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J. Math. Anal. Appl. 291 (2004) 775–793
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Bifurcations in an epidemic model with constant removal rate of the infectives

Wendi Wang [a,1] and Shigui Ruan [b,*,2]

[a] Department of Mathematics, Southwest Normal University, Chongqing, 400715, PR China
[b] Department of Mathematics, The University of Miami, PO Box 249085, Coral Gables, FL 33124-4250, USA

Received 10 November 2003

Submitted by G.F. Webb

Abstract

An epidemic model with a constant removal rate of infective individuals is proposed to understand the effect of limited resources for treatment of infectives on the disease spread. It is found that it is unnecessary to take such a large treatment capacity that endemic equilibria disappear to eradicate the disease. It is shown that the outcome of disease spread may depend on the position of the initial states for certain range of parameters. It is also shown that the model undergoes a sequence of bifurcations including saddle-node bifurcation, subcritical Hopf bifurcation, and homoclinic bifurcation.

Keywords: Epidemic; Constant removal rate; Bifurcation; Global analysis; Limit cycle

1. Introduction

The asymptotic behavior of epidemic models has been studied by many researchers (see [1,4–6,8–11,18,20] and the references cited therein). Periodic oscillations have been observed in the incidence of many infectious diseases, including measles, mumps, rubella, chickenpox, and influenza. In some locations, the incidence of some diseases, such as

* Corresponding author.
E-mail address: ruan@math.miami.edu (S. Ruan).
1 Research was supported by the Key Teachers Fund from the Minister of Education of China and the National Natural Science Fund of China.
2 Research was partially supported by the NSERC of Canada and by a start-up fund from the College of Arts and Sciences at the University of Miami. On leave from Dalhousie University, Halifax, Canada.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2003.11.043

chickenpox, mumps, and poliomyelitis, goes up and down every year (see Hethcote and Levin [8]). Because of the observed periodicity in the incidence of many diseases, there has been great interest in determining how periodic solutions can arise in epidemiological models. Hethcote et al. [9] found that a single population epidemiological model with bilinear incidence rates, constant population size and constant parameter values can have periodic solutions if and only if the model is cyclic of SIRS or SEIRS type and individuals can be "significantly delayed" in the removed class by mechanisms such as a large constant period of temporary immunity. Liu, Hethcote, and Levin [15], Liu, Levin, and Iwasa [16] proved that the nonlinear incidence rate of $\beta I^p S^q$ type can lead to periodic solutions in SIRS models. Lizana and Rivero [17] found that such a model admits codimension 2 bifurcations.

Treatment including isolation or quarantine is an important method to decrease the spread of diseases such as measles, AIDS, tuberculosis, and flu (Feng and Thieme [7], Wu and Feng [21], Hyman and Li [12]). In classical epidemic models, the removal rate of infectives is assumed to be proportional to the number of the infectives. This is unsatisfactory because the resources for treatment should be quite large. In fact, every community should have a suitable capacity for treatment. If it is too large, the community pays for unnecessary cost. If it is too small, the community has the risk of the outbreak of the disease. Thus, it is important to determine a suitable capacity for the treatment of a disease. In this paper, we suppose that the capacity for the treatment of a disease in a community is a constant r. In order to easily understand its effect, we consider a case that the removal rate of infectives equals r. This means that we use the maximal treatment capacity to cure or isolate infectives so that the disease is eradicated. This can occur if the disease is so dangerous that we hope to wipe out it quickly, or the disease spreads rapidly so that the treatment capacity is insufficient for treatment in a period (flu, for example).

The model to be studied takes the following form:

$$\begin{aligned}
\frac{dS}{dt} &= A - dS - \lambda SI, \\
\frac{dI}{dt} &= \lambda SI - (d+\gamma)I - h(I), \\
\frac{dR}{dt} &= \gamma I + h(I) - dR,
\end{aligned} \tag{1.1}$$

where $S(t)$, $I(t)$, and $R(t)$ denote the numbers of susceptible, infective, and recovered individuals at time t, respectively, A is the recruitment rate of the population, d is the natural death rate of the population, γ is the natural recovery rate of the infective individuals. We adopt a bilinear incidence rate in (1.1). A good alternative for this is a modified standard incidence rate $\lambda SI/(S+I)$ (see [7,21]). In (1.1), $h(I)$ is the removal rate of infective individuals due to the treatment of infectives. We suppose that the treated infectives become recovered when they are treated in treatment sites. We also suppose that

$$h(I) = \begin{cases} r, & \text{for } I > 0, \\ 0, & \text{for } I = 0, \end{cases} \tag{1.2}$$

where $r > 0$ is a constant and represents the capacity of treatment for infectives. This means that we use a constant removal rate for the infectives until the disease disappears.

Suppose that $(S(t), I(t), R(t))$ is a solution of (1.1). If $S(t) > 0$, $I(t) > 0$, $R(t) > 0$ for $0 \leqslant t < t_0$ and $I(t_0) = 0$, it is natural to assume that $(S(t), I(t), R(t))$ satisfies

$$\begin{cases} \dfrac{dS}{dt} = A - dS, \\ I(t) = 0, & \text{for } t \geqslant t_0. \\ \dfrac{dR}{dt} = -dR, \end{cases}$$

Consequently, $\mathbb{R}^3_+$ is positively invariant for system (1.1).

The purpose of this paper is to show that this removal rate has significant effects on the dynamics of (1.1). We will prove that (1.1) undergoes a sequence of bifurcations including saddle-node bifurcation, subcritical Hopf bifurcation, and homoclinic bifurcation. We will also present a global analysis of the model and discuss the existence and nonexistence of limit cycles. Optimal capacity for treatment can be chosen according to our results. Before going into any detail, we simplify the model. Since the first two equations are independent of the third one and its dynamic behavior is trivial when $I(t_0) = 0$ for some $t_0 > 0$, it suffices to consider the first two equations with $I > 0$. Thus, we restrict our attention to the following reduced model:

$$\begin{aligned} \frac{dS}{dt} &= A - dS - \lambda SI, \\ \frac{dI}{dt} &= \lambda SI - (d+\gamma)I - r. \end{aligned} \tag{1.3}$$

It is assumed that all the parameters are positive constants.

The organization of this paper is as follows. In the next section, we study the bifurcations of (1.3). In Section 3 we present a global analysis of the model. The paper ends with a brief discussions in Section 4.

2. Bifurcations

In this section, we first consider the equilibria of (1.3) and their local stability. Then we study the Hopf bifurcation and the Bogdanov–Takens bifurcation of (1.3).

