

## Periodic orbits near heteroclinic cycles in a cyclic replicator system

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**Abstract** A species is semelparous if every individual reproduces only once in its life and dies immediately after the reproduction. While the reproduction opportunity is unique per year and the individual's period from birth to reproduction is just  $n$  years, the individuals that reproduce in the  $i$ th year (modulo  $n$ ) are called the  $i$ th year class,  $i = 1, 2, \dots, n$ . The dynamics of the  $n$  year-class system can be described by a differential equation system of Lotka–Volterra type. For the case  $n = 4$ , there is a heteroclinic cycle on the boundary as shown in previous works. In this paper, we focus on the case  $n = 4$  and show the existence, growth and disappearance of periodic orbits near the heteroclinic cycle, which is a part of the conjecture by Diekmann and van Gils (SIAM J Appl Dyn Syst 8:1160–1189, 2009). By analyzing the Poincaré map near the heteroclinic cycle and introducing a metric to measure the size of the periodic orbit, we show that (i) when the average competitive degree among subpopulations (year classes) in the system is weak, there exists an asymptotically stable periodic orbit near the heteroclinic cycle which is repelling; (ii) the periodic orbit grows in size when some competitive degree increases, and converges to the heteroclinic cycle when the average competitive degree tends to be strong; (iii) when the average competitive

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degree is strong, there is no periodic orbit near the heteroclinic cycle which becomes asymptotically stable. Our results provide explanations why periodic solutions expand and disappear and why all but one subpopulation go extinct.

**Keywords** Replicator equation · Periodic orbit · Heteroclinic cycle · Average competitive degree · Semelparous population

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## 1 Introduction

There has been a growing literature on the theoretical study of semelparous population dynamics in recent years. Examples of semelparous species include annual and biennial plants, butterflies, cicadas, mayflies, Pacific salmon, etc (Behncke 2000). If there is a unique reproduction opportunity per year and the length of the life cycle is just  $n$  years, then the population can be classified into different year classes: the individuals that reproduce in the  $i$ th year (modulo  $n$ ) are called the  $i$ th year class,  $i = 1, 2, \dots, n$ .

Various dynamical outcomes are likely in semelparous populations since interactions such as competition exist between different year classes as well as within the same year class, a specific year class may be driven to extinction. The species in which all year classes except one go extinct is called a periodical insect (Bulmer 1977). Interesting examples include the 13- and 17-year classes of some cicada species. Mathematically, the dynamics of semelparous populations can be described by a discrete-time nonlinear Leslie matrix model (Cushing 2006; Davydova 2004; Davydova et al. 2005; Diekmann and van Gils year; Drissche and Zeeman 1998; Kon 2005; Kon and Iwasa 2007; Mjølhus et al. 2005). The phenomenon of periodical insects can be explained by the invariance of coordinate axes and hyperplanes of the full life cycle map (Davydova et al. 2003) and the existence of heteroclinic cycles connecting different year cycles (Cushing 2009; Diekmann and van Gils 2009).

Diekmann and van Gils (2009) demonstrated that the  $n$ -dimensional discrete-time Leslie matrix model can be reduced to a cyclic replicator system on the  $(n - 1)$  dimensional simplex and classified the repertoire of the dynamical behavior for  $n = 2$  and 3. For  $n = 4$ , they derived almost all possible dynamical behaviors and identified some open problems about the dynamics near the heteroclinic cycles.

Consider the cyclic replicator system (Diekmann and van Gils 2009; Edalat A and Zeeman 1992)

$$\dot{u}_i = u_i \left( -(Bu)_i + \sum_{j=1}^n u_j (Bu)_j \right), \quad i = 1, 2, \dots, n, \quad (1.1)$$

where  $u$  is an  $n$ -dimensional vector in the simplex  $S_n$  which is defined by

$$S_n = \left\{ u \in R^n : \sum_{j=1}^n u_j = 1, u_j \geq 0, j = 1, 2, \dots, n \right\},$$

and  $(Bu)_i$  denotes the  $i$ th component of the vector  $Bu$ .  $B$  is a circulant matrix defined by

$$B = \begin{pmatrix} 0 & b_1 & \cdots & b_{n-1} \\ b_{n-1} & 0 & \cdots & b_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ b_1 & b_2 & \cdots & 0 \end{pmatrix},$$

where the rows of  $B$  are cyclic permutations of the first row. System (1.1) is derived from the cyclic competitive system

$$\dot{x}_i = x_i(1 - (Ax)_i), \quad x_i \geq 0, \quad i = 1, 2, \dots, n, \tag{1.2}$$

where  $x_i$  represents the population density of the  $i$ th year class of the semelparous species, and  $n$  represents the individuals' period from birth to reproduction while it is supposed that there is a unique reproduction opportunity per year (Cushing 2006, 2009; Kon and Iwasa 2007; Mjølhus et al. 2005). The individuals that reproduce in the  $i$ th year (modulo  $n$ ) are called the  $i$ th year class. In system (1.2), all year classes have the same intrinsic growth rate and the interaction matrix  $A$  is circulant. These features are shown naturally in the interesting derivation in section A.2 of Diekmann and van Gils (2009).

The circulant matrix  $A$  in (1.2) is defined by

$$A = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_n & a_1 & \cdots & a_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & \cdots & a_1 \end{pmatrix}, \quad a_i \geq 0, \quad i = 1, 2, \dots, n,$$

where  $\frac{1}{a_1}$  represents the carrying capacity of every year class and  $\frac{a_i}{a_1} (i \neq 1)$  represents the competitive degree from other year classes. As shown by Diekmann and van Gils (2009, p.1163), the derivation of (1.1) from (1.2) is mainly by the projection from  $R_+^n$  to  $S_n$  through  $u_i = x_i / \sum_{j=1}^n x_j$  and

$$b_i = a_{i+1} - a_1, \quad i = 1, 2, \dots, n - 1.$$

Since  $x_i$  denotes the population density of the  $i$ th year class, then  $u_i$  represents the fraction of the  $i$ th year class. As shown by Diekmann and van Gils (2009), the dynamical behavior of (1.1) with  $n = 2, 3$  is completely understood and presented while that of the case  $n = 4$  is given in an almost complete picture. Let  $n = 4$ , matrix  $B$  in (1.1) becomes

$$B = \begin{pmatrix} 0 & b_1 & b_2 & b_3 \\ b_3 & 0 & b_1 & b_2 \\ b_2 & b_3 & 0 & b_1 \\ b_1 & b_2 & b_3 & 0 \end{pmatrix}. \tag{1.3}$$

It is shown by [Diekmann and van Gils \(2009\)](#) that (i) when  $b_1 + b_3 \neq 0$ , Hopf bifurcations occur at  $b_2 = 0$  and small periodic orbits emerge when  $b_2$  is close to zero, and (ii) there exist heteroclinic cycles on the boundary of  $S_4$ . Since the stability change of heteroclinic cycles will lead to large periodic orbits, an intriguing conjecture is put forward by [Diekmann and van Gils \(2009, p.1180\)](#): “*We conjecture that the small periodic orbits grow in size and finally disappear in the heteroclinic cycles*”.

