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ON THE EXISTENCE OF AXISYMMETRIC TRAVELING FRONTS IN LOTKA-VOLTERRA COMPETITION-DIFFUSION SYSTEMS IN $\mathbb{R}^3$

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Dedicated to Professor Robert Stephen Cantrell on the occasion of his 60th birthday

Abstract. This paper is concerned with the following two-species Lotka-Volterra competition-diffusion system in the three-dimensional spatial space

$$\begin{cases} \frac{\partial}{\partial t} u_1(\mathbf{x},t) = \Delta u_1(\mathbf{x},t) + u_1(\mathbf{x},t)\left[1- \ u_1(\mathbf{x},t) - k_1 u_2(\mathbf{x},t)\right], \\ \frac{\partial}{\partial t} u_2(\mathbf{x},t) = d\Delta u_2(\mathbf{x},t) + r u_2(\mathbf{x},t)\left[1 - u_2(\mathbf{x},t) - k_2 u_1(\mathbf{x},t)\right], \end{cases}$$

where $\mathbf{x} \in \mathbb{R}^3$ and $t > 0$. For the bistable case, namely $k_1, k_2 > 1$, it is well known that the system admits a one-dimensional monotone traveling front $\mathbf{\Phi}(x+ct) = (\Phi_1(x+ct), \Phi_2(x+ct))$ connecting two stable equilibria $\mathbf{E}_u = (1,0)$ and $\mathbf{E}_v = (0,1)$, where $c \in \mathbb{R}$ is the unique wave speed. Recently, two-dimensional V-shaped fronts and high-dimensional pyramidal traveling fronts have been studied under the assumption that $c > 0$. In this paper it is shown that for any $s > c > 0$, the system admits axisymmetric traveling fronts

$$\mathbf{\Psi}(\mathbf{x}', x_3 + st) = \big(\Phi_1(\mathbf{x}', x_3 + st), \Phi_2(\mathbf{x}', x_3 + st)\big)$$

in $\mathbb{R}^3$ connecting $\mathbf{E}_u = (1,0)$ and $\mathbf{E}_v = (0,1)$, where $\mathbf{x}' \in \mathbb{R}^2$. Here an *axisymmetric* traveling front means a traveling front which is axially symmetric with respect to the x_3-axis. Moreover, some important qualitative properties of the axisymmetric traveling fronts are given. When s tends to c, it is proven that the axisymmetric traveling fronts converge locally uniformly to planar traveling wave fronts in $\mathbb{R}^3$. The existence of axisymmetric traveling fronts is obtained by constructing a sequence of pyramidal traveling fronts and taking its limit. The qualitative properties are established by using the comparison principle and appealing to the asymptotic speed of propagation for the resulting system. Finally, the nonexistence of axisymmetric traveling fronts with concave/convex level sets is discussed.

1. **Introduction.** In this paper we study the existence of traveling wave solutions in the following two-specis Lotka-Volterra competition-diffusion system in the three-dimensional spatial space:

$$\begin{cases} \frac{\partial}{\partial t} u_1(\mathbf{x},t) = \Delta u_1(\mathbf{x},t) + u_1(\mathbf{x},t)\left[1- \ u_1(\mathbf{x},t) - k_1 u_2(\mathbf{x},t)\right], \\ \frac{\partial}{\partial t} u_2(\mathbf{x},t) = d\Delta u_2(\mathbf{x},t) + r u_2(\mathbf{x},t)\left[1 - u_2(\mathbf{x},t) - k_2 u_1(\mathbf{x},t)\right], \end{cases} \quad \mathbf{x} \in \mathbb{R}^3, t > 0, \tag{1}$$

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where k_1, k_2, r and d are positive constants, the variables $u_1(\mathbf{x},t)$ and $u_2(\mathbf{x},t)$ are the population densities of two competing species. In the field of population biology the Lotka-Volterra competition system is known as a physiological model describing competing interactions of multiple species. The related kinetic system of (1) is as follows:

$$\begin{cases} \frac{d}{dt}u_1(t) = u_1(t)\left[1 - \ u_1(t) - k_1u_2(t)\right], \\ \frac{d}{dt}u_2(t) = ru_2(t)\left[1 - u_2(t) - k_2u_1(t)\right], \end{cases} \quad t > 0. \tag{2}$$

We note that systems (1) and (2) are normalized so that they have the equilibrium solutions $\mathbf{E}_u = (1,0)$ and $\mathbf{E}_v = (0,1)$. Obviously, $\mathbf{E}_0 = (0,0)$ is also an equilibrium of systems (1) and (2). When $k_1, k_2 < 1$ or $k_1, k_2 > 1$, there exists the fourth equilibrium (co-existence state) $\mathbf{E}_* = (E_{*,1}, E_{*,2})$, where

$$E_{*,1} = \frac{k_1 - 1}{k_1k_2 - 1}, \quad E_{*,2} = \frac{k_2 - 1}{k_1k_2 - 1}.$$

In general, the species u_1 is called a *strong (weak, resp.) competitor* if $k_2 > 1$ ($k_2 < 1$, resp.). For the case $0 < k_1 < 1 < k_2$ (or $0 < k_2 < 1 < k_1$), one species is superior than the other. In this case, there is only one stable equilibrium and it is called the *monostable* case. For the case $k_1, k_2 > 1$, both $\mathbf{E}_u$ and $\mathbf{E}_v$ are stable and it is called the *bistable* (*strong competition*) case. We call the case when $k_1, k_2 < 1$ the *weak competition* (*coexistence*) case. See Guo and Wu [15] and Morita and Tachibana [37].

Traveling wave solutions of (1) in the *one-dimensional* spatial space have been extensively studied in the literature. We refer to a nice survey by Guo and Wu [15], see also Li et al. [32], Lin and Li [34], Morita and Tachibana [37], Zhao and Ruan [56] and the references therein. In this paper we are interested in the bistable case (or the strong competition case), namely, $k_1, k_2 > 1$. As mentioned above, in this case both $\mathbf{E}_u$ and $\mathbf{E}_v$ are stable. Following from Conley and Gardner [7], Gardner [11], Kan-on [27], Kan-on and Fang [30], and Volpert et al. [50], we know that in the bistable case system (1) admits a one-dimensional traveling wave front $\mathbf{\Phi}(x+ct) = (\Phi_1(x+ct), \Phi_2(x+ct))$ connecting $\mathbf{E}_v$ and $\mathbf{E}_u$, where $x \in \mathbb{R}$, $t > 0$, and the wave speed $c \in \mathbb{R}$. The traveling wave front $\mathbf{\Phi}(\xi) = (\Phi_1(\xi), \Phi_2(\xi))$ with $\xi = x + ct$ is unique up to translation and satisfies $\Phi_1'(\xi) > 0$ and $\Phi_2'(\xi) < 0$ for any $\xi \in \mathbb{R}$. In particular, the wave speed c is also unique. It should be pointed out that though the existence of traveling wave fronts is well known, there are few conclusions on the sign of the wave speed c of traveling wave solutions of (1) in the bistable case since it is difficult to determine the sign of c. Recently, Guo and Lin [14] gave some sufficient criteria about the sign of the wave speed under some parameter restrictions by using the result of Kan-on [27] (see also Alcahrani et al. [1] for the sign of wave speed for near-degenerate bistable competition models). For the positive stationary solutions in the bistable case, we refer to Kan-on [29] for the instability of stationary solutions and Kan-on [28] for the standing waves. Other than traveling wave solutions, there are solutions with two fronts approaching each other from both ends of the real line, which are called entire solutions and are constructed by Morita and Tachibana [37]. Here an entire solution means a solution defined for all $t \in \mathbb{R}$. Moreover, we refer to Guo and Wu [16] for a two-component lattice dynamical system arising in strong competition models and Lin and Li [34] for a Lotka-Volterra competition-diffusion system with nonlocal delays, respectively.