In order to find endemic equilibria of (1.3), we substitute $S = A/(d + \lambda I)$ into $\lambda SI - (d+\gamma)I - r = 0$ to obtain the quadratic equation

$$-\lambda(d+\gamma)I^2 + \big(\lambda A - r\lambda - \gamma d - d^2\big)I - rd = 0. \tag{2.1}$$

Set

$$R_0 = \frac{\lambda A}{d(d+\gamma)}, \qquad H = \frac{\lambda r}{d(d+\gamma)}.$$

Then (2.1) can be written as

$$\frac{\lambda}{d}I^2 - (R_0 - 1 - H)I + \frac{r}{d+\gamma} = 0. \tag{2.2}$$

$R_0 = \lambda A/(d(d+\gamma))$ is the reproduction number of (1.3) in the absence of the removal rate. It is evident that (2.2) does not have a positive solution if $R_0 \leqslant 1$. If $R_0 > 1$, it is easy to see that (2.2) does not have a positive solution if

$$\left(\sqrt{R_0}-1\right)^2 < H, \tag{2.3}$$

admits a positive solution if

$$\left(\sqrt{R_0}-1\right)^2 = H, \tag{2.4}$$

and has two positive solutions if

$$\left(\sqrt{R_0}-1\right)^2 > H > 0. \tag{2.5}$$

Thus, (1.3) does not have a positive equilibrium if $R_0 \leqslant 1$ or (2.3) holds. Furthermore, (2.4) implies that (1.3) has one endemic equilibrium and (2.5) implies that (1.3) has two endemic equilibria. If $N_1 = S + I$, we have

$$\frac{dN_1}{dt} = A - r - dN_1 - \gamma I \leqslant A - r - dN_1.$$

It follows that positive solutions of (1.3) are bounded. Note that the nonnegative I-axis repels positive solutions of (1.3) and that there is no equilibrium on the nonnegative S-axis. If $R_0 \leqslant 1$ or (2.3) holds, it follows that $I(t)$ becomes 0 in finite time, i.e., the disease disappears in a finite time.

Now, we propose the following assumption:

(H1) $R_0 > 1$ and $0 < H < (\sqrt{R_0}-1)^2$.

Let (H1) hold. Then (1.3) admits two endemic equilibria: $E_1 = (S_1, I_1)$ and $E_2 = (S_2, I_2)$, where

$$I_1 = \frac{d}{2\lambda}\left(R_0 - 1 - H - \sqrt{(R_0-1-H)^2 - 4H}\right), \qquad S_1 = A/(d+\lambda I_1),$$
$$I_2 = \frac{d}{2\lambda}\left(R_0 - 1 - H + \sqrt{(R_0-1-H)^2 - 4H}\right), \qquad S_2 = A/(d+\lambda I_2).$$

Although the endemic equilibria occur under the assumption (H1), we will show that the disease can disappear in a range of the parameters. This means that it is unnecessary to increase the removal rate r to $H > (\sqrt{R_0}-1)^2$ to make the disease disappear. We begin by analyzing the stability of these two equilibria. The Jacobian matrix of (1.3) at (S_1, I_1) is

$$J_1 = \begin{bmatrix} -d-\lambda I_1 & -\lambda S_1 \\ \lambda I_1 & \lambda S_1 - d - \gamma \end{bmatrix}.$$

Note that $A - dS_1 = \lambda S_1 I_1 = (d+\gamma)I_1 + r$. We have $S_1 = (A - (d+\gamma)I_1 - r)/d$. Thus, we have

$$\begin{aligned}\det(J_1) &= -d\lambda S_1 + d^2 + \gamma d + \lambda d I_1 + \lambda\gamma I_1 \\ &= -\lambda A + 2\lambda(d+\gamma)I_1 + \lambda r + d(d+\gamma)\end{aligned}$$

$$= d(d+\gamma)\left[-R_0 + \frac{2\lambda}{d} I_1 + H + 1\right]$$
$$= -d(d+\gamma)\sqrt{(R_0 - 1 - H)^2 - 4H} < 0.$$

It follows that (S_1, I_1) is a saddle point. The Jacobian matrix of (1.3) at (S_2, I_2) is

$$J_2 = \begin{bmatrix} -d - \lambda I_2 & -\lambda S_2 \\ \lambda I_2 & \lambda S_2 - d - \gamma \end{bmatrix}.$$

By the same argument, we obtain $\det(J_2) = d(d+\gamma)\sqrt{(R_0 - 1 - H)^2 - 4H} > 0$. Thus, (S_2, I_2) is a focus, a node, or a center. The stability of this equilibrium is stated in the following theorem.

Theorem 2.1. *Let* (H1) *hold. Then*

(i) *E_2 is stable if either*

$$\lambda A - 3d^2 - d\gamma - 2d^3/\gamma \leqslant \lambda r \tag{2.6}$$

or

$$\lambda r < \lambda A - 3d^2 - d\gamma - 2d^3/\gamma \quad \textit{and}$$
$$\lambda r < \frac{1}{2}\left[2\lambda A + (2d+\gamma)(d+\gamma)\left(1 - \sqrt{1 + \frac{4\lambda A}{(\gamma+d)^2}}\right)\right]. \tag{2.7}$$

(ii) *E_2 is unstable if*

$$\lambda r < \lambda A - 3d^2 - d\gamma - 2d^3/\gamma \quad \textit{and}$$
$$\lambda r > \frac{1}{2}\left[2\lambda A + (2d+\gamma)(d+\gamma)\left(1 - \sqrt{1 + \frac{4\lambda A}{(\gamma+d)^2}}\right)\right]. \tag{2.8}$$

Proof. Since $S_2 = (A - (d+\gamma)I_2 - r)/d$, we see that the trace of J_2 is

$$\operatorname{tr}(J_2) = -2d - \lambda I_2 + \lambda S_2 - \gamma = -\frac{(2d\lambda + \gamma\lambda)}{d} I_2 - \frac{2d^2 - \lambda A + r\lambda + \gamma d}{d}. \tag{2.9}$$

Thus, the trace is negative if $2d^2 - \lambda A + r\lambda + \gamma d \geqslant 0$. Suppose

$$2d^2 - \lambda A + r\lambda + \gamma d < 0. \tag{2.10}$$

Let us find the conditions under which $\operatorname{tr}(J_2) = 0$. Set

$$D_1 \triangleq -\frac{d(d+\gamma)}{\lambda(2d+\gamma)}\left(\frac{d}{d+\gamma} + 1 - R_0 + H\right).$$

(2.9) implies that $\operatorname{tr}(J_2) = 0$ is equivalent to

$$I_2 = -\frac{2d^2 - \lambda A + r\lambda + \gamma d}{\lambda(2d+\gamma)} = D_1. \tag{2.11}$$

If

$$D_2 \triangleq -\frac{2d}{2d+\gamma} + \frac{\gamma}{2d+\gamma}(R_0 - 1 - H),$$

it follows from the definition of I_2 that $\mathrm{tr}(J_2) = 0$ is equivalent to

$$D_2 = \sqrt{(R_0 - 1 - H)^2 - 4H}. \tag{2.12}$$

Thus, the set of $\mathrm{tr}(J_2) = 0$ is empty if

$$H \geqslant R_0 - 1 - \frac{2d}{\gamma}. \tag{2.13}$$

Suppose

$$H < R_0 - 1 - \frac{2d}{\gamma}. \tag{2.14}$$

Taking squares on both sides of (2.12) and simplifying the resulting equation, we obtain

$$D_3 \triangleq r^2\lambda + \left(-3\gamma d - 2\lambda A - \gamma^2 - 2d^2\right)r + \lambda A^2 - dA\gamma - 2Ad^2 = 0. \tag{2.15}$$

Hence,

$$r = \frac{1}{2\lambda}\left[2\lambda A + (2d+\gamma)(d+\gamma)\left(1 \pm \sqrt{1 + \frac{4\lambda A}{(\gamma+d)^2}}\right)\right].$$

In view of (2.10), we have

$$r = \frac{1}{2\lambda}\left[2\lambda A + (2d+\gamma)(d+\gamma)\left(1 - \sqrt{1 + \frac{4\lambda A}{(\gamma+d)^2}}\right)\right]. \tag{2.16}$$

As a consequence, we see that (H1), (2.14) and (2.16) are the necessary and sufficient conditions for $\mathrm{tr}(J_2) = 0$.