In this paper, we consider the case  $(b_1 + b_3)b_2 < 0$  and  $b_1b_3 < 0$  while we focus on the case  $b_1 + b_3 < 0$ ,  $b_2 > 0$  and  $b_1b_3 < 0$  because of the symmetry of system (1.1). Our aim is to establish the existence, uniqueness, growth and disappearance of periodic orbits near the heteroclinic cycle, which is part of the conjecture of [Diekmann and van Gils \(2009\)](#). By analyzing the Poincaré map near the heteroclinic cycle and introducing a metric to measure the size of periodic orbits, we show that (i) when  $b_1 + b_2 + b_3$  is slightly less than zero, there exists a periodic orbit near the heteroclinic cycle and the periodic orbit is asymptotically stable while the heteroclinic cycle is repelling; (ii) the periodic orbit grows in size when  $b_2$  increases and tends to the heteroclinic cycle when  $b_1 + b_2 + b_3$  converges to zero; (iii) when  $b_1 + b_2 + b_3$  is slightly larger than zero, there is no periodic orbit near the heteroclinic cycle which becomes asymptotically stable. While the expression  $b_1 + b_2 + b_3$  corresponds to the average competitive degree among different year classes, our results provide explanations why periodic solutions expand and disappear, and why all but one year class go extinct: when the average competitive degree is weak, the year classes in the system coexist and exhibit periodic oscillations; the magnitude of the periodic oscillations grows in size when the competitive degree  $\frac{a_3}{a_1}$  increases; when the average competitive degree is strong, the periodic oscillations disappear and all but one year class go extinct. Numerical simulations are also given to show the existence, growth and disappearance of periodic orbits.

The paper is organized as follows. In Sect. 2, we recall some results by [Diekmann and van Gils \(2009\)](#) on the cyclic replicator system. In Sect. 3, we establish the existence and uniqueness of periodic orbits. The growth and disappearance of periodic orbit is studied in Sect. 4. In Sect. 5, we give some ecological applications of our results and present some discussions.

## 2 The cyclic replicator system

In this section, some previous results about (1.1) for  $n = 4$  are recalled. Since the dynamical behavior of (1.1) for the case  $(b_1 + b_3)b_2 > 0$  is given by [Diekmann and van Gils \(2009\)](#), we analyze the case  $(b_1 + b_3)b_2 < 0$  and  $b_1b_3 < 0$  in this paper. While system (1.1) is invariant under  $(B, t) \rightarrow (-B, -t)$ , we focus on the case

$$b_1 + b_3 < 0, b_2 > 0, b_1b_3 < 0.$$

Results for the case  $b_1 + b_3 > 0, b_2 < 0$  and  $b_1b_3 < 0$  are similar: the directions of the orbits are just opposite to those in the case  $b_1 + b_3 < 0, b_2 > 0$  and  $b_1b_3 < 0$ .

Without loss of generality, we suppose  $b_3 < 0$ . When both of the left and right sides of (1.1) are divided by  $-b_3$  and a time rescaling  $t \rightarrow t/(-b_3)$  is applied, matrix  $B$

of (1.1) as shown in (1.3) becomes the following form while  $b_1/(-b_3)$  and  $b_2/(-b_3)$  are still denoted by  $b_1$  and  $b_2$ , respectively:

$$B = \begin{pmatrix} 0 & b_1 & b_2 & -1 \\ -1 & 0 & b_1 & b_2 \\ b_2 & -1 & 0 & b_1 \\ b_1 & b_2 & -1 & 0 \end{pmatrix}.$$

Then the case we focus on becomes

$$0 < b_1 < 1, \quad b_2 > 0, \quad b_3 = -1. \tag{2.1}$$

Let  $S$  be a circular matrix defined by

$$S = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

**Lemma 2.1** (Diekmann and van Gils 2009) *The replicator system (1.1) is equivariant with respect to  $S$ , i.e., if  $u$  is a solution of (1.1), then  $Su$  is also a solution of (1.1).*

Let  $E_i$  denote the equilibrium with  $i$  coexisting species. When conditions in (2.1) hold, the equilibria of (1.1) (modulo cyclic permutation) are:

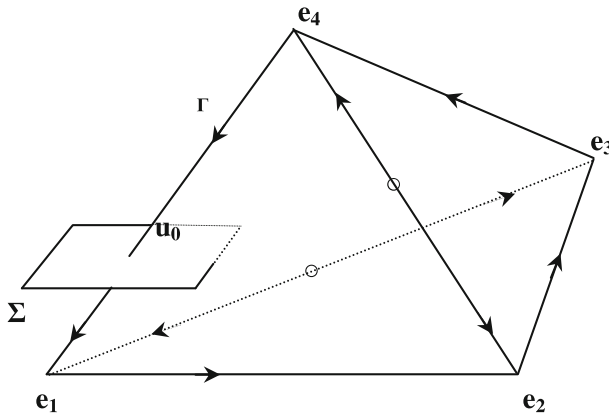
$$E_1 = (1, 0, 0, 0), \quad E_{2s} = \left(\frac{1}{2}, 0, \frac{1}{2}, 0\right), \quad E_4 = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right).$$

**Lemma 2.2** (Diekmann and van Gils 2009)

- (i) *The equilibrium  $E_1$  has eigenvalues  $-b_1, 1, -b_2$  with corresponding eigenvectors  $(1, 0, 0, -1), (-1, 1, 0, 0)$  and  $(-1, 0, 1, 0)$ , respectively;*
- (ii) *The equilibrium  $E_{2s}$  has eigenvalues  $\frac{1}{2}b_2, \frac{1}{2}(b_2 - b_1 + 1), \frac{1}{2}(b_2 - b_1 + 1)$  with corresponding eigenvectors  $(1, 0, -1, 0), (0, 1, \frac{1}{b_1}(-1 - b_1), \frac{1}{b_1})$  and  $(1, 0, -\frac{1}{b_1}, \frac{1}{b_1}(1 - b_1))$ , respectively;*
- (iii) *The equilibrium  $E_4$  has eigenvalues  $\frac{1}{4}(b_1 - 1 - b_2), \frac{1}{4}[b_2 \pm i(1 + b_1)]$  and the eigenvalue  $\frac{1}{4}(b_1 - 1 - b_2)$  has an eigenvector  $(-1, 1, -1, 1)$ .*

**Lemma 2.3** (Diekmann and van Gils 2009) *Let (2.1) hold.*

- (i) *A supercritical Hopf bifurcation occurs at  $b_2 = 0$ , i.e., there exist small stable periodic orbits near  $E_4$  when  $b_2$  is slightly larger than zero;*
- (ii) *There exists a heteroclinic cycle  $\Gamma$ , which connects the four equilibria  $E_1$  (i.e.,  $E_1, SE_1, S^2E_1, S^3E_1$ ) in cyclic order. When  $b_1 + b_2 - 1 > 0$ ,  $\Gamma$  is asymptotically stable.*