The above results on the existence of traveling wave solutions of (1) are only about one-dimensional traveling wave solutions (or planar traveling wave solutions in high-dimensional spaces). However, it is observed that in high-dimensional spaces propagating wave fronts may change shape and evolve to new nonplanar traveling waves. Therefore, it is interesting but challenging to study possible nonplanar traveling waves. In the past decade, multi-dimensional traveling fronts have attracted a lot of attention and new types of nonplanar traveling waves have been obtained for the *scalar* reaction-diffusion equation

$$\frac{\partial}{\partial t}u(\mathbf{x},t) = d\Delta u(\mathbf{x},t) + f(u(\mathbf{x},t)),\ \mathbf{x} \in \mathbb{R}^m,\ t > 0 \tag{3}$$

with various nonlinearities. For the combustion nonlinearity, see Bonnet and Hamel [3], Hamel and Monneau [17], and Hamel et al. [18]. For the Fisher-KPP case, see Brazhnik and Tyson [4], Hamel and Roquejoffre [21] and Huang [26]. For the bistable case (in particular the Allen-Cahn equation), see Fife [10], Hamel et al. [19, 20], Ninomiya and Taniguchi [39, 40] and Gui [13] for V-form front solutions with $m = 2$, Chen et al. [6], Hamel et al. [19, 20] and Taniguchi [48] for cylindrically symmetric traveling fronts with $m \geq 3$, and Taniguchi [46, 47, 48] and Kurokawa and Taniguchi [31] for traveling fronts with pyramidal shapes with $m \geq 3$. For traveling fronts with V-shape, pyramidal shape and conical shape for a bistable reaction-diffusion equation with time-periodic nonlinearity, we refer to Wang and Wu [54], Sheng et al. [44] and Wang [52], respectively. For non-connected traveling fronts and non-convex traveling fronts, we refer to del Pino et al. [41, 42]. We also refer to a survey by Witelski et al. [55] on axisymmetric traveling waves of semi-linear elliptic equations. Other related works can be found in Chapuisat [5], El Smaily et al. [9], Fife [10], Hamel and Roquejoffre [22], and Morita and Ninomiya [36].

Recently, there have been important progresses on the study of nonplanar traveling wave solutions in *systems* of reaction-diffusion equations. By using bifurcation theory, Haragus and Scheel [23, 24, 25] studied almost planar waves (V-form waves) in reaction-diffusion systems in which the interface region is close to hyperplanes (the angle of the interface is close to π). By developing the arguments of Ninomiya and Taniguchi [39, 40], Wang [51] established the existence and stability of two-dimensional V-form curved fronts for bistable reaction-diffusion systems for any admissible wave speed. In particular, the result of Wang [51] are applicable to system (1) with $k_1, k_2 > 1$ in $\mathbb{R}^2$. Furthermore, the existence, uniqueness and stability of pyramidal traveling fronts of bistable reaction-diffusion systems in $\mathbb{R}^3$ were established in Wang et al. [53] by extending the arguments of Taniguchi [46, 47]. The result of [53] is also applicable to system (1) with $k_1, k_2 > 1$ in $\mathbb{R}^3$. At the same time, Ni and Taniguchi [38] also established the existence of pyramidal traveling fronts of (1) with $k_1, k_2 > 1$ in $\mathbb{R}^m$ ($m \geq 3$). We refer to [53, 38] and the next section for details on the pyramidal traveling fronts of (1).

In this paper we are interested in the axisymmetric traveling wave solutions of (1) with $k_1, k_2 > 1$ in $\mathbf{x} \in \mathbb{R}^3$. Though axisymmetric traveling wave solutions in scalar bistable reaction-diffusion equations have been studied before (see Hamel et al. [19, 20] and Taniguchi [48]), here we would like to emphasize that there is no result about the axisymmetric traveling wave solutions of bistable reaction-diffusion systems in $\mathbb{R}^3$ up to now (according to our best knowledge). The purpose of the current paper is to establish the existence of axisymmetric traveling fronts with wave speed $s > c > 0$ for (1) in $\mathbb{R}^3$ and to show some important qualitative properties of the axisymmetric traveling fronts. In addition, we will show the nonexistence of axisymmetric traveling fronts. Now we state the main results of this paper.

Theorem 1.1. *Assume that $k_1, k_2 > 1$ and $c > 0$, where c is the wave speed of the planar traveling wave front $\mathbf{\Phi}(\mathbf{x}\cdot\mathbf{e}+ct) = (\Phi_1(\mathbf{x}\cdot\mathbf{e}+ct), \Phi_2(\mathbf{x}\cdot\mathbf{e}+ct))$ of (1) with $\mathbf{\Phi}(-\infty) = \mathbf{E}_v$, $\mathbf{\Phi}(+\infty) = \mathbf{E}_u$, $\mathbf{e} \in \mathbb{R}^3$ and $|\mathbf{e}| = 1$. Then for any $s > c$, there exists a function $\mathbf{\Psi}(\mathbf{x}) = (\Psi_1(\mathbf{x}), \Psi_2(\mathbf{x})) \in C^2(\mathbb{R}^3)$ satisfying*

$$\begin{cases} \Delta\Psi_1(\mathbf{x}) - s\frac{\partial}{\partial x_3}\Psi_1(\mathbf{x}) + \Psi_1(\mathbf{x})[1-\Psi_1(\mathbf{x}) - k_1\Psi_2(\mathbf{x})] = 0, \\ d\Delta\Psi_2(\mathbf{x}) - s\frac{\partial}{\partial x_3}\Psi_2(\mathbf{x}) + r\Psi_2(\mathbf{x})[1-\Psi_2(\mathbf{x}) - k_2\Psi_1(\mathbf{x})] = 0, \end{cases} \quad \mathbf{x} \in \mathbb{R}^3. \tag{4}$$

In addition, one has

(i) $\mathbf{\Psi}(\mathbf{x}_1', x_3) = \mathbf{\Psi}(\mathbf{x}_2', x_3)$, $\forall\, \mathbf{x}_1', \mathbf{x}_2' \in \mathbb{R}^2$ *with* $|\mathbf{x}_1'| = |\mathbf{x}_2'|$, $x_3 \in \mathbb{R}$;

(ii) for any $(\mathbf{x}_0', x_3') \in \mathbb{R}^2 \times \mathbb{R}$ *with* $x_3' \geq m_* |\mathbf{x}_0'|$,

$$\Psi_1(\mathbf{x}' + \mathbf{x}_0', x_3) \leq \Psi_1(\mathbf{x}', x_3 + x_3'), \quad \forall\, (\mathbf{x}', x_3) \in \mathbb{R}^2 \times \mathbb{R}$$

and

$$\Psi_2(\mathbf{x}' + \mathbf{x}_0', x_3) \geq \Psi_2(\mathbf{x}', x_3 + x_3'), \quad \forall\, (\mathbf{x}', x_3) \in \mathbb{R}^2 \times \mathbb{R},$$

where $m_* = \sqrt{\frac{s^2-c^2}{c}}$;

(iii) $\frac{\partial}{\partial x_3}\Psi_1(\mathbf{x}) > 0$ *and* $\frac{\partial}{\partial x_3}\Psi_2(\mathbf{x}) < 0$ *for any* $\mathbf{x} \in \mathbb{R}^3$;

(iv) $\frac{\partial}{\partial x_i}\Psi_1(\mathbf{x}) > 0$ *on* $x_i \in (0,\infty)$, $\frac{\partial}{\partial x_i}\Psi_2(\mathbf{x}) < 0$ *on* $x_i \in (0,\infty)$, $i = 1, 2$;

(v) $\lim_{x_3\to+\infty} \|\mathbf{\Psi}(\cdot, x_3) - \mathbf{E}_u\|_{C(\mathbb{R}^2)} = 0$ *and* $\lim_{x_3\to-\infty} \|\mathbf{\Psi}(\cdot, x_3) - \mathbf{E}_v\|_{C_{loc}(\mathbb{R}^2)} = 0$;

(vi) $\frac{\partial}{\partial \nu}\Psi_1(\mathbf{x}) > 0$ *and* $\frac{\partial}{\partial \nu}\Psi_2(\mathbf{x}) < 0$ *for any* $\mathbf{x} \in \mathbb{R}^3$, *where*

$$\nu = \frac{1}{\sqrt{1+\nu_1^2+\nu_2^2}}(\nu_1, \nu_2, 1)$$

satisfies $\sqrt{\nu_1^2+\nu_2^2} \leq \frac{1}{m_*}$.