Now, we show that E_2 is stable if (2.6) is valid. The previous discussions show that the stability of E_2 does not change if (2.13) holds. Note that (2.13) is equivalent to $\lambda A - 3d^2 - d\gamma - 2d^3/\gamma < \lambda r$. By the definitions of D_1 and D_2, we have

$$\mathrm{tr}(J_2) = -\frac{(2d\lambda + \gamma\lambda)}{d}(I_2 - D_1) = -\frac{(2d+\gamma)}{2}\left(\sqrt{(R_0 - 1 - H)^2 - 4H} - D_2\right). \tag{2.17}$$

Thus, (2.13) implies that $\mathrm{tr}(J_2) < 0$. Therefore, E_2 is stable if (2.6) holds.

Notice that

$$\left[(R_0 - 1 - H)^2 - 4H\right] - D_2^2 = \frac{4\lambda}{d(2d+\gamma)^2(d+\gamma)}D_3.$$

It follows that $\mathrm{tr}(J_2) < 0$ if (2.7) is valid and that $\mathrm{tr}(J_2) > 0$ if (2.8) holds. □

Now, we discuss some implications of Theorem 2.1. If we draw the stability region of the endemic equilibrium E_2 in the (λ, r)-plane by fixing A and γ, we see that the region becomes smaller as d becomes larger. If we draw the stability region of the endemic

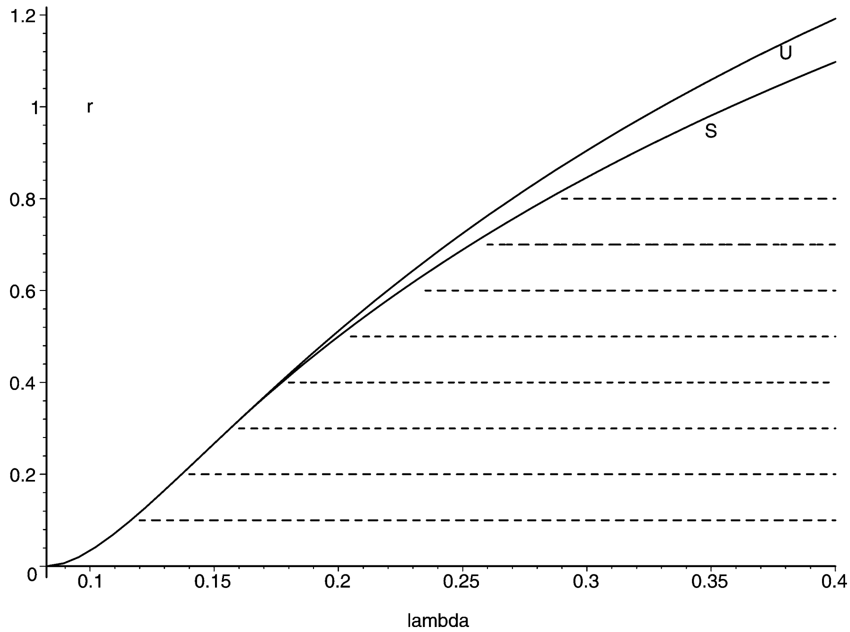

Fig. 1. Stability region of E_2 when $A = 4$, $\gamma = 0.8$, $d = 0.3$.

equilibrium E_2 in the (λ, r)-plane by fixing A and d, we see that the region becomes smaller as γ becomes larger. If we draw the stability region of the endemic equilibrium E_2 in the (λ, r)-plane by fixing d and γ, we see that the region becomes larger as A becomes larger. Thus, we may say that d and γ destabilize the endemic equilibrium E_2 and that A stabilizes the endemic equilibrium E_2. A typical stability region for E_2 is shown in Fig. 1. The region with dashed lines is the stable region and the region between the curve S and the curve U is the unstable region. E_2 disappears above the curve U. From this figure, we see that there is a $\lambda_0 > 0$ such that if $\lambda > \lambda_0$, E_2 undergoes stable state, unstable state and disappears at last as h increases from 0. This suggests a possibility that (1.3) admits a Hopf bifurcation.

Let us now verify the existence of a Hopf bifurcation in (1.3) and determine its direction. Set

$$h_0 = \frac{1}{2\lambda}\left[2\lambda A + (2d+\gamma)(d+\gamma)\left(1 - \sqrt{1 + \frac{4\lambda A}{(\gamma+d)^2}}\right)\right].$$

Theorem 2.2. *Let* (H1) *hold. Assume further that*

$$r < \left[\lambda A - 3d^2 - d\gamma - 2d^3/\gamma\right]/\lambda. \tag{2.18}$$

Then there is a family of unstable limit cycles if r is less than and near h_0, i.e., a subcritical Hopf bifurcation occurs when r passes through h_0.

Proof. Suppose $r = h_0$. Then $\mathrm{tr}(J_2) = 0$. It follows from (2.11) that

$$I_2 = \frac{d+\gamma}{\lambda}\left(-\frac{d}{d+\gamma} - \frac{1}{2} + \frac{1}{2}\sqrt{1+\frac{4\lambda A}{(d+\gamma)^2}}\right),$$
$$S_2 = \frac{d+\gamma}{2\lambda}\left(1+\sqrt{1+\frac{4\lambda A}{(d+\gamma)^2}}\right).$$

Set

$$\omega = \sqrt{\det(J_2)} = \sqrt{d(d+\gamma)\sqrt{(R_0-1-H)^2-4H}}.$$

Then the eigenvalues of J_2 are $\lambda_1 = \omega i$ and $\lambda_2 = -\omega i$.

Perform coordinate transformations by $x = S - S_2$, $y = I - I_2$. Then system (1.3) becomes

$$\frac{dx}{dt} = -(d+\lambda I_2)x - \lambda S_2 y - \lambda xy,$$
$$\frac{dy}{dt} = \lambda I_2 x + (\lambda S_2 - d - \gamma)y + \lambda xy. \tag{2.19}$$

Setting $x = -\lambda S_2 v$, $y = \omega u + (d+\lambda I_2)v$ and using $\operatorname{tr}(J_2) = \lambda S_2 - 2d - \gamma - \lambda I_2 = 0$, $\omega^2 = \det(J_2) = -d\lambda S_2 + d^2 + \gamma d + \lambda d I_2 + \lambda\gamma I_2$, we obtain

$$\frac{du}{dt} = -\omega v + f(u,v), \qquad \frac{dv}{dt} = \omega u + g(u,v), \tag{2.20}$$

where

$$f = \frac{\lambda v(-\lambda S_2 + \lambda I_2 + d)(\omega u + dv + \lambda I_2 v)}{\omega}, \qquad g = -\lambda v(\omega u + dv + \lambda I_2 v).$$

Using the fact that $\lambda S_2 - 2d - \gamma - \lambda I_2 = 0$, we obtain $g = \omega f/(d+\gamma)$. If

$$\mu = \frac{1}{16}[f_{uuu} + f_{uvv} + g_{uuv} + g_{vvv}]$$
$$+ \frac{1}{16\omega}\big[f_{uv}(f_{uu}+f_{vv}) - g_{uv}(g_{uu}+g_{vv}) - f_{uu}g_{uu} + f_{vv}g_{vv}\big],$$

by some tedious calculations, we obtain

$$\mu = -\frac{\lambda^2(-\lambda S_2 + \lambda I_2 + d)^2(d+\lambda I_2)(-3d^2 - 4\gamma d - \gamma^2 + \omega^2 - 2d\lambda I_2 - 2\gamma\lambda I_2)}{8\omega^2(d+\gamma)^2}.$$