**Fig. 1** The heteroclinic cycle  $\Gamma$  of (1.1) is the connection of  $e_i$  in the order of  $e_1 \rightarrow e_2 \rightarrow e_3 \rightarrow e_4 \rightarrow e_1$ .  $u_0$  is a trivial point on  $\Gamma$  and  $\Sigma$  is a hyperplane perpendicular to  $\Gamma$  at  $u_0$ . The empty circles denote the unstable equilibria ( $E_{2s}$ ) on the boundary

*Remark 2.4* While system (1.1) comes from (1.2) where  $a_i \geq 0$  for  $1 \leq i \leq n$ , the constraints on  $b_i$ ,  $1 \leq i \leq n - 1$  in (1.1) for  $n = 4$  are

$$\begin{aligned}
 b_1 + b_3 &\leq 1 - b_2, \\
 3b_1 - b_3 &\geq b_2 - 1, \\
 3b_3 - b_1 &\geq b_2 - 1, \\
 3b_2 - b_1 &\geq b_3 - 1.
 \end{aligned}
 \tag{2.2}$$

Since the analysis in this paper is limited in a parameter region  $|b_1 + b_2 + b_3| < \epsilon$  where  $\epsilon$  is sufficiently small, the constraints in (2.2) are not specifically considered. However, parameters are given under the constraints in numerical simulations.

### 3 Existence and uniqueness of periodic orbits

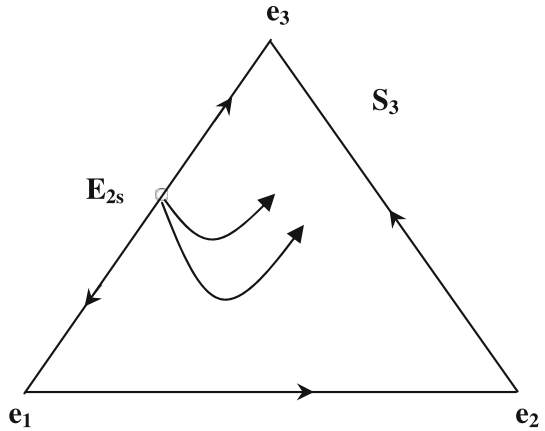
In this section, we establish the existence and uniqueness of a periodic orbit of (1.1) near the heteroclinic cycle  $\Gamma$  by analyzing the fixed point of the Poincaré map near  $\Gamma$ . In order to study how the dynamic behavior of (1.1) changes when the parameter  $b_2$  varies, we fix the other parameters  $b_1$  and  $b_3$ .

Let  $e_1 = (1, 0, 0, 0)^T$  and  $e_i = S^{i-1}e_1$ ,  $i = 2, 3, 4$ . Then the heteroclinic cycle  $\Gamma$  is the connection of  $e_i$  in the order of  $e_1 \rightarrow e_2 \rightarrow e_3 \rightarrow e_4 \rightarrow e_1$ , which lies on the boundary of  $S_4$  as shown in Fig. 1. The boundary of  $S_4$  is denoted by  $\text{bd}S_4$ , and the following result shows the asymptotical stability of  $\Gamma$  on  $\text{bd} S_4$ .

**Lemma 3.1** *Let (2.1) hold. The heteroclinic cycle  $\Gamma$  is asymptotically stable on  $\text{bd}S_4$ .*

*Proof* It follows from Lemma 2.1 that we only need to show  $\Gamma$  is asymptotically stable on the two-dimensional surface  $S_3 = \{u \in S_4 : u_4 = 0 \text{ and } \sum_{i=1}^3 u_i = 1\}$ . By Lemma 2.2, there are four equilibria of (1.1) on  $S_3$ . Equilibrium  $e_1$  has eigenvalues

**Fig. 2** On  $S_3$ , equilibria  $e_1$  and  $e_2$  are saddles,  $e_3$  is a stable node and  $E_{2s}$  is an unstable node. Both equilibrium  $e_3$  and the orbit  $\overline{e_1 e_2} \cup \overline{e_2 e_3} \subset \Gamma$  are asymptotically stable on  $S_3$



$-b_2$  and 1 with corresponding eigenvectors  $(-1, 0, 1, 0)$  and  $(-1, 1, 0, 0)$ , respectively. It follows from  $e_2 = Se_1$  and Lemma 2.2(i) that equilibrium  $e_2$  has eigenvalues  $-b_1$  and 1 with corresponding eigenvectors  $(-1, 1, 0, 0)$  and  $(0, -1, 1, 0)$ , respectively. Then equilibria  $e_1$  and  $e_2$  are saddles on  $S_3$ . Similarly,  $e_3$  has eigenvalues  $-b_1$  and  $-b_2$  with corresponding eigenvectors  $(0, -1, 1, 0)$  and  $(1, 0, -1, 0)$ , respectively. Then  $e_3$  is a stable node. Since all the eigenvalues of  $E_{2s}$  are positive,  $E_{2s}$  is an unstable node on  $S_3$ . While there is no interior equilibrium on the planar surface  $S_3$ , we can draw the phase portraits of (1.1) on  $S_3$  as shown in Fig. 2, where  $\Gamma$  is asymptotically stable on  $S_3$ . □

The Poincaré map near  $\Gamma$  can be approximated by the following projection (Hofbauer 1987; Hofbauer and Sigmund 1998, 15, p. 226, equation(17.20)):

$$L : z \rightarrow Pz + q, \tag{3.1}$$

where  $z = (z_1, z_2)^T$ . The heteroclinic cycle  $\Gamma$  corresponds to  $z_1, z_2 \rightarrow +\infty$ , and the periodic orbit of (1.1) near  $\Gamma$  corresponds to the fixed point  $\hat{z} = (\hat{z}_1, \hat{z}_2)^T$  of  $L$  if  $\hat{z}$  is positive and large, i.e.,  $\hat{z} = (I - P)^{-1}q$  is positive and large when the matrix  $I - P$  is invertible.

$P$  and  $q$  in (3.1) are obtained as follows. At the equilibrium  $e_1$ , eigenvalues  $-b_1$  and 1 have eigenvectors on  $\Gamma$ , and eigenvalue  $-b_2$  has an eigenvector  $(-1, 0, 1, 0)$ , which is transversal to  $\Gamma$  as shown in Fig. 1. By Lemma 2.1, similar discussions can be given for equilibria  $e_i, i = 2, 3, 4$ . That is, at  $e_i$ , eigenvalues  $-b_1$  and 1 have eigenvectors on  $\Gamma$ , and eigenvalue  $-b_2$  has an eigenvector which is transversal to  $\Gamma, i = 2, 3, 4$ . Hence, the matrix  $P$  comes from the combination of eight projections and can be expressed as the product of four matrices  $C$  (Hofbauer and Sigmund 1998, p. 226), i.e.  $P = C^4$  and

$$C = \begin{pmatrix} b_2 & 1 \\ b_1 & 0 \end{pmatrix},$$

then

$$P = \begin{pmatrix} (b_1 + b_2^2)^2 + b_1 b_2^2 & b_2 (2b_1 + b_2^2) \\ b_1 b_2 (2b_1 + b_2^2) & b_1 (b_1 + b_2^2) \end{pmatrix}. \tag{3.2}$$