Since $\mathbf{\Psi}(\mathbf{x})$ satisfies (i)-(v) of Theorem 1.1, we call the function $\mathbf{\Psi}(\mathbf{x})$ an *axisymmetric traveling front* of (1). As stated previously, it is difficult to determine the sign of the wave speed c of traveling wave solutions of (1) with $k_1, k_2 > 1$. For the reader's convenience, here we recall some sufficient conditions from Guo and Lin [14] to ensure $c > 0$. That is, *one has $c > 0$ if one of the following conditions holds:*

1) $r = d$ *and* $k_2 > k_1 > 1$;
2) $r < d$, $k_1 > 1$ *and* $k_2 \geq \left(\frac{d}{r}\right)^2 k_1$;
3) $r = \frac{d}{4}$, $k_2 \geq \frac{4}{3}$ *and* $1 < k_1 \leq \frac{5}{4}$ *except* $(k_1, k_2) = \left(\frac{5}{4}, \frac{4}{3}\right)$.

In the following we evaluate the limit of axisymmetric traveling fronts $\mathbf{\Psi}(\mathbf{x})$ with speed $s > c$ as $s \to c$ and the nonexistence of axisymmetric traveling fronts.

Theorem 1.2. *Let $s > c$ and denote $\mathbf{\Psi}(\mathbf{x})$ defined in Theorem 1.1 by $\mathbf{\Psi}^s(\mathbf{x})$. Let $\Psi_2^s(\mathbf{0}) = \Phi_2(0)$. Then one has*

$$\lim_{s\to c} \|\mathbf{\Psi}^s(\mathbf{x}) - \mathbf{\Phi}(x_3)\|_{C^2_{loc}(\mathbb{R}^3)} = 0.$$

Theorem 1.3. *For $s > c$, there is no axisymmetric traveling front $\mathbf{\Psi}(\mathbf{x})$ of (1) satisfying (4), $\lim_{x_3\to+\infty} \mathbf{\Psi}(\mathbf{0}, x_3) = \mathbf{E}_u$, $\lim_{x_3\to-\infty} \mathbf{\Psi}(\mathbf{0}, x_3) = \mathbf{E}_v$ and*

$$\frac{\partial^2}{\partial x_i^2}\Psi_1(\mathbf{x})\Big|_{\mathbf{x}'=\mathbf{0}} \leq 0, \quad \frac{\partial^2}{\partial x_i^2}\Psi_2(\mathbf{x})\Big|_{\mathbf{x}'=\mathbf{0}} \geq 0, \quad \frac{\partial}{\partial x_3}\Psi_1(\mathbf{x}) \geq 0, \quad \frac{\partial}{\partial x_3}\Psi_2(\mathbf{x}) \leq 0,$$

where $i = 1, 2$.

Theorem 1.4. *For $s < c$, there is no axisymmetric traveling front $\mathbf{\Psi}(\mathbf{x})$ of (1) satisfying (4), $\lim_{x_3\to+\infty}\mathbf{\Psi}(\mathbf{0},x_3)=\mathbf{E}_u$, $\lim_{x_3\to-\infty}\mathbf{\Psi}(\mathbf{0},x_3)=\mathbf{E}_v$ and*

$$\frac{\partial^2}{\partial x_i^2}\Psi_1(\mathbf{x})\bigg|_{\mathbf{x}'=\mathbf{0}} \geq 0,\ \ \frac{\partial^2}{\partial x_i^2}\Psi_2(\mathbf{x})\bigg|_{\mathbf{x}'=\mathbf{0}} \leq 0,\ \ \frac{\partial}{\partial x_3}\Psi_1(\mathbf{x}) \geq 0,\ \ \frac{\partial}{\partial x_3}\Psi_2(\mathbf{x}) \leq 0,$$

where $i = 1, 2$.

The result of Theorem 1.3 corresponds to Remark 1.7 of Hamel et al. [19] for the scalar bistable equation, see also Hamel and Monneau [17, Theorems 1.1 and 1.6] for the combustion equation. As reported by Hamel et al. [19], in the terminology of Haragus and Scheel [24], there is no exterior corner for system (1), while the solutions given in Theorem 1.1 are interior corners. Theorem 1.4 further implies that there must be $s > c$ for an interior corner. See also Haragus and Scheel [24, Theorem 1.1].

In this paper we prove Theorems 1.1-1.4 by using the results of Wang et al. [53] and Ni and Taniguchi [38] on the pyramidal traveling fronts of (1). Set $u_2^* = 1 - u_2$ and transform system (1) into (for the sake of simplicity, we drop the symbol $*$)

$$\begin{cases} \frac{\partial}{\partial t}u_1 = \Delta u_1 + u_1(\mathbf{x},t)\left[1-k_1-u_1(\mathbf{x},t)+k_1u_2(\mathbf{x},t)\right], \\ \frac{\partial}{\partial t}u_2 = d\Delta u_2 + r\left(1-u_2(\mathbf{x},t)\right)\left[k_2u_1(\mathbf{x},t)-u_2(\mathbf{x},t)\right], \end{cases} \mathbf{x}\in\mathbb{R}^3, t>0. \tag{5}$$

Correspondingly, the equilibria $\mathbf{E}_u = (1,0)$, $\mathbf{E}_v = (0,1)$, $\mathbf{E}_0 = (0,0)$ and $\mathbf{E}_* = (E_{*,1}, E_{*,2})$ become $\mathbf{E}^1 = (1,1)$, $\mathbf{E}^0 = (0,0)$, $\mathbf{E}^v = (0,1)$ and $\mathbf{E}^* = (E_1^*, E_2^*)$, respectively, where

$$E_1^* = \frac{k_1-1}{k_1k_2-1}, \quad E_2^* = \frac{k_2(k_1-1)}{k_1k_2-1}.$$

Let $U_1(\xi) = \Phi_1(\xi)$ and $U_2(\xi) = 1 - \Phi_2(\xi)$. Then $\mathbf{U}(\xi) = (U_1(\xi), U_2(\xi))$ is a traveling wave front of (5) connecting $\mathbf{E}^1 = (1,1)$ and $\mathbf{E}^0 = (0,0)$, where $\xi = \mathbf{x}\cdot\mathbf{e}+ct$, $\mathbf{e}\in\mathbb{R}^3$ with $|\mathbf{e}| = 1$. In particular, $\mathbf{U}(-\infty) = \mathbf{E}^0$, $\mathbf{U}(+\infty) = \mathbf{E}^1$, $U_1'(\xi) > 0$ and $U_2'(\xi) > 0$ for any $\xi \in \mathbb{R}$. To complete the proof of Theorem 1.1, we need only to prove that there exists an axisymmetric traveling front $\mathbf{W}(\mathbf{x})$ of (5) satisfying (14) and (ii)-(vii) of Theorem 3.1 in Section 3. In order to obtain such a function $\mathbf{W}(\mathbf{x})$, we use the results of Wang et al. [53] to construct a sequence of pyramidal traveling fronts of (5), and then take a limit for the sequence of pyramidal traveling fronts. Thus, the limit function is just the expected solution. This step is similar to that in Taniguchi [48]. However, due to the effect of the coupled nonlinearity, we cannot use the arguments of Taniguchi [48] and Hamel et al. [19] to prove qualitative properties of the axisymmetric traveling front $\mathbf{W}(\mathbf{x})$ (namely (ii)-(vii) of Theorem 3.1). Therefore, in this paper we develop a new method to show (ii)-(vii) of Theorem 3.1 in Section 3, where a crucial procedure is to use the comparison principle and appeal to the spreading speed of solutions of the resulting equations (systems). The proof of Theorem 1.2 can be completed by using the result of Theorem 1.1. In Section 4, we prove Theorems 1.3 and 1.4, which imply the nonexistence of axisymmetric traveling fronts. Before giving the proofs of Theorems 1.1-1.4, we first show the existence results and some qualitative properties of pyramidal traveling fronts of (5) in Section 2. In Section 5, we give the proofs of two important lemmas, which are listed in Section 4. Finally, in Section 6 we give a discussion on the obtained results of this paper.