Note that $-\lambda S_2 + \lambda I_2 + d = -d - \gamma$. We have

$$\mu = \frac{\lambda^2(d+\lambda I_2)(2d^2 + 3\gamma d + \gamma^2 + \lambda S_2 d + d\lambda S_2 + \gamma\lambda I_2)}{8\omega^2} > 0.$$

The conclusion of this theorem follows from [13, Theorem 3.4.2 and formula (3.4.11)]. □

As an example, we fix $A = 8$, $d = 0.1$, $\lambda = 1$, $\gamma = 1$. Then $(\sqrt{R_0} - 1)^2 d(d+\gamma) = 6.2338$, $\lambda A - 3d^2 - d\gamma - 2d^3/\gamma = 7.868$ and $h_0 = 5.2023$ (we always keep 4 decimal places for a real number in this paper). Then Theorem 2.2 shows that there is an unstable limit cycle when r decreases from 5.2023, which is shown in Fig. 2.

At this time, the local stability of the equilibria of model (1.3) is clear. In order to determine the global dynamics of the model, we investigate its global bifurcation. Suppose

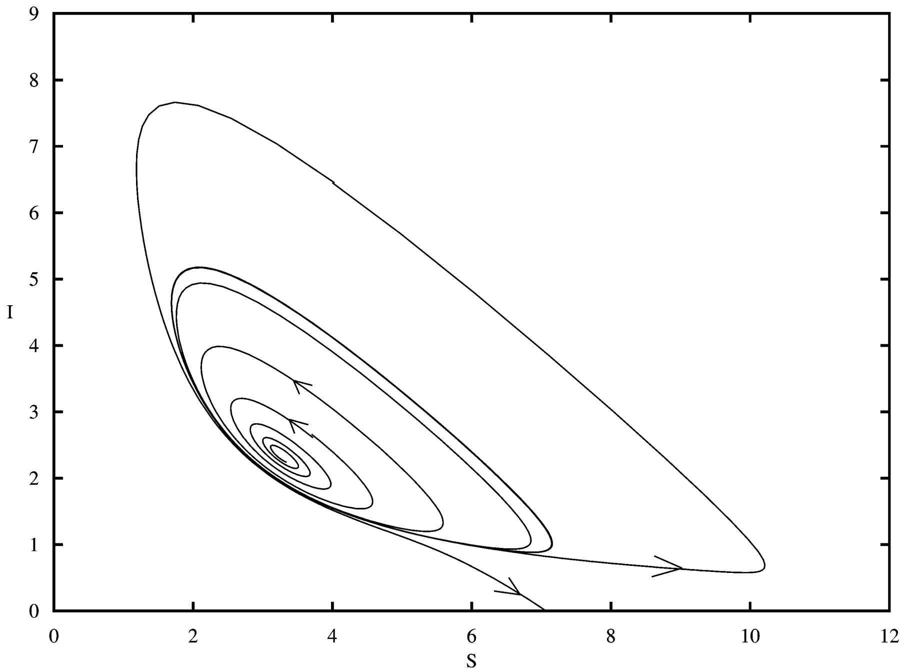

Fig. 2. An unstable periodic solution exists when $A = 8$, $d = 0.1$, $\lambda = 1$, $\gamma = 1$, $r = 5.1$.

(H2) $R_0 > 1$ and $H = (\sqrt{R_0} - 1)^2$.

Then (1.3) has one unique positive equilibrium (S^*, I^*) where

$$I^* = \frac{d}{\lambda}\bigl(\sqrt{R_0} - 1\bigr), \qquad S^* = \frac{A}{d\sqrt{R_0}}. \tag{2.21}$$

The Jacobian matrix of (1.3) at this point is

$$J_0 = \begin{bmatrix} -d - \lambda I^* & -\lambda S^* \\ \lambda I^* & \lambda S^* - d - \gamma \end{bmatrix}.$$

Suppose

(H3) $\sqrt{R_0} = 1 + d/\gamma$.

By (2.21), we have

$$\det(J_0) = -d\lambda S^* + d^2 + \gamma d + \lambda d I^* + \lambda\gamma I^* = \frac{-\lambda A + d^2 R_0 + \gamma d R_0}{\sqrt{R_0}} = 0.$$

Furthermore, (H3) implies that

$$\operatorname{tr}(J_0) = -2d - \lambda I^* + \lambda S^* - \gamma = -\frac{d^2\sqrt{R_0} + d^2 R_0 - \lambda A + \gamma d\sqrt{R_0}}{d\sqrt{R_0}} = 0.$$

Thus, (H2) and (H3) imply that the Jacobian matrix has a zero eigenvalue with multiplicity 2. This suggests that (1.3) may admit a Bogdanov–Takens bifurcation. We confirm this by giving the following theorem.

Theorem 2.3. *Suppose that* (H2) *and* (H3) *hold. Then the equilibrium* (S^*, I^*) *of* (1.3) *is a cusp of codimension* 2, *i.e., it is a Bogdanov–Takens singularity.*

Proof. Introduce the change of variables $x = S - S^*$, $y = I - I^*$. Then (1.3) becomes

$$\begin{aligned} \frac{dx}{dt} &= -(d + \lambda I^*)x - \lambda S^* y - \lambda xy, \\ \frac{dy}{dt} &= \lambda I^* x + (\lambda S^* - d - \gamma)y + \lambda xy. \end{aligned} \tag{2.22}$$

Notice that $\operatorname{tr}(J_0) = 0$ and $\det(J_0) = 0$ imply that

$$d + \lambda I^* = \lambda S^* - d - \gamma, \qquad \lambda^2 S^* I^* = (\lambda S^* - d - \gamma)^2. \tag{2.23}$$

Let $X = x$, $Y = -(\lambda S^* - d - \gamma)x - (\lambda S^* - d - \gamma)^2 y/(\lambda I^*)$. Then (2.22) becomes

$$\begin{aligned} \frac{dX}{dt} &= Y + a_{11}X^2 + a_{12}XY, \\ \frac{dY}{dt} &= (d + \gamma)a_{11}X^2 + (d + \gamma)a_{12}XY, \end{aligned} \tag{2.24}$$

where

$$a_{11} = \frac{\lambda S^* - d - \gamma}{S^*}, \qquad a_{12} = \frac{1}{S^*}.$$

Change the variables one more time by letting $x = X - a_{12}X^2/2$, $y = Y + a_{11}X^2$, we have

$$\begin{aligned} \frac{dx}{dt} &= y + P_1(x, y), \\ \frac{dy}{dt} &= (d + \gamma)a_{11}x^2 + \big((d + \gamma)a_{12} + 2a_{11}\big)xy + P_2(x, y), \end{aligned} \tag{2.25}$$

where P_i are smooth functions in (x, y) at least of the third order.

Note that $a_{11} > 0$ and $a_{12} > 0$. It follows from [2,3,19] that (1.3) admits a Bogdanov–Takens bifurcation. □

In the following, we will find the versal unfolding in terms of the original parameters in (1.3). In this way, we will know the approximate homoclinic bifurcation curve. We choose A and r as bifurcation parameters. Fix $d = d_0$, $\lambda = \lambda_0$, and $\gamma = \gamma_0$. Let $A = A_0 + \lambda_1$ and $r = r_0 + \lambda_2$, where λ_1 and λ_2 are parameters which vary in a small neighborhood of the origin.