Similarly, the vector  $q$  comes from the combination of eight projections and can be expressed as the sum of four vectors as follows:

$$q = C^3\alpha + C^2\alpha + C\alpha + \alpha,$$

where  $\alpha = (\alpha_1, \alpha_2)^T$  and

$$\alpha_1 = -\int_0^1 c_1(u_2)du_2, \quad \alpha_2 = -\int_0^1 c_2(u_2)du_2.$$

The function  $c_1(u_2)$  is given by

$$\begin{aligned} c_1(u_2) &= \lim_{u_3, u_4 \rightarrow 0} \frac{\dot{u}_3}{u_3 \dot{u}_2} + b_2 \frac{1}{u_2} - \frac{1}{b_1} \frac{1}{1-u_2} \\ &= \lim_{u_1 \rightarrow 1-u_2} \frac{-b_2 u_1 + u_2 + (b_1 - 1)u_1 u_2}{u_1 u_2 (1 + (b_1 - 1)u_2)} + b_2 \frac{1}{u_2} - \frac{1}{b_1} \frac{1}{1-u_2} \\ &= -b_2 \left( \frac{1}{u_2} + \frac{1-b_1}{1+(b_1-1)u_2} \right) + \left( \frac{1}{b_1} \frac{1}{1-u_2} + \frac{b_1-1}{b_1} \frac{1}{1+(b_1-1)u_2} \right) \\ &\quad + \frac{b_1-1}{1+(b_1-1)u_2} + b_2 \frac{1}{u_2} - \frac{1}{b_1} \frac{1}{1-u_2} \\ &= (b_1 - 1) \frac{1 + b_1 + b_1 b_2}{b_1 (1 + (b_1 - 1)u_2)}. \end{aligned}$$

Thus, we have

$$\alpha_1 = (1 + b_1 + b_1 b_2) \bar{b}_1, \quad \bar{b}_1 = \frac{-\log b_1}{b_1} > 0.$$

Similarly,

$$\begin{aligned} c_2(u_2) &= \lim_{u_3, u_4 \rightarrow 0} \frac{\dot{u}_4}{u_4 \dot{u}_2} + b_1 \frac{1}{u_2} + \frac{b_2}{b_1} \frac{1}{1-u_2} \\ &= \lim_{u_1 \rightarrow 1-u_2} \frac{-b_1 u_1 - b_2 u_2 + (b_1 - 1)u_1 u_2}{u_1 u_2 (1 + (b_1 - 1)u_2)} + b_1 \frac{1}{u_2} + \frac{b_2}{b_1} \frac{1}{1-u_2} \\ &= -b_1 \left( \frac{1}{u_2} + \frac{1-b_1}{1+(b_1-1)u_2} \right) - b_2 \left( \frac{1}{b_1} \frac{1}{1-u_2} + \frac{b_1-1}{b_1} \frac{1}{1+(b_1-1)u_2} \right) \end{aligned}$$

$$\begin{aligned}
 & + \frac{b_1 - 1}{1 + (b_1 - 1)u_2} + b_1 \frac{1}{u_2} + \frac{b_2}{b_1} \frac{1}{1 - u_2} \\
 & = (b_1 - 1) \frac{b_1 + b_1^2 - b_2}{b_1(1 + (b_1 - 1)u_2)}.
 \end{aligned}$$

Therefore, we have

$$\alpha_2 = (b_1 + b_1^2 - b_2)\bar{b}_1.$$

Let

$$\mu = b_1 + b_2 - 1. \tag{3.3}$$

Then  $P = P(\mu)$  and  $q = q(\mu)$  are functions of  $\mu$  since  $b_1$  is fixed and  $b_2 = 1 - b_1 + \mu$ .

We now show that  $I - P(\mu)$  is invertible. Since the matrix  $P$  in (3.2) is positive, it follows from Perron–Frobenius Theorem that there is a principal eigenvalue  $\rho(\mu) > 0$  with corresponding left and right eigenvectors  $v(\mu)$  and  $w(\mu)$ , which satisfy

$$\rho(\mu)^{-k} P^k z \rightarrow cw(\mu), \quad \forall z > 0, \quad k \rightarrow +\infty, \tag{3.4}$$

where  $c = \sum_{i=1}^4 v_i z_i$  and  $b_2 = 1 - b_1 + \mu$ . A straightforward computation shows that  $\rho(0) = 1$ . Since the parameter  $b_1$  in  $\mu$  is fixed, all entries of matrix  $P$  in (3.2) are positive and strictly increasing functions of  $\mu$  (i.e.,  $b_2$ ). Then  $\rho(\mu)$  is a strictly increasing function of  $\mu$  [21, p. 4]. Thus we have

$$\frac{d\rho}{d\mu} > 0.$$

Hence, if  $\mu < 0$  then  $\rho < 1$ , and it follows from (3.4) that  $P^k z \rightarrow 0+$ , i.e.,  $\Gamma$  is repelling. Similarly, if  $\mu > 0$  then  $\rho > 1$  and  $\Gamma$  is asymptotically stable.

Since  $\frac{d\rho(\mu)}{d\mu} > 0$  and  $\rho(0) = 1$ , there is  $\mu_1 > 0$  such that if  $-\mu_1 < \mu < 0$ , both eigenvalues of  $P(\mu)$  are less than 1; if  $0 < \mu < \mu_1$ , only one of the eigenvalues is larger than 1. Thus, we have proved the following results.

**Lemma 3.2** *There is  $\mu_1 > 0$  such that when  $|\mu| < \mu_1$  and  $\mu \neq 0$ ,*

$$\det(I - P(\mu))\mu < 0.$$

Hence,  $I - P(\mu)$  is invertible as  $|\mu| < \mu_1$  and  $\mu \neq 0$ .

Next, we show that  $(I - P(\mu))^{-1}q(\mu)$  is positive and large. When  $|\mu| < \mu_1$  and  $\mu \neq 0$ , the projection  $L$  has a fixed point  $\hat{z}(\mu)$ :

$$\hat{z}(\mu) = (I - P(\mu))^{-1}q(\mu) = \frac{adj(I - P(\mu))q(\mu)}{\det(I - P(\mu))}. \tag{3.5}$$

Since

$$q(\mu) = (q_1(\mu), q_2(\mu))^T = \sum_{j=0}^3 C^j \alpha,$$

$$\sum_{j=0}^3 C^j = \begin{pmatrix} 1 + b_1 + b_2 + 2b_1b_2 + b_2^2 + b_2^3 & 1 + b_1 + b_2 + b_2^2 \\ b_1 + b_1b_2 + b_1^2 + b_1b_2^2 & 1 + b_1 + b_1b_2 \end{pmatrix},$$

where  $b_2 = 1 - b_1 + \mu$ , a long but straightforward computation shows that

$$q_1(\mu) = \left[ 1 + 13b_1 - 10b_1^2 + 6b_1^3 - 3b_1^4 + b_1^5 \right. \\ \left. + (20b_1 - 20b_1^2 + 12b_1^3 - 4b_1^4) \mu + O(\mu^2) \right] \bar{b}_1,$$

$$q_2(\mu) = \left[ -1 + 3b_1 + 10b_1^2 - 6b_1^3 + 3b_1^4 - b_1^5 \right. \\ \left. + (-1 + 10b_1^2 - 8b_1^3 + 3b_1^4) \mu + O(\mu^2) \right] \bar{b}_1,$$

where  $O(\mu^k)$  denotes that  $\mu^k O(\mu^k)$  is a bounded function of  $\mu$  as  $\mu \rightarrow 0$  for  $k \geq 1$ .