2. Preliminaries.

In this section we state the existence results on the pyramidal traveling fronts of (5) in $\mathbb{R}^3$, which has been established by Ni and Taniguchi [38] and

Wang et al. [53]. Then we show some properties of the pyramidal traveling fronts which are very important to establish the axisymmetric traveling wave solutions in next section. Suppose $k_1, k_2 > 1$. Let $\mathbf{U}(\mathbf{x}\cdot\mathbf{e}+ct) = (U_1(\mathbf{x}\cdot\mathbf{e}+ct), U_2(\mathbf{x}\cdot\mathbf{e}+ct))$ be the planar traveling wave front of (5) connecting $\mathbf{E}^0$ and $\mathbf{E}^1$. Assume $c > 0$.

For two vectors $\mathbf{c} = (c_1, c_2)$ and $\mathbf{d} = (d_1, d_2)$, the symbol $\mathbf{c} \ll \mathbf{d}$ means $c_i < d_i$ for each $i = 1, 2$ and $\mathbf{c} \leq \mathbf{d}$ means $c_i \leq d_i$ for each $i = 1, 2$. The interval $[\mathbf{c}, \mathbf{d}]$ denotes the set of $\mathbf{q} \in \mathbb{R}^2$ with $\mathbf{c} \leq \mathbf{q} \leq \mathbf{d}$. For $\mathbf{c} = (c_1, c_2)$, we denote $|\mathbf{c}| = \sqrt{c_1^2 + c_2^2}$. For any $\mathbf{u} \in BC\left(\mathbb{R}^3, \mathbb{R}^2\right)$, we define

$$\|\mathbf{u}\|_{C(\mathbb{R}^3)} = \sup_{\mathbf{x}\in\mathbb{R}^3} |\mathbf{u}(\mathbf{x})|,$$

where $BC\left(\mathbb{R}^3, \mathbb{R}^2\right)$ denotes the set of bounded and continuous functions defined on $\mathbb{R}^3$. Fix $s > c > 0$. Assume that the solutions travel towards the $-x_3$ direction without loss of generality. Take

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{v}(\mathbf{x}', x_3 + st, t), \ \mathbf{x}' = (x_1, x_2), \ \mathbf{x} = (\mathbf{x}', x_3) = (x_1, x_2, x_3).$$

Then we have the following initial value problem

$$\begin{cases} \frac{\partial}{\partial t} v_1(\mathbf{x}, t) = \Delta v_1(\mathbf{x}, t) - s\frac{\partial}{\partial x_3} v_1(\mathbf{x}, t) + v_1(\mathbf{x}, t)\left[1 - k_1 - v_1(\mathbf{x}, t) + k_1 v_2(\mathbf{x}, t)\right], \\ \frac{\partial}{\partial t} v_2(\mathbf{x}, t) = d\Delta v_2(\mathbf{x}, t) - s\frac{\partial}{\partial x_3} v_2(\mathbf{x}, t) + r\left(1 - v_2(\mathbf{x}, t)\right)\left[k_2 v_1(\mathbf{x}, t) - v_2(\mathbf{x}, t)\right], \\ v_1(\mathbf{x}, 0) = v_1^0(\mathbf{x}), \ v_2(\mathbf{x}, 0) = v_2^0(\mathbf{x}), \end{cases} \tag{6}$$

where $\mathbf{x} \in \mathbb{R}^3$, $t > 0$.

Let $n \geq 3$ be a given integer and

$$m_* = \frac{\sqrt{s^2 - c^2}}{c}.$$

Let $\{\mathbf{A}_j = (A_j, B_j)\}_{j=1}^n$ be a set of unit vectors in $\mathbb{R}^2$ such that

$$A_j B_{j+1} - A_{j+1} B_j > 0, j = 1, 2, \cdots, n-1; \ A_n B_1 - A_1 B_n > 0.$$

Now $(m_* \mathbf{A}_j, 1) \in \mathbb{R}^3$ is the normal vector of $\left\{\mathbf{x} \in \mathbb{R}^3 \middle| -x_3 = m_*\left(\mathbf{A}_j, \mathbf{x}'\right)\right\}$. Set

$$h_j\left(\mathbf{x}'\right) = m_*\left(\mathbf{A}_j, \mathbf{x}'\right) \text{ and } h\left(\mathbf{x}'\right) = \max_{1\leq j\leq n} h_j\left(\mathbf{x}'\right) = m_* \max_{1\leq j\leq n}\left(\mathbf{A}_j, \mathbf{x}'\right)$$

for $\mathbf{x}' \in \mathbb{R}^2$. We call $\left\{\mathbf{x} = (\mathbf{x}', x_3) \in \mathbb{R}^3 \middle| -x_3 = h\left(\mathbf{x}'\right)\right\}$ a 3-*dimensional pyramid* in $\mathbb{R}^3$. Letting

$$\Omega_j = \left\{\mathbf{x}' \in \mathbb{R}^2 \middle| h\left(\mathbf{x}'\right) = h_j\left(\mathbf{x}'\right)\right\}$$

for $j = 1, \cdots, n$, we have $\mathbb{R}^2 = \cup_{j=1}^n \Omega_j$. Denote the boundary of Ω_j by $\partial\Omega_j$. Let

$$E = \cup_{j=1}^n \partial\Omega_j.$$

Now we set

$$S_j = \left\{\mathbf{x} \in \mathbb{R}^3 \middle| -x_3 = h_j\left(\mathbf{x}'\right) \text{ for } \mathbf{x}' \in \Omega_j\right\}$$

for $j = 1, \cdots, n$, and call $\cup_{j=1}^n S_j \subset \mathbb{R}^3$ the *lateral faces of the pyramid*. Put

$$\Gamma_j = S_j \cap S_{j+1}, \ \Gamma_n = S_n \cap S_1, \ j = 1, \cdots, n-1.$$

Then $\Gamma := \cup_{j=1}^n \Gamma_j$ represents the set of all edges of a pyramid. Define

$$\mathbf{v}^-(\mathbf{x}) = \mathbf{U}\left(\frac{c}{s}\left(x_3 + h\left(\mathbf{x}'\right)\right)\right) = \max_{1\leq j\leq n} \mathbf{U}\left(\frac{c}{s}\left(x_3 + h_j\left(\mathbf{x}'\right)\right)\right).$$

Define

$$D(\gamma) = \left\{\mathbf{x} \in \mathbb{R}^3 \middle| \operatorname{dist}\left(\mathbf{x}, \cup_{j=1}^n \Gamma_j\right) > \gamma\right\} \quad \text{for } \gamma > 0.$$

Let $\mathbf{v}\left(\mathbf{x},t;\mathbf{v}^{-}\right)=\left(v_1\left(\mathbf{x},t;\mathbf{v}^{-}\right),v_2\left(\mathbf{x},t;\mathbf{v}^{-}\right)\right)$ be the solution of (6) with $\mathbf{v}^0=\mathbf{v}^{-}$. Then there exists a function $\mathbf{V}(\mathbf{x})\in C^2\left(\mathbb{R}^3\right)$ such that

$$\mathbf{V}(\mathbf{x})=\lim_{t\to\infty}\mathbf{v}\left(\mathbf{x},t;\mathbf{v}^{-}\right).$$

The following theorem is obtained by Wang et al. [53, Theorem 1.1], see also Ni and Taniguchi [38].