Suppose that $A = A_0$, $d = d_0$, $\lambda = \lambda_0$, $\gamma = \gamma_0$, and $r = r_0$ satisfy (H2) and (H3). Then by the transformations of $x = S - S^*$, $y = I - I^*$, (1.3) becomes

$$\begin{aligned} \frac{dx}{dt} &= \lambda_1 - (d_0 + \lambda_0 I^*)x - \lambda_0 S^* y - \lambda_0 xy, \\ \frac{dy}{dt} &= -\lambda_2 + \lambda_0 I^* x + (\lambda_0 S^* - d_0 - \gamma_0)y + \lambda_0 xy. \end{aligned} \tag{2.26}$$

Note that (H2) and (H3) imply that

$$S^* = \frac{(d_0+\gamma_0)^2}{\lambda_0\gamma_0}, \qquad I^* = \frac{d_0^2}{\lambda_0\gamma_0}. \tag{2.27}$$

Now, we transform (2.26) by setting

$$X = x, \qquad Y = \lambda_1 - (d_0 + \lambda_0 I^*)x - \lambda_0 S^* y - \lambda_0 xy.$$

If we rewrite X, Y as x, y, respectively, by means of (2.27) we obtain

$$\begin{aligned} \frac{dx}{dt} &= y,\\ \frac{dy}{dt} &= c_0 + (-\lambda_1+\lambda_2)\lambda_0 x - \frac{\lambda_1}{S^*}y + \lambda_0 d_0 x^2 + c_1 xy + \frac{1}{S^*}y^2 + R_1(x,y), \end{aligned} \tag{2.28}$$

where $R_1(x,y)$ is a smooth function of x and y at least of order three and

$$\begin{aligned} c_0 &= \frac{(d_0+\gamma_0)(\lambda_2\gamma_0 - d_0\lambda_1 + d_0\lambda_2)}{\gamma_0},\\ c_1 &= \frac{\lambda_0(2d_0^4 + 7d_0^3\gamma_0 + 9d_0^2\gamma_0^2 + 5d_0\gamma_0^3 + \gamma_0^4 + \lambda_0\gamma_0^2\lambda_1)}{(d_0+\gamma_0)^4}. \end{aligned}$$

Next, introduce a new time variable τ by $dt = (1 - x/S^*)\,d\tau$. Rewriting τ as t, we obtain

$$\begin{aligned} \frac{dx}{dt} &= y\left(1 - \frac{x}{S^*}\right),\\ \frac{dy}{dt} &= \left(1 - \frac{x}{S^*}\right)\Bigg(c_0 + (-\lambda_1+\lambda_2)\lambda_0 x - \frac{\lambda_1}{S^*}y + \lambda_0 d_0 x^2 + c_1 xy + \frac{1}{S^*}y^2\\ &\qquad + R_1(x,y)\Bigg). \end{aligned} \tag{2.29}$$

Let $X = x$, $Y = y(1 - x/S^*)$ and rename X and Y as x and y, we have

$$\begin{aligned} \frac{dx}{dt} &= y,\\ \frac{dy}{dt} &= c_0 + c_2 x - \frac{\lambda_1}{S^*}y + c_3 x^2 + c_4 xy + R_2(x,y), \end{aligned} \tag{2.30}$$

where $R_2(x,y)$ is a smooth function of x and y at least of order three and

$$\begin{aligned} c_2 &= -\frac{2c_0 - \lambda_0\lambda_2 S^* + \lambda_0\lambda_1 S^*}{S^*},\\ c_3 &= \frac{c_0 + \lambda_0 d_0 S^{*2} - 2\lambda_0\lambda_2 S^* + 2\lambda_0\lambda_1 S^*}{S^{*2}},\\ c_4 &= \frac{c_1 S^{*2} + \lambda_1}{S^{*2}}. \end{aligned}$$

Make the change of variable $x = X + \lambda_1/(c_4 S^*)$ and rewrite X as x, we obtain

$$\frac{dx}{dt} = y,$$
$$\frac{dy}{dt} = c_5 + c_6 x + c_3 x^2 + c_4 xy + R_3(x, y), \tag{2.31}$$

where $R_3(x, y)$ is a smooth function of x, y, λ_1, and λ_2 at least of order three and

$$c_5 = \frac{c_0 c_4^2 S^{*2} + c_2 \lambda_1 c_4 S^* + c_3 \lambda_1^2}{c_4^2 S^{*2}}, \qquad c_6 = \frac{c_2 c_4 S^* + 2c_3 \lambda_1}{c_4 S^*}.$$

Notice that $c_4 > 0$ and $c_3 > 0$ when λ_i are small. Make the change of variables one more time by setting

$$X = c_4^2 x / c_3, \qquad Y = c_4^3 y / c_3^2, \qquad \tau = c_3 t / c_4$$

and denoting them by x, y, t, respectively. Then we obtain

$$\frac{dx}{dt} = y,$$
$$\frac{dy}{dt} = \tau_1 + \tau_2 x + x^2 + xy + R_4(x, y), \tag{2.32}$$

where $R_4(x, y)$ is a smooth function of x, y, λ_1, and λ_2 at least of order three and

$$\tau_1 = \frac{c_5 c_4^4}{c_3^3}, \qquad \tau_2 = \frac{c_6 c_4^2}{c_3^2}.$$

By the theorems in Bogdanov [2,3] and Takens [19] or Kuznetsov [14], we obtain the following local representations of the bifurcation curves in a small neighborhood of the origin:

Theorem 2.4. *Suppose that A_0, d_0, λ_0, γ_0, and r_0 satisfy* (H2) *and* (H3). *Then* (1.3) *admits the following bifurcation behavior*:

(1) *There is a saddle-node bifurcation curve* $SN = \{(\lambda_1, \lambda_2)\colon\ 4c_3 c_5 = c_6^2 + o(|(\lambda_1, \lambda_2)|^2)\}$.
(2) *There is a Hopf bifurcation curve* $H = \{(\lambda_1, \lambda_2)\colon\ c_5 + o(|(\lambda_1, \lambda_2)|^2) = 0,\ c_6 < 0\}$.
(3) *There is a homoclinic bifurcation curve* $HL = \{(\lambda_1, \lambda_2)\colon\ 25c_3 c_5 + 6c_6^2 = o(|(\lambda_1, \lambda_2)|^2)\}$.