Let  $D = (d_{ij})_{2 \times 2}$  denote  $\text{adj}(I - P(\mu))$ . Then

$$D = \begin{pmatrix} 1 - b_1(b_1 + b_2^2) & b_2(2b_1 + b_2^2) \\ b_1b_2(2b_1 + b_2^2) & 1 - (b_1 + b_2^2)^2 - b_1b_2^2 \end{pmatrix}.$$

We have

$$(\text{adj}(I - P(\mu))q(\mu))_1 = d_{11}q_1 + d_{12}q_2 \\ = \left[ 16(b_1 - b_1^2 + b_1^3 - b_1^4) + 4(-1 + 8b_1 - 10b_1^2 \right. \\ \left. + 8b_1^3 - b_1^4) \mu + O(\mu^2) \right] \bar{b}_1,$$

$$(\text{adj}(I - P(\mu))q(\mu))_2 = d_{21}q_1 + d_{22}q_2 \\ = \left[ 16b_1^2(1 - b_1 + b_1^2 - b_1^3) + 4(1 - 4b_1 + 10b_1^2 \right. \\ \left. - 12b_1^3 + 9b_1^4) \mu + O(\mu^2) \right] \bar{b}_1. \tag{3.6}$$

Since

$$b_1 - b_1^2 + b_1^3 - b_1^4 > 0, \quad 1 - b_1 + b_1^2 - b_1^3 > 0,$$

it follows from (3.6) that there is  $\mu_2 > 0$  ( $\mu_2 < \mu_1$ ) such that when  $|\mu| < \mu_2$ , we have

$$(\text{adj}(I - P(\mu))q(\mu))_1 \geq 8(b_1 - b_1^2 + b_1^3 - b_1^4) \bar{b}_1 > 0, \\ (\text{adj}(I - P(\mu))q(\mu))_2 \geq 8b_1^2(1 - b_1 + b_1^2 - b_1^3) \bar{b}_1 > 0. \tag{3.7}$$

By Lemma 3.2, we have  $\det(I - P(\mu)) > 0$  as  $-\mu_2 < \mu < 0$  and  $\det(I - P(0)) = 0$ . Since

$$\hat{z}(\mu) = \frac{\text{adj}(I - P(\mu))q(\mu)}{\det(I - P(\mu))},$$

it follows from (3.7) and the continuity of  $\det(I - P(\mu))$  that when  $-\mu_2 < \mu < 0$ , we have  $\hat{z}(\mu) > 0$  and  $\hat{z}(\mu) \rightarrow \infty$  as  $\mu \rightarrow 0^-$ . That is, there is a periodic orbit near  $\Gamma$  as  $\mu$  is slightly less than zero, and the periodic orbit tends to  $\Gamma$  as  $\mu$  converges to zero. We still denote the small interval by  $-\mu_2 < \mu < 0$ .

Finally, we show the periodic orbit of (1.1) is stable. By changing the variables  $z = Z + \hat{z}(\mu)$  in the projection  $L$ , it becomes

$$L : Z \rightarrow PZ.$$

Since  $\rho(\mu) < 1$  as  $-\mu_2 < \mu < 0$ , it follows from (3.4) that  $P^k Z \rightarrow 0$  as  $k \rightarrow +\infty$ . That is,  $\hat{z}$  is asymptotically stable, which means that the periodic orbit is asymptotically stable. Then we have the following result on the stability of the periodic orbit.

**Theorem 3.3** *Let (2.1) hold. There is  $\mu_2 > 0$  ( $\mu_2 < \mu_1$ ) such that when  $-\mu_2 < \mu < 0$ , there exists a periodic orbit near the heteroclinic cycle  $\Gamma$ . The periodic orbit is asymptotically stable and tends to  $\Gamma$  as  $\mu \rightarrow 0$ .*

We show the periodic orbit in Theorem 3.3 is unique in some region. On  $S_4$ , let  $u_0$  be a trivial point on  $\Gamma$  and  $\Sigma$  a hyperplane perpendicular to  $\Gamma$  at  $u_0$  as shown in Fig. 1. While system (1.1) is defined on  $S_4$ , we still denote  $\Sigma \cap S_4$  by  $\Sigma$ . Let  $O(u_0, \delta_\mu) = \{u : \|u - u_0\| < \delta_\mu\}$ . It follows from Lemma 3.1 that for any  $\mu$  with  $|\mu| < \mu_2$ , there is  $\delta_\mu > 0$  such that the transversal section  $O(u_0, \delta_\mu) \cap \Sigma$  satisfies that for any point  $u \in O(u_0, \delta_\mu) \cap \Sigma$ , the solution of (1.1) through  $u$  will cross  $\Sigma$  again at a point near  $u_0$ . That is, the Poincaré map can be defined on  $O(u_0, \delta_\mu) \cap \Sigma$ .

Since  $\delta_\mu$  could be chosen such that it is a continuous function of  $\mu$ , we have  $\delta_0 = \min_{|\mu| \leq \mu_2/2} \delta_\mu > 0$ . By Theorem 3.3, the periodic orbit tends to  $\Gamma$  as  $\mu \rightarrow 0^-$ . Then for  $\delta_0 > 0$ , there is  $\mu_3 > 0$  ( $\mu_3 < \mu_2/2$ ) such that for any  $\mu$  with  $-\mu_3 < \mu < 0$ , there is a fixed point of (3.1) on  $O(u_0, \delta_0) \cap \Sigma$ . Since the matrix  $I - P(\mu)$  is invertible as  $-\mu_3 < \mu < 0$ , the periodic orbit of (1.1) that crosses  $O(u_0, \delta_0) \cap \Sigma$  is unique. Thus, we have just proved the following conclusion.

**Theorem 3.4** *There are  $\delta_0 > 0$  and  $\mu_3 > 0$  ( $\mu_3 < \mu_2/2$ ) such that as  $-\mu_3 < \mu < 0$ , the periodic orbit of (1.1) that crosses  $O(u_0, \delta_0) \cap \Sigma$  is unique.*

#### 4 Growth and disappearance of periodic orbits

In this section, we introduce a metric to measure the size of periodic orbits. Then we show that the periodic orbit, which is given in Theorems 3.3 and 3.4, grows in size as  $\mu$  increases and disappears as  $\mu$  is slightly larger than zero.