Theorem 2.1. *For each $s>c>0$, there exists a solution $\mathbf{u}(\mathbf{x},t)=\mathbf{V}(\mathbf{x}',x_3+st)$ of (5) satisfying $\mathbf{V}(\mathbf{x})>\mathbf{v}^{-}(\mathbf{x})$, $\lim_{\gamma\to\infty}\sup_{\mathbf{x}\in D(\gamma)}\left|\mathbf{V}(\mathbf{x})-\mathbf{v}^{-}(\mathbf{x})\right|=0$ and*

$$\begin{cases}\Delta V_1-s\frac{\partial}{\partial x_3}V_1+V_1(\mathbf{x})\left[1-k_1-V_1(\mathbf{x})+k_1V_2(\mathbf{x})\right]=0,\\ d\Delta V_2-s\frac{\partial}{\partial x_3}V_2+r\left(1-V_2(\mathbf{x})\right)\left[k_2V_1(\mathbf{x})-V_2(\mathbf{x})\right]=0,\end{cases}\quad \mathbf{x}\in\mathbb{R}^3. \tag{7}$$

Moreover, for any $\mathbf{u}^0\in C\left(\mathbb{R}^3,\mathbb{R}^N\right)$ with $\mathbf{u}^0(\mathbf{x})\in\left[\mathbf{E}^0,\mathbf{E}^1\right]$ for $\mathbf{x}\in\mathbb{R}^3$ and

$$\lim_{\gamma\to\infty}\sup_{\mathbf{x}\in D(\gamma)}\left|\mathbf{u}^0(\mathbf{x})-\mathbf{V}(\mathbf{x})\right|=0, \tag{8}$$

the solution $\mathbf{u}(\mathbf{x},t;\mathbf{u}^0)$ of (5) with initial value $\mathbf{u}^0$ satisfies

$$\lim_{t\to\infty}\left\|\mathbf{u}(\cdot,\cdot,t;\mathbf{u}^0)-\mathbf{V}(\cdot,\cdot+st)\right\|_{C(\mathbb{R}^3)}=0. \tag{9}$$

The next two lemmas show the monotonicity of the pyramidal traveling front $\mathbf{V}$.

Lemma 2.2. *For any $(x_0,y_0,z_0)\in\mathbb{R}^3$ with $z_0\geq h(x_0,y_0)$, one has*

$$\mathbf{V}(x_1+x_0,x_2+y_0,x_3)\leq\mathbf{V}(x_1,x_2,x_3+z_0)\quad\text{for any}\quad(x_1,x_2,x_3)\in\mathbb{R}^3.$$

Lemma 2.3. *Let*

$$\nu=\frac{1}{\sqrt{1+\nu_1^2+\nu_2^2}}\begin{pmatrix}\nu_1\\ \nu_2\\ 1\end{pmatrix}$$

be a given constant vector with $\sqrt{\nu_1^2+\nu_2^2}\leq\frac{1}{m_*}$. Then one has

$$\frac{\partial}{\partial\nu}\mathbf{V}(\mathbf{x})\gg\mathbf{0},\quad\forall\ \mathbf{x}\in\mathbb{R}^3.$$

The proofs of the above lemmas are similar to those of Taniguchi [48, Lemma 2.5] and Taniguchi [48, Lemma 3.4], respectively. Here we omit them.

It follows from Lemma 2.3 that $\frac{\partial}{\partial x_3}\mathbf{V}(\mathbf{x})\gg\mathbf{0}$ for any $\mathbf{x}\in\mathbb{R}^3$. In the following we further show that if the initial value $\mathbf{v}^0$ is even in x_1, then the solution $\mathbf{v}(\mathbf{x},t;\mathbf{v}^0)$ is also even in x_1. Furthermore, if $\mathbf{v}^0$ is nondecreasing in $x_1\geq0$, then the solution $\mathbf{v}(\mathbf{x},t;\mathbf{v}^0)$ is also nondecreasing in $x_1\geq0$. Here we use a method which is different from that in Taniguchi [48, Lemma 3.5].

Lemma 2.4. *Assume that $f(x)\in BC\left([0,\infty),\mathbb{R}\right)$ is nondecreasing. Then the function*

$$F(x,t)=\frac{1}{\sqrt{4\pi kt}}\int_0^\infty\left(e^{-(x-y)^2/4kt}+e^{-(x+y)^2/4kt}\right)f(y)dy,\qquad x\geq0,t>0$$

is nondecreasing on $x\in[0,\infty)$, where $k>0$.

Proof. Let $\varepsilon > 0$. We show that $F(x+\varepsilon,t) - F(x,t) \geq 0$. We have

$$
\begin{aligned}
&F(x+\varepsilon,t) - F(x,t) \\
&\quad = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left(e^{-((x+\varepsilon)-y)^2/4kt} + e^{-((x+\varepsilon)+y)^2/4kt}\right) f(y)dy \\
&\qquad - \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left(e^{-(x-y)^2/4kt} + e^{-(x+y)^2/4kt}\right) f(y)dy \\
&\quad = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{x+\varepsilon} e^{-y^2/4kt} f((x+\varepsilon)-y)\,dy \\
&\qquad + \frac{1}{\sqrt{4\pi kt}} \int_{x+\varepsilon}^{\infty} e^{-y^2/4kt} f(y-(x+\varepsilon))\,dy \\
&\qquad - \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{x} e^{-y^2/4kt} f(x-y)\,dy - \frac{1}{\sqrt{4\pi kt}} \int_{x}^{\infty} e^{-y^2/4kt} f(y-x)\,dy \\
&\quad = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{x} e^{-y^2/4kt} \left[f((x+\varepsilon)-y) - f(x-y)\right] dy \\
&\qquad + \frac{1}{\sqrt{4\pi kt}} \int_{x+\varepsilon}^{\infty} e^{-y^2/4kt} \left[f(y-(x+\varepsilon)) - f(y-x)\right] dy \\
&\qquad + \frac{1}{\sqrt{4\pi kt}} \int_{x}^{x+\varepsilon} e^{-y^2/4kt} \left[f((x+\varepsilon)-y) - f(y-x)\right] dy.
\end{aligned}
$$

By direct calculations, we have

$$
\begin{aligned}
&\frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{x} e^{-y^2/4kt} \left[f((x+\varepsilon)-y) - f(x-y)\right] dy \\
&\qquad = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty e^{-(x-y)^2/4kt} \left[f(\varepsilon+y) - f(y)\right] dy, \\
&\frac{1}{\sqrt{4\pi kt}} \int_{x+\varepsilon}^{\infty} e^{-y^2/4kt} \left[f(y-(x+\varepsilon)) - f(y-x)\right] dy \\
&\qquad = -\frac{1}{\sqrt{4\pi kt}} \int_0^\infty e^{-(x+\varepsilon+y)^2/4kt} \left[f(\varepsilon+y) - f(y)\right] dy
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{\sqrt{4\pi kt}} \int_{x}^{x+\varepsilon} e^{-y^2/4kt} \left[f((x+\varepsilon)-y) - f(y-x)\right] dy \\
&\quad = \frac{1}{\sqrt{4\pi kt}} \int_{x}^{x+\frac{\varepsilon}{2}} e^{-y^2/4kt} \left[f((x+\varepsilon)-y) - f(y-x)\right] dy \\
&\qquad + \frac{1}{\sqrt{4\pi kt}} \int_{x+\frac{\varepsilon}{2}}^{x+\varepsilon} e^{-y^2/4kt} \left[f((x+\varepsilon)-y) - f(y-x)\right] dy \\
&\quad = \frac{1}{\sqrt{4\pi kt}} \int_0^{\frac{\varepsilon}{2}} e^{-(x+y)^2/4kt} \left[f(\varepsilon-y) - f(y)\right] dy \\
&\qquad - \frac{1}{\sqrt{4\pi kt}} \int_0^{\frac{\varepsilon}{2}} e^{-(x+\varepsilon-y)^2/4kt} \left[f(\varepsilon-y) - f(y)\right] dy \\
&\quad = \frac{1}{\sqrt{4\pi kt}} \int_0^{\frac{\varepsilon}{2}} \left(e^{-(x+y)^2/4kt} - e^{-(x+\varepsilon-y)^2/4kt}\right) \left[f(\varepsilon-y) - f(y)\right] dy.
\end{aligned}
$$

In view of $x \geq 0$, we obtain

$$\begin{aligned}
&F(x+\varepsilon,t)-F(x,t)\\
&\quad=\frac{1}{\sqrt{4\pi kt}}\int_0^\infty\left(e^{-(x-y)^2/4kt}-e^{-(x+\varepsilon+y)^2/4kt}\right)[f(\varepsilon+y)-f(y)]\,dy\\
&\quad+\frac{1}{\sqrt{4\pi kt}}\int_0^{\frac{\varepsilon}{2}}\left(e^{-(x+y)^2/4kt}-e^{-(x+\varepsilon-y)^2/4kt}\right)[f(\varepsilon-y)-f(y)]\,dy\\
&\quad\geq 0.
\end{aligned}$$

This completes the proof. □

Lemma 2.5. *Assume that* $\mathbf{v}^0(\mathbf{x}) \in C\left(\mathbb{R}^3, \left[\mathbf{E}^0, \mathbf{E}^1\right]\right)$ *is even in* x_1*, uniformly continuous in* $\mathbf{x} \in \mathbb{R}^3$*, and nondecreasing in* $x_1 \in [0,\infty)$*. Then there exists a unique solution* $\mathbf{v}(\mathbf{x},t) \in C\left(\mathbb{R}^3 \times [0,\infty), \left[\mathbf{E}^0, \mathbf{E}^1\right]\right) \bigcap C^{2,1}\left(\mathbb{R}^3 \times (0,\infty), \left[\mathbf{E}^0, \mathbf{E}^1\right]\right)$ *of* (6) *such that* $\mathbf{v}(\mathbf{x},t)$ *is even in* x_1 *and nondecreasing in* $x_1 \in [0,\infty)$*.*

Proof. The proof is divided into three steps.