Theorem 2.4 gives us a global picture on the dynamical behavior of (1.3) near the degenerate equilibrium. In order to illustrate the results, let us consider an example. First, we express the three curves by the original parameters. After some calculations, we obtain the saddle-node bifurcation curve:

$$\begin{aligned} &4d_0^2(d_0+\gamma_0)^2\lambda_1 - 4d_0(d_0+\gamma_0)^3\lambda_2 + \lambda_0\gamma_0(\gamma_0+5d_0)\lambda_1^2 \\ &\quad - 2\lambda_0\gamma_0(3\gamma_0+5d_0)\lambda_1\lambda_2 + 5\lambda_0\gamma_0(d_0+\gamma_0)\lambda_2^2 + o\big(\lambda_1^2+\lambda_2^2\big) = 0, \end{aligned}$$

the Hopf bifurcation curve:

$$\begin{aligned} &(d_0+\gamma_0)^3 d_0(\gamma_0+2d_0)^2\lambda_1 - (d_0+\gamma_0)^4(\gamma_0+2d_0)^2\lambda_2 + \lambda_0\gamma_0^2\big(\gamma_0^2-2d_0^2\big)\lambda_1^2 \\ &\quad + \lambda_0\gamma_0^2(\gamma_0+2d_0)(d_0+\gamma_0)\lambda_1\lambda_2 + o\big(\lambda_1^2+\lambda_2^2\big) = 0, \end{aligned}$$

and the homoclinic bifurcation curve:

$$\begin{aligned}&25d_0^2(d_0+\gamma_0)^2(\gamma_0+2d_0)^2\lambda_1-25d_0(d_0+\gamma_0)^3(\gamma_0+2d_0)^2\lambda_2\\&\quad-\lambda_0\big(6\gamma_0^3-93d_0\gamma_0^2-150d_0^2\gamma_0-76d_0^3\big)\gamma_0\lambda_1^2\\&\quad-\lambda_0\big(62\gamma_0^2+113d_0\gamma_0+76d_0^2\big)\gamma_0(\gamma_0+2d_0)\lambda_1\lambda_2\\&\quad+19\lambda_0\gamma_0(\gamma_0+2d_0)^2(d_0+\gamma_0)\lambda_2^2+o\big(\lambda_1^2+\lambda_2^2\big)=0.\end{aligned}$$

If $d_0 = 0.1$, $\lambda_0 = 0.2$, $\gamma_0 = 4.0$, $A_0 = 2.1538$, $r_0 = 0.0013$, it is easy to see that $A = A_0$, $d = d_0$, $\lambda = \lambda_0$, $\gamma = \gamma_0$, and $r = r_0$ satisfy (H2) and (H3). By the previous formulae, we see that the saddle-node bifurcation curve is

$$-1.025\lambda_1+42.025\lambda_2-5.4878\lambda_1^2+30.4878\lambda_1\lambda_2-25.0\lambda_2^2+o\big(\lambda_1^2+\lambda_2^2\big)=0,$$

the Hopf bifurcation curve is

$$-1.025\lambda_1+42.025\lambda_2-0.4311\lambda_1^2-0.4646\lambda_1\lambda_2+o\big(\lambda_1^2+\lambda_2^2\big)=0,$$

and the homoclinic bifurcation curve is

$$-1.025\lambda_1+42.025\lambda_2+2.5344\lambda_1^2+48.2211\lambda_1\lambda_2-15.2{\lambda_2}^2+o\big(\lambda_1^2+\lambda_2^2\big)=0.$$

The (λ_1, λ_2)-plane near the origin is divided into 4 regions by these bifurcation curves, as shown in Fig. 3. Fix $\lambda_1 > 0$ and increase λ_2 from 0. When (λ_1, λ_2) lies in the region I which is below the curve *HL*, there is no limit cycle or homoclinic orbit and E_2 is stable. When (λ_1, λ_2) lies in the region II which is between the curve H and the curve *HL*, E_2 remains stable and there is a unique unstable limit cycle inside which the positive orbits of

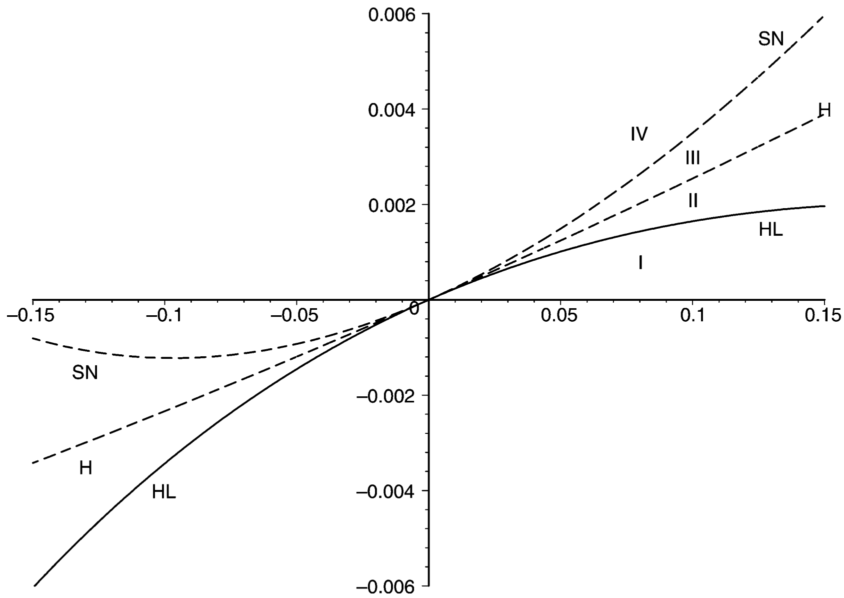

Fig. 3. Bifurcation curves and the four typical regions. The horizontal axis is the λ_1-axis and the vertical axis is the λ_2-axis.

(1.3) tend to E_2 as t tends to infinity. Thus, the disease is persistent inside the cycle. When (λ_1, λ_2) lies in the region III which is between the curve *SN* and the curve H, E_1 is a saddle, E_2 is an unstable node and the limit cycle disappears. By the results of the next section, we see that any positive orbit of (1.3) except the two equilibria E_1, E_2 and the stable manifolds of E_1 intersects the positive S-axis in finite time, i.e., the disease becomes extinct in finite time. When (λ_1, λ_2) lies in the region IV which is above the curve *SN*, (1.3) does not have a positive equilibrium, which implies that any positive orbit of (1.3) meets the positive S-axis in finite time, and therefore, the disease will disappear. Since the increase of λ_2 corresponds to the increase of the removal rate r, the above discussions indicate that it is sufficient to increase r to the extent where E_2 becomes unstable in order to wipe out the disease.

3. Global analysis

The objective of this section is to study the global structure of (1.3). We always suppose that (H1) holds in this section. If system (1.3) does not have a limit cycle, it is easy to classify its dynamical behavior. If E_2 is unstable, any positive semi-orbit except the two equilibria and the stable manifolds of E_1 intersects the positive S-axis in finite time. A typical phase portrait is shown in Fig. 4. If E_2 is stable, there is a region whose boundary includes the two stable manifolds of E_1 such that any positive semi-orbit inside this region

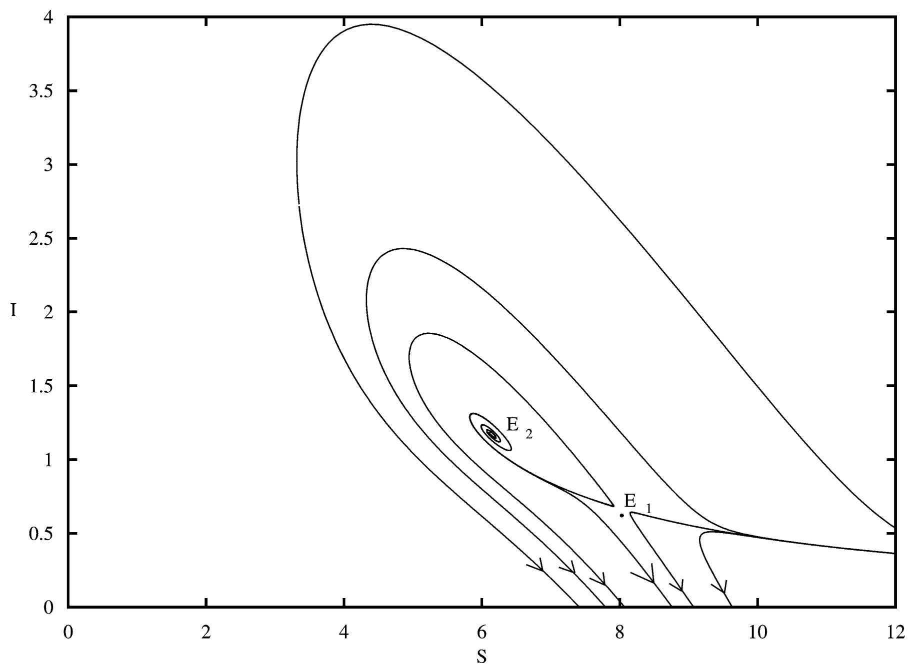

Fig. 4. Extinction of the disease, where $A = 4$, $d = 0.3$, $\gamma = 0.8$, $\lambda = 0.3$, and $r = 0.87$.