The size of the periodic orbit can be measured by  $\|\hat{z}\| = |\hat{z}_1| + |\hat{z}_2|$ . The reason is that while the heteroclinic cycle  $\Gamma$  corresponds to  $z_1, z_2 \rightarrow +\infty$ , then the larger the

$||\hat{z}||$  the closer the periodic orbit to  $\Gamma$ , which implies the larger the size of the periodic orbit. We show that the metric  $||\hat{z}(\mu)||$  is a monotonously increasing function of  $\mu$ . In fact, it follows from Lemma 3.2 that  $\det(I - P(\mu)) > 0$  as  $-\mu_3 < \mu < 0$ . Then by (3.5), we need to show that (i)  $\frac{d(\det(I - P(\mu)))}{d\mu} < 0$  and (ii)  $\frac{d||adj(I - P(\mu))q(\mu)||}{d\mu} > 0$  when  $-\mu_3 < \mu < 0$ .

Since

$$\det(I - P) = \det \begin{pmatrix} 1 - (b_1 + b_2^2)^2 - b_1 b_2^2 & -b_2 (2b_1 + b_2^2) \\ -b_1 b_2 (2b_1 + b_2^2) & 1 - b_1 (b_1 + b_2^2) \end{pmatrix},$$

we have

$$\begin{aligned} \frac{d(\det(I - P(\mu)))}{d\mu} &= \frac{d(\det(I - P(\mu)))}{db_2} \\ &= 2b_2 \left( 3b_1 (b_1 + b_2^2)^2 - 2 (b_1 + b_2^2) + 2b_1^2 b_2^2 + b_1^3 - 2b_1 \right. \\ &\quad \left. - b_1 (2b_1 + b_2^2)^2 - 2b_1 b_2^2 (2b_1 + b_2^2) \right), \end{aligned}$$

where  $b_2 = 1 - b_1 + \mu$ . Let  $\mu = 0$ , i.e.,  $b_2 = 1 - b_1$ , we have

$$\frac{d(\det(I - P(\mu)))}{d\mu} \Big|_{\mu=0} = -4(1 - b_1)(1 + b_1^2) < 0.$$

By the continuity of  $\frac{d(\det(I - P(\mu)))}{d\mu}$ , we have

**Lemma 4.1** *There is  $\mu_4 > 0$  ( $\mu_4 < \mu_3$ ) such that when  $-\mu_4 < \mu < 0$ ,  $\frac{d(\det(I - P(\mu)))}{d\mu} < 0$ .*

It follows from (3.7) that when  $-\mu_4 < \mu < 0$ , we have

$$(adj(I - P(\mu))q(\mu))_1 > 0, (adj(I - P(\mu))q(\mu))_2 > 0,$$

and

$$\begin{aligned} ||adj(I - P(\mu))q(\mu)|| &= (adj(I - P(\mu))q(\mu))_1 + (adj(I - P(\mu))q(\mu))_2 \\ &= [16(b_1 - b_1^2 + b_1^3 - b_1^4) + 16b_1^2(1 - b_1 + b_1^2 - b_1^3) \\ &\quad + 16(b_1 - b_1^3 + 2b_1^4)\mu + O(\mu^2)]\bar{b}_1, \end{aligned}$$

that is,

$$\frac{d||adj(I - P(\mu))q(\mu)||}{d\mu} \Big|_{\mu=0} = 16(b_1 - b_1^3 + 2b_1^4)\bar{b}_1 > 0.$$

It follows from the continuity of  $\frac{d||adj(I - P(\mu))q(\mu)||}{d\mu}$  that there is  $\mu_5 > 0$  ( $\mu_5 < \mu_4$ ) such that when  $-\mu_5 < \mu < 0$ ,  $\frac{d||adj(I - P(\mu))q(\mu)||}{d\mu} > 0$ . Hence, when  $-\mu_5 < \mu < 0$

and  $\mu$  increases,  $\|\hat{z}\| = \frac{\|adj(I-P(\mu))q(\mu)\|}{det(I-P(\mu))}$  increases, which means that the periodic orbit in Theorems 3.3 and 3.4 increases in size. Then we have the following result.

**Theorem 4.2** *Let (2.1) hold. There is  $\mu_5 > 0$  ( $\mu_5 < \mu_4$ ) such that as  $-\mu_5 < \mu < 0$ , the periodic orbit near the heteroclinic cycle  $\Gamma$  increases in size as  $\mu$  increases.*

Finally, we will show that the large periodic orbit near  $\Gamma$  disappears when  $\mu$  is slightly larger than zero. As discussed in the proof of Theorem 3.3, the fixed point of (3.1) as shown in (3.5) is positive when  $-\mu_5 < \mu < 0$ . However, when  $0 < \mu < \mu_5$ , it follows from Lemma 3.2 that  $det(I - P(\mu)) < 0$ . By (3.7), we have  $adj(I - P(\mu))q(\mu) > 0$ . Hence,  $\hat{z}(\mu) = \frac{adj(I-P(\mu))q(\mu)}{det(I-P(\mu))}$  is negative. That is, there is no positive fixed point of (3.1) when  $0 < \mu < \mu_5$ . Then there is no periodic orbit of (1.1) which would cross through  $O(u_0, \delta_0) \cap \Sigma$ . Hence, the following conclusion holds.

**Theorem 4.3** *Let (2.1) hold. When  $0 < \mu < \mu_5$ , there is no periodic orbit of (1.1) that would cross through  $O(u_0, \delta_0) \cap \Sigma$ .*

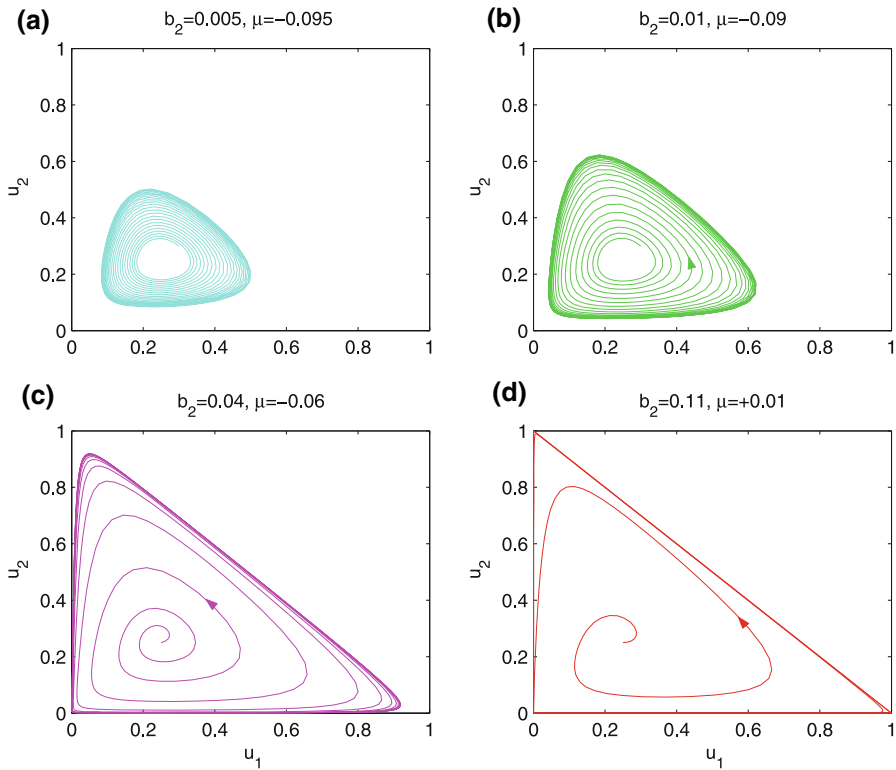
Therefore, it follows from Theorems 3.3, 3.4, 4.2 and 4.3 that the unique periodic orbit of (1.1), which crosses through  $O(u_0, \delta_0) \cap \Sigma$  when  $-\mu_5 < \mu < 0$ , grows in size as  $b_2$  increases, tends to  $\Gamma$  when  $\mu \rightarrow 0^-$ , and disappears when  $0 < \mu < \mu_5$ .