Step 1. Let $X = BUC\left(\mathbb{R}^3, \mathbb{R}^2\right)$ with norm $\|\cdot\|_{C(\mathbb{R}^3,\mathbb{R}^2)}$, where $BUC\left(\mathbb{R}^3, \mathbb{R}^2\right)$ denotes the set of bounded and uniformly continuous functions defined on $\mathbb{R}^3$. Define

$$\mathbf{T}(t) = \begin{pmatrix} \Gamma(t) & 0 \\ 0 & \Gamma(dt) \end{pmatrix}$$

by $\mathbf{T}(t)\mathbf{u} = \mathrm{diag}(\Gamma(t)u_1, \Gamma(dt)u_2)$ for $\mathbf{u}(\mathbf{x}) \in X$, where

$$\Gamma(t)u(\mathbf{x}) = \iiint_{\mathbb{R}^3} \frac{1}{\left(\sqrt{4\pi t}\right)^3} e^{-\frac{(x_1-x)^2+(x_2-y)^2+(x_3-z)^2}{4t}} u(x,y,z)\,dxdydz.$$

By results in Daners and McLeod [8] we know that $\mathbf{T}(t)$ is a strongly continuous analytic semigroup of contractions on X which is generated by $\mathbf{D}\Delta_X$, where $\mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$ and Δ_X is the X-realisation of Δ. In particular, for any $\mathbf{u}(\mathbf{x}) \in X$ we have

$$\mathbf{T}(t)\mathbf{u} \to \mathbf{u} \quad \text{as } t \to 0^+ \text{ in } X. \tag{10}$$

We identify an element $\mathbf{w} \in C([0,\infty), X)$ as a function from $\mathbb{R}^3 \times [0,\infty)$ to $\mathbb{R}^2$ defined by $\mathbf{w}(\mathbf{x},t) = \mathbf{w}(t)(\mathbf{x})$. Let

$$\mathcal{S}_\infty = \left\{ \mathbf{w}(t) \in C([0,\infty), X) \;\middle|\; \begin{array}{c} \mathbf{w}(0) = \mathbf{v}^0,\ \mathbf{E}^0 \leq \mathbf{w}(\mathbf{x},t) \leq \mathbf{E}^1, \forall \mathbf{x} \in \mathbb{R}^3, t > 0; \\ \mathbf{w}(\mathbf{x},t) \text{ is nondecreasing in } x_1 \in [0,\infty); \\ \mathbf{w}(\mathbf{x},t) \text{ is even in } x_1 \in \mathbb{R}. \end{array} \right\}.$$

For any $\mathbf{w}(t) \in \mathcal{S}_\infty$, consider the following initial-valued problem

$$\begin{cases} \frac{\partial}{\partial t}u_1 - \Delta u_1 + M_1 u_1 = M_1 w_1 + f_1(w_1, w_2), \\ \frac{\partial}{\partial t}u_2 - d\Delta u_2 + M_2 u_2 = M_2 w_2 + f_2(w_1, w_2), \\ u_1(0) = v_1^0,\ u_2(0) = v_2^0, \end{cases} \tag{11}$$

where $f_1(w_1, w_2) = w_1(1 - k_1 - w_1 + k_1 w_2)$, $f_2(w_1, w_2) = r(1 - w_2)(k_2 w_1 - w_2)$, the positive constants M_1 and M_2 are large enough so that $M_i w_i + f_i(w_1, w_2)$ are nondecreasing on w_i in $\left[\mathbf{E}^0, \mathbf{E}^1\right]$, $i = 1, 2$. The existence and uniqueness of solutions of (11) are well known. In particular, the solution $\mathbf{u}(\mathbf{x},t)$ of (11) satisfies the following integral equation

$$\mathbf{u}(t) = e^{-\mathbf{M}t}\mathbf{T}(t)\mathbf{v}^0 + \int_0^t e^{-\mathbf{M}(t-s)}\mathbf{T}(t-s)\mathbf{H}(\mathbf{w}(s))\,ds,$$

where $e^{-\mathbf{M}t}$=diag$\left(e^{-M_1 t}, e^{-M_2 t}\right)$, $\mathbf{H}(\mathbf{w}) = \left(H^1(\mathbf{w}), H^2(\mathbf{w})\right)$ and

$$H^i(\mathbf{w}) = M_i w_i + f_i(\mathbf{w}).$$

Since $H^i(\mathbf{w})$ are nondecreasing on $w_j (j = 1, 2)$, it is easy to show that

$$\mathbf{E}^0 \le \mathbf{u}(\mathbf{x}, t) \le \mathbf{E}^1, \quad \forall \mathbf{x} \in \mathbb{R}^3,\ t > 0.$$

By virtue of (10), we further obtain that $\mathbf{u}(t) \in C([0, \infty), X)$.

In view of the evenness of $\mathbf{v}^0(\mathbf{x})$ and $\mathbf{w}(\mathbf{x}, t)$ in x_1, we know that the solution $\mathbf{u}(\mathbf{x}, t)$ of (11) is also even on $x_1 \in \mathbb{R}$. In particular, we have

$$\frac{\partial}{\partial x_1}\mathbf{u}(0, x_2, x_3, t) = 0, \quad \forall (x_2, x_3) \in \mathbb{R}^2,\ t > 0.$$

Consequently, we have that the solution $\mathbf{u}(\mathbf{x}, t)$ of (11) satisfies the following parabolic problem

$$\begin{cases} \frac{\partial}{\partial t}\overline{u}_1 - \Delta\overline{u}_1 + M_1\overline{u}_1 = M_1 w_1 + w_1[1 - k_1 - w_1 + k_1 w_2],\ \mathbf{x} \in \Omega,\ t > 0, \\ \frac{\partial}{\partial t}\overline{u}_2 - d\Delta\overline{u}_2 + M_2\overline{u}_2 = M_2 w_2 + r(1 - w_2)[k_2 w_1 - w_2],\ \mathbf{x} \in \Omega,\ t > 0, \\ \frac{\partial}{\partial \mathbf{n}}\overline{u}_1(\mathbf{x}) = \frac{\partial}{\partial \mathbf{n}}\overline{u}_2(\mathbf{x}) = 0,\ \mathbf{x} \in \partial\Omega,\ t > 0, \\ \overline{u}_1(\mathbf{x}, 0) = v_1^0(\mathbf{x}),\ \overline{u}_2(\mathbf{x}, 0) = v_2^0(\mathbf{x}),\ x \in \overline{\Omega}, \end{cases} \tag{12}$$

where $\Omega = \left\{\mathbf{x} \in \mathbb{R}^3, x_1 > 0\right\}$, $\frac{\partial}{\partial \mathbf{n}}$ denotes the outside normal derivative. It is known that the solution $\overline{\mathbf{u}}(\mathbf{x}, t) = (\overline{u}_1(\mathbf{x}, t), \overline{u}_2(\mathbf{x}, t))$ of (12) satisfies the following integral equation