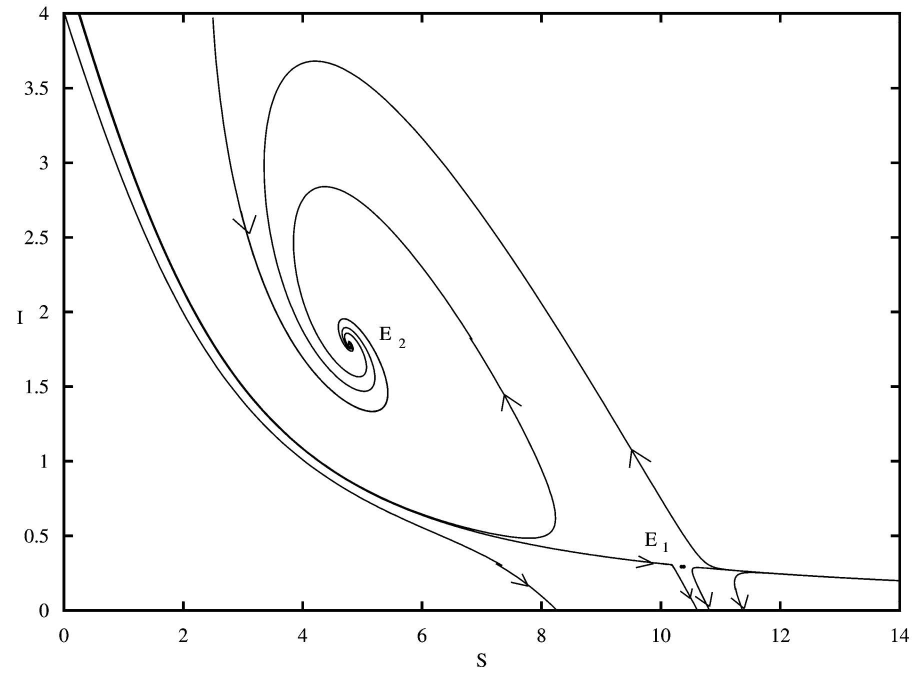

Fig. 5. Persistence of the disease, where $A = 4$, $d = 0.3$, $\gamma = 0.8$, $\lambda = 0.3$, and $r = 0.6$.

tends to E_2 as t tends to infinity and any positive semi-orbit outside this region meets the positive S-axis in a finite time. A typical phase portrait is shown in Fig. 5.

When (1.3) admits a limit cycle, more complicated dynamical behavior will occur, as is suggested by the Bogdanov–Takens bifurcation in Section 2. For this reason, we consider the existence and nonexistence of limit cycles in (1.3). Let $x = S - S_2$, $y = I - I_2$. Then (1.3) becomes

$$\begin{aligned} \frac{dx}{dt} &= -(d + \lambda I_2)x - \lambda S_2 y - \lambda xy, \\ \frac{dy}{dt} &= \lambda xy + \lambda I_2 x + (\lambda S_2 - d - \gamma)y. \end{aligned} \tag{3.1}$$

Set $X = -dx - (d + \gamma)y$, $Y = x + y$ and rewrite X, Y as x, y, respectively. (3.1) becomes

$$\begin{aligned} \frac{dx}{dt} &= (-2d - \gamma + S_2\lambda - I_2\lambda)x + \left(-d\gamma - d^2 + dS_2\lambda - \lambda I_2\gamma - dI_2\lambda\right)y \\ &\quad + \frac{\lambda}{\gamma}x^2 + \frac{\lambda(2d + \gamma)}{\gamma}xy + \frac{\lambda d(d + \gamma)}{\gamma}y^2, \\ \frac{dy}{dt} &= x. \end{aligned} \tag{3.2}$$

If $\Delta = (R_0 - 1 - H)^2 - 4H$, it is easy to see that

$$\left(-d\gamma - d^2 + dS_2\lambda - \lambda I_2\gamma - dI_2\lambda\right) = -\det(J_2) = -d(d + \gamma)\sqrt{\Delta} < 0,$$

$(-2d-\gamma+S_2\lambda-I_2\lambda)=\mathrm{tr}(J_2).$

Make the change of the variables $X=x$, $Y=(d(d+\gamma))^{1/2}\Delta^{1/4}y$, $\theta=(d(d+\gamma))^{1/2}\Delta^{1/4}t$ and rewrite X, Y and θ as x, y and t, respectively. System (3.2) becomes

$$\begin{aligned}\frac{dx}{dt}&=-y+\delta x+lx^2+mxy+ny^2,\\ \frac{dy}{dt}&=x,\end{aligned}\tag{3.3}$$

where

$$\begin{aligned}&\delta=\bigl(d(d+\gamma)\bigr)^{-1/2}\Delta^{-1/4}\,\mathrm{tr}(J_2),\qquad l=\frac{\lambda}{\gamma}\bigl(d(d+\gamma)\bigr)^{-1/2}\Delta^{-1/4},\\ &m=\frac{\lambda(2d+\gamma)}{\gamma d(d+\gamma)}\Delta^{-1/2},\qquad n=\lambda\bigl(d(d+\gamma)\bigr)^{-1/2}\Delta^{-3/4}/\gamma.\end{aligned}\tag{3.4}$$

Now, we can state the following result.

Theorem 3.1. *Let* (H1) *hold. Then there is no limit cycle in* (1.3) *if one of the following holds*:

(i) (2.8) *holds*;
(ii) (2.6) *is valid.*

Proof. It suffices to prove that (3.3) does not have a limit cycle. First, we suppose that assumption (i) is valid. By the discussions in the proof of Theorem 2.1, we see that $\mathrm{tr}(J_2)>0$. Hence $\delta>0$, and therefore, $\delta m(l+n)>0$. It follows from [22, Theorem 12.5] that there is no limit cycle in (3.3).