Numerical simulations in Fig. 3 show that the periodic orbit of (1.1) grows in size when  $b_2$  increases and disappears when  $\mu > 0$ . We select  $(u_1, u_2)$  in the solution  $u = (u_1, u_2, u_3, u_4)$  as an example while similar principles can be shown for  $(u_3, u_4)$ . In the simulations, we fix  $b_1 = 0.1$  and  $b_3 = -0.2$ . Then it follows from (2.2) that  $0 < b_2 < 0.3$ . Let  $b_2 = 0.005, 0.01, 0.04, 0.11$ , respectively, then  $\mu = b_1 + b_2 + b_3 = -0.095, -0.09, -0.06, +0.01$ , respectively. When  $\mu = -0.095 < 0$ , there is a small periodic orbit. The periodic orbit grows in size as  $b_2$  increases from 0.005 to 0.01 to 0.04, and tends to  $\Gamma$  monotonously. When  $\mu = 0.01 > 0$ , the periodic orbit disappears and  $\Gamma$  is asymptotically stable. In Fig. 4, we select the first three components  $(u_1(t), u_2(t), u_3(t))$  of solution  $u(t)$  while  $u_4(t) = 1 - u_1(t) - u_2(t) - u_3(t)$ . Numerical simulations show that the periodic orbit of (1.1) is asymptotically stable in the three-dimensional space.

In Sect. 2, we changed the variables as follows:  $\frac{b_1}{-b_3} \rightarrow b_1, \frac{b_2}{-b_3} \rightarrow b_2$  and  $b_3 \rightarrow -1$ . Now we take the reverse changes of variables and return to the original parameters  $b_1, b_2$  and  $b_3$ . Let  $\hat{\mu} = -b_3\mu_5$ . Then we can restate our results about (1.1) as follows.

**Corollary 4.4** *Suppose  $b_1 + b_3 < 0, b_2 > 0$  and  $b_1b_3 < 0$ . For  $\hat{\mu} > 0$  and  $\delta_0 > 0$ , we have*

- (i) *When  $-\hat{\mu} < b_1 + b_2 + b_3 < 0$ , there is a unique periodic orbit of (1.1) that crosses through the transversal sector  $O(u_0, \delta_0) \cap \Sigma$ . The periodic orbit is asymptotically stable.*
- (ii) *The periodic orbit grows in size when  $b_2$  increases in the interval  $(-(b_1 + b_3) - \hat{\mu}, -(b_1 + b_3))$  and tends to the heteroclinic cycle  $\Gamma$  as  $b_2 \rightarrow -(b_1 + b_3)$ .*
- (iii) *There is no periodic orbit of (1.1) that would cross through  $O(u_0, \delta_0) \cap \Sigma$  when  $0 < b_1 + b_2 + b_3 < \hat{\mu}$ .*
- (iv) *The heteroclinic cycle  $\Gamma$  is repelling as  $b_1 + b_2 + b_3 < 0$  while it is asymptotically stable as  $b_1 + b_2 + b_3 > 0$ .*



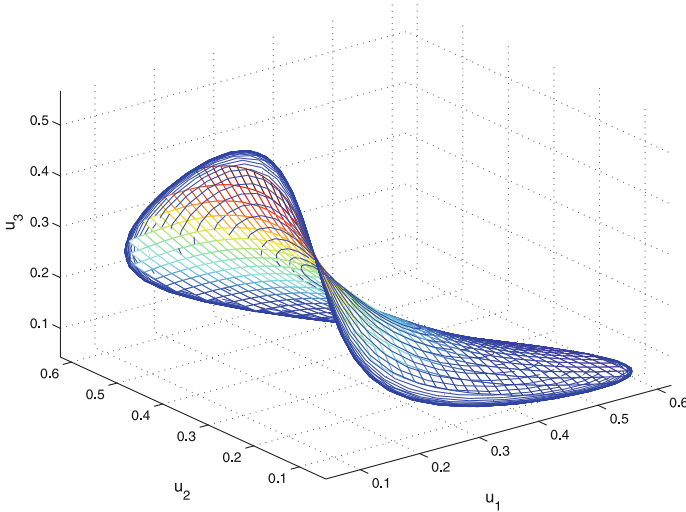
**Fig. 3** In the computations,  $b_1 = 0.1$  and  $b_3 = -0.2$ . Take  $b_2 = 0.005, 0.01, 0.04, 0.11$ , respectively, then  $\mu = b_1 + b_2 + b_3 = -0.095, -0.09, -0.06, +0.01$ , respectively. **a** When  $\mu = -0.095 < 0$ , there is a small periodic orbit. **b** The periodic orbit grows in size as  $b_2$  increases from 0.005 to 0.01 to 0.04 and **c** tends to  $\Gamma$  monotonically. **d** When  $\mu = 0.01 > 0$ , the periodic orbit disappears and  $\Gamma$  is asymptotically stable

## 5 Applications and discussions

In this section, we apply our results to some ecological examples. As mentioned in Sect. 1, there are many semelparous species in natural environments. While different year classes of the species are identical except for their reproduction time, some of them go asymptotically extinct. The most interesting periodical species are the 13th and 17th year cicadas of eastern North America. What are the mechanisms that result in both the persistence of only one brood and the remaining of this brood? How could the year classes coexist?