$$\begin{aligned} \overline{u}_i(\mathbf{x}, t) \;=\; & e^{-M_i t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{1}{\left(\sqrt{4\pi D_i t}\right)^3}e^{-\frac{(x_2-y)^2+(x_3-z)^2}{4D_i t}} \\ & \times\left(e^{-\frac{(x_1-x)^2}{4D_i t}} + e^{-\frac{(x_1+x)^2}{4D_i t}}\right)v_i^0(x, y, z)\,dxdydz \\ & + \int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{e^{-M_i(t-s)}}{\left(\sqrt{4\pi D_i(t-s)}\right)^3}e^{-\frac{(x_2-y)^2+(x_3-z)^2}{4D_i(t-s)}} \\ & \times\left(e^{-\frac{(x_1-x)^2}{4D_i(t-s)}} + e^{-\frac{(x_1+x)^2}{4D_i(t-s)}}\right)H^i(\mathbf{w}(x, y, z, s))\,dxdydzds, \end{aligned}$$

where $i = 1, 2$. By the uniqueness of solutions of (12), we have $\overline{\mathbf{u}}(\mathbf{x}, t) \equiv \mathbf{u}(\mathbf{x}, t)$ on $\overline{\Omega} \times [0, \infty)$. Following Lemma 2.4, we know that the solution $\mathbf{u}(\mathbf{x}, t)$ of (11) is nondecreasing on $x_1 \in (0, \infty)$. Define an operator $\mathcal{A}$ by $\mathbf{u}(t) = \mathcal{A}\mathbf{w}(t)$. Then we know that $\mathcal{A}$ maps $\mathcal{S}_\infty$ into $\mathcal{S}_\infty$.

Step 2. Fix $T_0 > 0$. Denote

$$\mathcal{S}_{T_0} = \left\{\mathbf{w}(t) \in C([0, T_0], X) \;\middle|\; \begin{array}{c} \mathbf{w}(\mathbf{x}, 0) = \mathbf{v}^0(\mathbf{x}),\ \forall\, \mathbf{x} \in \mathbb{R}^3; \\ \mathbf{E}^0 \le \mathbf{w}(\mathbf{x}, t) \le \mathbf{E}^1,\ \forall\, \mathbf{x} \in \mathbb{R}^3,\ t \in [0, T_0]; \\ \mathbf{w}(\mathbf{x}, t) \text{ is nondecreasing on } x_1 \in [0, \infty); \\ \mathbf{w}(\mathbf{x}, t) \text{ is even in } x_1 \in \mathbb{R}. \end{array}\right\}.$$

It is obvious that $\mathcal{A}$ maps $\mathcal{S}_{T_0}$ into $\mathcal{S}_{T_0}$. Let $L > 0$ such that

$$|\mathbf{H}(\phi) - \mathbf{H}(\psi)| \le L|\phi - \psi|$$

for any $\phi, \psi \in \mathbb{R}^2$. Take $T_0 < \frac{1}{2L}$. By virtue of

$$\mathbf{u}(t) = e^{-\mathbf{M}t}\mathbf{T}(t)\mathbf{v}^0 + \int_0^t e^{-\mathbf{M}(t-s)}\mathbf{T}(t-s)\mathbf{H}(\mathbf{w}(s))\,ds, \quad 0 < t \le T_0,$$

we can show that $\mathcal{A}$ mapping $\mathcal{S}_{T_0}$ into $\mathcal{S}_{T_0}$ is a contraction map. Thus, the contraction mapping principle implies that there exists a unique $\mathbf{u}(t) \in \mathcal{S}_{T_0}$ such that $\mathbf{u}(\cdot) = \mathcal{A}\mathbf{u}(\cdot)$, namely,

$$\begin{cases} \frac{\partial}{\partial t}u_1(\mathbf{x},t) = \Delta u_1(\mathbf{x},t) + u_1(\mathbf{x},t)\left[1 - k_1 - u_1(\mathbf{x},t) + k_1 u_2(\mathbf{x},t)\right], \\ \frac{\partial}{\partial t}u_2(\mathbf{x},t) = d\Delta u_2(\mathbf{x},t) + r\left(1 - u_2(\mathbf{x},t)\right)\left[k_2 u_1(\mathbf{x},t) - u_2(\mathbf{x},t)\right], \\ u_1(\mathbf{x},0) = v_1^0(\mathbf{x}),\ u_2(\mathbf{x},0) = v_2^0(\mathbf{x}), \end{cases} \tag{13}$$

on $(\mathbf{x},t) \in \mathbb{R}^3 \times (0,T_0]$. Consequently, repeating the above procedure on $t \in [T_0, 2T_0]$, $\cdots$, $[mT_0,(m+1)T_0], \cdots$, we know that there exists $\mathbf{u}(t) \in \mathcal{S}_\infty$ such that $\mathbf{u}(\cdot) = \mathcal{A}\mathbf{u}(\cdot)$, namely, there exists a unique

$$\mathbf{u}(\mathbf{x},t) \in C\left(\mathbb{R}^3 \times [0,\infty), \left[\mathbf{E}^0,\mathbf{E}^1\right]\right) \bigcap C^{2,1}\left(\mathbb{R}^3 \times (0,\infty), \left[\mathbf{E}^0,\mathbf{E}^1\right]\right)$$

satisfying (13).

Step 3. Let $\mathbf{v}(\mathbf{x},t) := \mathbf{u}(x_1,x_2,x_3 - st,t)$ for any $\mathbf{x} \in \mathbb{R}^3$ and $t > 0$, where $\mathbf{u}(\mathbf{x},t)$ is the solution of (13). Then it is easy to see that $\mathbf{v}(\mathbf{x},t)$ is the solution of (6). Since $\mathbf{u} \in \mathcal{S}_\infty$, we have that $\mathbf{v}(\mathbf{x},t)$ is symmetric on x_1 and is nondecreasing in $x_1 \in [0,+\infty)$. This completes the proof. □

Corollary 1. *Suppose that* $\mathbf{v}^-(\mathbf{x})$ *is even in* $x_1 \in \mathbb{R}$ *and* $x_2 \in \mathbb{R}$*, respectively. Then the pyramidal traveling front* $\mathbf{V}(\mathbf{x})$ *defined by Theorem 2.1 satisfies*

$$\begin{aligned} \mathbf{V}(x_1,x_2,x_3) = \mathbf{V}(-x_1,x_2,x_3), &\qquad \mathbf{V}(x_1,x_2,x_3) = \mathbf{V}(x_1,-x_2,x_3),\ \forall\ \mathbf{x} \in \mathbb{R}^3, \\ \frac{\partial}{\partial x_1}\mathbf{V}(\mathbf{x}) \gg \mathbf{0}, &\qquad \forall\ \mathbf{x} \in (0,+\infty) \times \mathbb{R}^2, \\ \frac{\partial}{\partial x_2}\mathbf{V}(\mathbf{x}) \gg \mathbf{0}, &\qquad \forall\ \mathbf{x} \in \mathbb{R} \times (0,+\infty) \times \mathbb{R}. \end{aligned}$$