If the assumption (ii) holds, we transform (3.3) by $X=mx$, $Y=my$ to obtain

$$\begin{aligned}\frac{dX}{dt}&=-Y+\delta X+\frac{l}{m}X^2+XY+\frac{n}{m}Y^2,\\ \frac{dY}{dt}&=X.\end{aligned}\tag{3.5}$$

Following the proof of Theorem 2.1, we see that the assumption (ii) implies that $\mathrm{tr}(J_2)<0$. As a consequence, we have $\delta<0$. Since $l/m>0$ and $n/m>0$, it follows from [22, Lemma 12.1] that there is no limit cycle in (3.5) if

$$\delta+\frac{m}{2n}\leqslant 0.\tag{3.6}$$

By (3.4), we see that (3.6) is equivalent to

$$\mathrm{tr}(J_2)\leqslant-\frac{2d+\gamma}{2}\Delta^{1/2}.\tag{3.7}$$

By (2.9) and the definition of I_2, we see that (3.7) is equivalent to

$$\gamma(R_0-1-H)-2d\leqslant 0.\tag{3.8}$$

It is easy to verify that (3.8) is equivalent to (2.6). Consequently, there is no limit cycle in (1.3). □

Now, we know the global structure of (1.3) in almost all the cases. If (2.8) is valid, since E_2 is unstable and there is no limit cycle, any orbit except the two endemic equilibria and the stable manifolds of E_1 meets the positive S-axis in finite time, i.e., the disease becomes extinct in finite time. If the assumption (ii) of Theorem 3.1 holds, since there is no limit cycle in (1.3) and E_2 is stable, there is a region D whose boundary includes the two stable manifolds of E_1 such that any positive orbit inside D tends to E_2 as t tends to infinity and any positive orbit outside D intersects the positive S-axis in finite time.

By Theorem 3.1, the significant change of dynamical behavior of (1.3) can only occur in the case where (2.7) holds. Since a homoclinic orbit is important in determining the asymptotic behavior of (1.3), we now present a different way to show the existence of a homoclinic orbit in (1.3) where the homoclinic orbit may not be in a small neighborhood of a degenerate equilibrium. Choose $(A_0, d_0, \lambda_0, \gamma_0, r_0)$ such that (2.18) holds when $A = A_0$, $d = d_0$, $\lambda = \lambda_0$, $\gamma = \gamma_0$, $r = r_0$, and

$$h_0 = \frac{1}{2}\left[2\lambda_0 A_0 + (2d_0+\gamma)(d_0+\gamma_0)\left(1-\sqrt{1+\frac{4\lambda_0 A_0}{(\gamma_0+d_0)^2}}\right)\right]. \tag{3.9}$$

We fix $d = d_0$, $\lambda = \lambda_0$, $\gamma = \gamma_0$ and set

$$\Delta_0 = \left(\frac{\lambda_0 A_0}{d_0(d_0+\gamma_0)} - 1 - \frac{\lambda_0 r_0}{d_0(d_0+\gamma_0)}\right)^2 - 4\frac{\lambda_0 r_0}{d_0(d_0+\gamma_0)}.$$

Vary r and A by

$$r = r_0 - \theta, \qquad \frac{\lambda_0 A}{d_0(d_0+\gamma_0)} = 1 + \frac{\lambda_0 r}{d_0(d_0+\gamma_0)} + \sqrt{\Delta_0 + 4\frac{\lambda_0 r}{d_0(d_0+\gamma_0)}}. \tag{3.10}$$

We can see that Δ is invariant as θ varies. As a consequence, by (3.4) we can see that l, m, n are invariant as θ varies. Furthermore, by (2.9) and the definition of I_2, we have

$$\begin{aligned}\operatorname{tr}(J_2) &= \frac{\gamma_0}{2}(R_0 - 1 - H) - d_0 - \frac{2d_0+\gamma_0}{2}\sqrt{\Delta} \\ &= \frac{\gamma_0}{2}\sqrt{\Delta_0 + 4\frac{\lambda_0 r}{d_0(d_0+\gamma_0)}} - d_0 - \frac{2d_0+\gamma_0}{2}\sqrt{\Delta_0}.\end{aligned}$$

It follows that $\operatorname{tr}(J_2)$ is decreasing, and therefore, δ is decreasing, as θ increases. Now, it is easy to check that (3.3) is a rotated vector field with respect to parameter θ. If we increase θ from 0, it follows from Theorem 2.2 that an unstable limit cycle is produced due to Hopf bifurcation and this limit cycle expands as θ increases. Moreover, (H1) holds as θ varies because we have (3.10). When θ is increased to r_0, the equilibrium E_1 of (1.3) becomes $(A/d_0, 0)$ (a disease-free equilibrium) and the equilibrium E_2 of (1.3) becomes

$$(I_2, S_2) = \left(\frac{d_0}{\lambda_0}\sqrt{\Delta_0}, \frac{A}{d_0(1+\sqrt{\Delta_0})}\right).$$

Since E_2 is globally stable in model (1.3) when $r = 0$, it follows that the trivial equilibrium $(0, 0)$ of (3.3) is globally stable at this time. Hence, the unstable limit cycle must meet a homoclinic orbit before $\theta = r_0$. This means that there exists a homoclinic orbit in the following system

$$\frac{dS}{dt} = A - dS - \lambda SI, \qquad \frac{dI}{dt} = \lambda SI - (d + \gamma)I - r,$$

which is considered in the R^2 plane. By the form of the above system, a homoclinic orbit starting from the interior of R^2_+ cannot meet the nonnegative S-axis and the positive I-axis. This shows that the homoclinic orbit starting from the interior of R^2_+ must lie in the interior of R^2_+. Therefore, we can state the following result.

Theorem 3.2. *Let* $(A_0, d_0, \lambda_0, \gamma_0, r_0)$ *satisfy* (2.18) *and* (3.9)*. Then there exist* $r < r_0$ *and* A *which satisfy* (3.10) *such that* (3.3) *admits a homoclinic orbit, and therefore,* (1.3) *has a homoclinic orbit.*

4. Discussion

In this paper, we have proposed an epidemic model with a constant removal rate of the infective individuals to understand the effect of the treatment capacity on the disease transmission. If the parameters satisfy (2.8), Theorems 2.1 and 3.1 imply that the disease becomes extinct in a finite time because the endemic equilibrium E_2 is unstable and there is no limit cycle in (1.3). Thus, it is unnecessary to take such a large treatment capacity that the endemic equilibria disappear to eradicate the disease. If the parameters satisfy (2.6), there is a region such that the number of infectives tends to I_2 if the initial position lies in the region and the disease dies out if the initial position lies outside this region. If the parameters satisfy (2.7), the disease is persistent if the initial position lies in the region and the disease becomes extinct if the initial position lies outside this region. Since the eventual behavior is related to the initial positions, this model may be more realistic and useful.

We have shown that the model exhibits Bogdanov–Takens bifurcations, i.e., there are saddle-node bifurcation, subcritical Hopf bifurcation, and homoclinic bifurcation in the system, even though the incidence rate is bilinear. Since the model is globally stable in the absence of the removal rate, this suggests that a constant removal rate of the infectives induces the periodic oscillations of diseases. In contrast, the previous studies show that periodic coefficients, corresponding to periodic environment, time delays and nonlinear incidence rates of $\beta I^p S^q$ type are the causes of periodicity of diseases.

By carrying out the bifurcation analysis, we have obtained a clear picture about the dynamic behavior of the model near the degenerate equilibrium and obtained the approximate homoclinic bifurcation curve. We have also carried out a global qualitative analysis of the model. The result on the nonexistence of a limit cycle in (1.3) gives us the global structure of the model and indicates that complicated behavior of the model can only occur when (2.7) holds. Theorem 3.2 presents the existence of a homoclinic orbit in (1.3) in a large range of parameters.

The model we have studied in this paper is of SIR type, which is applicable for diseases such as measles, AIDS, flu, etc. Our analysis can be adapted to an SI model, which is used for sexually transmitted diseases or bacterial infections.

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