The novel model (1.1) derived by Diekmann and van Gils (2009) provides a way to explore the mechanisms. Based on the model and basic results by Diekmann and van Gils (2009), our analysis gives some answers to the questions under conditions in Corollary 4.4. In fact, conditions in (2.1) have interesting ecological meanings. Consider the first equation of (1.2) for  $n = 4$ :

$$\dot{x}_1 = x_1(1 - a_1x_1 - a_2x_2 - a_3x_3 - a_4x_4).$$



**Fig. 4** In the computations,  $b_1 = 0.1, b_2 = 0.01$  and  $b_3 = -0.2$ . The first three components  $(u_1(t), u_2(t), u_3(t))$  of a solution  $u(t)$  are selected while  $u_4(t) = 1 - u_1(t) - u_2(t) - u_3(t)$ . It is shown that the periodic orbit is asymptotically stable in the three-dimensional space

Then  $\frac{a_i}{a_1}$  represents the competitive degree from the  $i$ th year class to the 1th year class,  $i = 2, 3, 4$ . As shown by [Adamic and Huberman \(2000\)](#) and [Lopez and Sanjuan \(2001\)](#), the competitive degree is called strong (weak) if  $\frac{a_i}{a_1} > 1 (< 1)$ . By (2.1), we have  $b_1 > 0$  and  $b_2 > 0$ , i.e.,  $\frac{a_i}{a_1} > 1, i = 2, 3$ . That is, the competition from the  $i$ th year class to the 1st year class is strong,  $i = 2, 3$ . By (2.1), we have  $b_3 < 0$ , i.e.,  $\frac{a_4}{a_1} < 1$ . That is, the competition from the 4th year class to the 1st year class is weak. Since  $b_1 = a_2 - a_1, b_2 = a_3 - a_1$  and  $b_3 = a_4 - a_1$ , condition  $-(b_1 + b_3) - \hat{\mu} < b_2 < -(b_1 + b_3)$  in Corollary 4.4(i) can be rewritten as

$$1 - \frac{\hat{\mu}}{3a_1} < \frac{1}{3} \left( \frac{a_2}{a_1} + \frac{a_3}{a_1} + \frac{a_4}{a_1} \right) < 1.$$

That is, the average competitive degree  $\frac{1}{3}(\frac{a_2}{a_1} + \frac{a_3}{a_1} + \frac{a_4}{a_1})$  from the other three year classes is slightly weak while two of them are strong. Corollary 4.4 shows that when the average competitive degree is slightly weak, the year classes coexist and present periodic oscillations. The oscillation magnitude grows monotonically as the parameter  $\frac{a_3}{a_1}$  increases, where  $\frac{a_3}{a_1}$  represents the competitive degree from the year class which is next to the next year class of the present one. In fact, as shown in Fig. 3a–c, orbits from the interior of the cone gradually converge to the periodic solution, which means the population densities of the broods change in a recurrent way and all of the broods are persistent as defined by [Hofbauer and Sigmund \(1998\)](#). Furthermore, the sizes of the periodic solutions in Fig. 3a–c increase monotonically as  $\frac{a_3}{a_1}$  increases, which means the population densities of the broods vary in enlarged regions and the oscillations become more and more fierce. Here, the increase of  $\frac{a_3}{a_1}$  is equivalent to the increase of  $b_2$  since  $b_2 = a_3 - a_1$  and  $a_1$  is fixed in the simulations. Hence, the weak average

competitive degrees guarantee the coexistence of the broods, and the coexistence is in the form of periodic oscillations while the competitive degree  $\frac{a_3}{a_1}$  plays an important role in the magnitude of the oscillations.

Corollary 4.4 also shows that when the average competitive degree becomes strong, i.e.,

$$1 < \frac{1}{3} \left( \frac{a_2}{a_1} + \frac{a_3}{a_1} + \frac{a_4}{a_1} \right) < 1 + \frac{\hat{\mu}}{3a_1},$$

the periodic solution disappears and  $\Gamma$  becomes asymptotically stable. Then the orbits near  $\Gamma$  converge to  $\Gamma$  gradually, which means the orbits spiral outward to the boundary of the cone in a recurrent and sequential movement near the equilibria  $e_i$  (with longer and longer time episodes). The ecological interpretation is as follows. When the orbits are not so close to  $\Gamma$ , the broods coexist and there is not a fixed brood that remains strong (dominant), i.e., the broods become the strong one in a cyclic way in the sense that one brood is growing when its predecessor declines while the dominant period of the broods becomes longer and longer. However, when the orbits gradually go close to  $e_i$  such that the quantities of other broods (i.e., not the  $i$ th brood) are less than one, all broods but the  $i$ th one go extinct. In fact, as shown in Fig. 3d, orbits initially converge to  $\Gamma$  in a recurrent way, which means that all the broods coexist while the strong brood is not the same one but changes in a cyclic way. When the orbits go close to  $e_i$  such that the quantities of all but the  $i$ th brood are less than one, only the  $i$ th brood could survive. Hence, it is the strong average competitive degree that results in the persistence of only one brood in the situation we considered.

The remaining of the particular year class can be discussed as follows. Since the orbit of (1.1) near  $\Gamma$  tends to  $\Gamma$  as  $t \rightarrow \infty$ , one of the year classes will go extinct as the first missing one when its number is less than one. Without loss of generality, let  $x_4 = 0$ . It follows from Lemma 3.1 that only the 3rd year classes can persist while other year classes will go extinct. Then by Lemma 2.1, if the  $i$ th year class is the first missing one, then the  $(i - 1)$ th (modulo 4) year class will be the unique persistent class. Hence, the persistent year class is determined by the first missing class. The first missing class sensitively depends on the distribution of the initial population densities since the orbits of (1.1) tend to  $\Gamma$  in a cyclic way.

In a two-species competitive Lotka–Volterra system where the axial equilibria are locally stable, it is shown by Hofbauer and Sigmund (1988) and Murry (1993) that almost all the orbits converge to the axial equilibria without periodic oscillations and the strong subpopulation that has the largest initial density would persist. The result has been extended to higher dimensions (Drissche and Zeeman 1998; Wang 2003; Wang and Wu 2011; Xiao and Li 2000; Zeeman 1993; Zeeman and Zeeman 2002). It shows that in some ecological models where each of the competition degrees between subpopulations (not the average competitive degree) is strong, only the strongest one could persist while others go extinct. Our analysis shows a different result. In the situation we consider, there are both strong and weak competitions ( $\frac{a_2}{a_1} > 1$ ,  $\frac{a_3}{a_1} > 1$  but  $\frac{a_4}{a_1} < 1$ ). While weak average competitive degrees imply coexistence with periodic oscillations, the strong average competitive degrees imply that there is only one persistent subpopulation, which is not necessarily the one that has the largest initial

density, but depends on the distribution of initial population densities in a sensitive way.

While the periodic orbits shown in this paper are close to the heteroclinic cycle on the boundary, the coexistence derived from their stability seems not so strong from the ecological viewpoint. However, our work demonstrates both the effect of the average competitive degrees on the coexistence and the effect of some particular factor on the increase of the magnitudes of the periodic oscillations, which is verified in large regions by numerical simulations. In fact, numerical simulations in Fig. 3 show that when the average competitive degree is weak as shown in Fig. 3a, there is a stable periodic orbit near the positive equilibrium ( $E_4$ ) where the coexistence is strong. Furthermore, the increase of the particular factor ( $\frac{a_3}{a_1}$ ) in large regions also leads to the increase of the magnitude of the periodic orbit as shown in Fig. 3a–c. While our result is restricted to a small region, it is possible that the result could be extended to large regions, which still remains a challenge in the conjecture by Diekmann and van Gils (2009).

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