asymptotically stable. Choose $\tau_1 = 0.325$, $\tau_2 = 0.525$, the computer simulations are depicted in Fig. 2(B1)–(B3).

6. Discussion

Recently, a few applicable sufficient criteria have been established for the stability of neural network models with delays (see Bélair [18], Bélair and Dufour [17], Gopalsamy and He [19], van den Driessche and Zou [20], Ye, Michel and Wang [21] and the references cited therein). Bifurcations in neural network models with a single delay have also been observed by many researchers (see Bélair, Campbell and van den Driessche [22], Gopalsamy and Leung [7], Marcus, Waugh and Westervelt [23], Olien and Bélair [8], Wu [24] and the references cited therein). However, there are few papers on the bifurcations of the neural network models with multiple delays (see Campbell [3]).

In this paper, we have considered a simple two-neuron network model with two delays. The characteristic equation of the linearized system at the zero solution is a transcendental equation involving exponential functions. As pointed out by Olien and Bélair [8], it is difficult to find all parameters for all the characteristic roots to have negative real parts. By using the technique in Cooke and Grossman [15] and Rouché's theorem, we have derived some sufficient conditions to ensure that all the characteristic roots have negative real parts. Hence, the zero solution of the model is asymptotically stable.

In the case when there is no self-connection, we have found that when the sum of the two delays, $\tau = \tau_1 + \tau_2$, varies, the zero solution loses its stability and a Hopf bifurcation occurs, that is, a family of periodic solutions bifurcate from the zero solution when τ passes a critical value, say τ_0. The stability and direction of the Hopf bifurcation are studied by applying the normal form theory and the center manifold theorem.

Neural networks with delays have very rich dynamics. From the point of view of nonlinear dynamics their analyses are useful in solving problems of both theoretical and practical importance. The two-neuron networks with two delays discussed above are quite simple, but they are potentially useful since the complexity found in these simple cases might be carried over to larger networks with multiple delays.

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