Proof. Since $\mathbf{v}^-(\mathbf{x})$ is even in x_1 and x_2, respectively, it is easy to see that $\mathbf{v}^-(\mathbf{x})$ is nondecreasing in $x_1 > 0$ and $x_2 > 0$, respectively. It follows from Lemma 2.5 that $\mathbf{v}(\mathbf{x},t;\mathbf{v}^-)$ is even in x_1 and x_2 and is nondecreasing in $x_1 > 0$ and $x_2 > 0$, respectively. Thus, $\mathbf{V}(\mathbf{x})$ is even in x_1 and x_2 and is nondecreasing in $x_1 > 0$ and $x_2 > 0$, respectively. By (7), we obtain

$$\begin{cases} \Delta\varphi_1(\mathbf{x}) - s\frac{\partial}{\partial x_3}\varphi_1(\mathbf{x}) - \left[k_1 + 2V_1(\mathbf{x})\right]\varphi_1(\mathbf{x}) \\ \qquad = -\left[1 + k_1 V_2(\mathbf{x})\right]\varphi_1(\mathbf{x}) - k_1 V_1(\mathbf{x})\varphi_2(\mathbf{x}), \\ d\Delta\varphi_2(\mathbf{x}) - s\frac{\partial}{\partial x_3}\varphi_2(\mathbf{x}) - 2V_2(\mathbf{x})\varphi_2(\mathbf{x}) \\ \qquad = -\left[1 + k_2 V_1(\mathbf{x})\right]\varphi_2(\mathbf{x}) - rk_2\left[1 - V_2(\mathbf{x})\right]\varphi_1(\mathbf{x}), \\ \varphi_1(0,x_2,x_3) = \varphi_2(0,x_2,x_3) = 0 \end{cases}$$

for $\mathbf{x} = (x_1,x_2,x_3) \in (0,+\infty) \times \mathbb{R}^2$, where

$$\varphi_1(\mathbf{x}) = \frac{\partial}{\partial x_1}V_1(\mathbf{x}) \geq 0 \text{ and } \varphi_2(\mathbf{x}) = \frac{\partial}{\partial x_1}V_2(\mathbf{x}) \geq 0,\ \forall\ \mathbf{x} \in (0,+\infty) \times \mathbb{R}^2.$$

Therefore, we have

$$\begin{cases} \Delta\varphi_1(\mathbf{x}) - s\frac{\partial}{\partial x_3}\varphi_1(\mathbf{x}) - \left[k_1 + 2V_1(\mathbf{x})\right]\varphi_1(\mathbf{x}) \leq 0, & \forall\ \mathbf{x} \in (0,+\infty) \times \mathbb{R}^2, \\ d\Delta\varphi_2(\mathbf{x}) - s\frac{\partial}{\partial x_3}\varphi_2(\mathbf{x}) - 2V_2(\mathbf{x})\varphi_2(\mathbf{x}) \leq 0, & \forall\ \mathbf{x} \in (0,+\infty) \times \mathbb{R}^2, \\ \varphi_1(0,x_2,x_3) = \varphi_2(0,x_2,x_3) = 0, & \forall\ (x_2,x_3) \in \mathbb{R}^2. \end{cases}$$

By Theorem 2.1 we have $\varphi_1(\mathbf{x}) = \frac{\partial}{\partial x_1}V_1(\mathbf{x}) \not\equiv 0$ and $\varphi_2(\mathbf{x}) = \frac{\partial}{\partial x_1}V_2(\mathbf{x}) \not\equiv 0$ in $\mathbf{x} \in (0,+\infty) \times \mathbb{R}^2$. Applying the maximum principle (Potter and Weinberger [43]) for the scalar equation yields

$$\frac{\partial}{\partial x_1}V_1(\mathbf{x})=\varphi_1(\mathbf{x})>0,\ \frac{\partial}{\partial x_1}V_2(\mathbf{x})=\varphi_2(\mathbf{x})>0,\ \forall\ \mathbf{x}\in(0,+\infty)\times\mathbb{R}^2.$$

This completes the proof. □

3. **Axisymmetric traveling fronts.** In this section we establish the existence of axisymmetric traveling fronts of (5) in $\mathbb{R}^3$. The method is to take the limit of a sequence of pyramidal traveling fronts. Consequently, we show some important qualitative properties of the axisymmetric traveling fronts.

Let

$$h^k\left(x_1,x_2\right)=m_*\max_{1\leq i\leq 2^k}\left\{x_1\cos\frac{2\left(i-1\right)\pi}{2^k}+x_2\sin\frac{2\left(i-1\right)\pi}{2^k}\right\},\qquad k=1,2,\cdots.$$

It is not difficult to show that the plane

$$x_3=m_*\left(x_1\cos\frac{2\left(i-1\right)\pi}{2^k}+x_2\sin\frac{2\left(i-1\right)\pi}{2^k}\right)$$

is tangent to the rotating surface

$$x_3=m_*\sqrt{x_1^2+x_2^2}$$

for any $k\in\mathbb{N}$ and $1\leq i\leq 2^k$. Replacing $h(\mathbf{x}')$ by $h^k\left(\mathbf{x}'\right)$ in Theorem 2.1, we obtain a sequence of pyramidal traveling fronts of (5), namely,

$$\mathbf{V}^1,\mathbf{V}^2,\cdots,\mathbf{V}^k,\cdots,$$

where

$$\mathbf{V}^k\left(\mathbf{x}\right)=\lim_{t\to\infty}\mathbf{v}\left(\mathbf{x},t;\ \mathbf{v}^{k,-}\right),\quad \mathbf{v}^{k,-}\left(\mathbf{x}\right)=\mathbf{U}\left(\frac{c}{s}\left(x_3+h^k\left(\mathbf{x}'\right)\right)\right).$$

Denote the edge of the pyramid $x_3=h^k\left(\mathbf{x}'\right)$ by Γ^k and

$$D^k\left(\gamma\right)=\left\{\mathbf{x}\in\mathbb{R}^3\middle|\ \mathrm{dist}\left(\mathbf{x},\cup_{j=1}^{2^k}\Gamma_j^k\right)>\gamma\right\}\quad\text{ for }\gamma>0.$$

Since $\mathbf{v}^{k,-}\left(\mathbf{x}\right)$ is nondecreasing in $x_1\in(0,\infty)$ and $x_2\in(0,\infty)$ and is even in $x_1\in\mathbb{R}$ and $x_2\in\mathbb{R}$, respectively, by Theorem 2.1, Lemma 2.3 and Corollary 1 we obtain

$$\begin{aligned}
&\mathbf{V}^1\leq\mathbf{V}^2\leq\cdots\leq\mathbf{V}^k\leq\cdots,\quad \forall\ \mathbf{x}\in\mathbb{R}^3,\\
&\frac{\partial}{\partial x_1}\mathbf{V}^k\left(\mathbf{x}\right)\gg\mathbf{0},\quad \forall\ \mathbf{x}\in(0,\infty)\times\mathbb{R}^2,\\
&\frac{\partial}{\partial x_2}\mathbf{V}^k\left(\mathbf{x}\right)\gg\mathbf{0},\quad \forall\ \mathbf{x}\in\mathbb{R}\times(0,\infty)\times\mathbb{R},\\
&\frac{\partial}{\partial \nu}\mathbf{V}^k\left(\mathbf{x}\right)\gg\mathbf{0},\quad \forall\ \mathbf{x}\in\mathbb{R}^3,
\end{aligned}$$

where $\nu=\frac{1}{\sqrt{1+\nu_1^2+\nu_2^2}}(\nu_1,\nu_2,1)$ satisfies $\sqrt{\nu_1^2+\nu_2^2}\leq\frac{1}{m_*}$. Since

$$h^k\left(x_1,x_2\right)=h^k\left(x_1\cos\frac{\pi}{2^{k-1}}+x_2\sin\frac{\pi}{2^{k-1}},-x_1\sin\frac{\pi}{2^{k-1}}+x_2\cos\frac{\pi}{2^{k-1}}\right),$$

we have

$$\mathbf{V}^k\left(\mathbf{x}\right)=\mathbf{V}^k\left(\mathbf{x}',x_3\right)=\mathbf{V}^k\left(\mathbf{B}_k\mathbf{x}',x_3\right),\ \forall\mathbf{x}\in\mathbb{R}^3,$$

where

$$\mathbf{B}_k=\begin{pmatrix}\cos\frac{\pi}{2^{k-1}} & \sin\frac{\pi}{2^{k-1}}\\ -\sin\frac{\pi}{2^{k-1}} & \cos\frac{\pi}{2^{k-1}}\end{pmatrix}.$$

Take $x_3^k \in \mathbb{R}$ such that $x_3^k \geq x_3^{k+1}$ and $V_2^k\left(0,0,x_3^k\right) = \theta_2$, where $\theta_2 \in (0, E_2^*)$ satisfies the following assumption

(**H**) Assume that $\theta_2 = E_2^* - \Lambda_2$, where Λ_2 satisfies

$$0 < \Lambda_2 < \min\left\{E_2^*, (k_1k_2-1)(1-E_2^*), \frac{\sqrt{(1-E_2^*)^2 + 4(1-E_2^*)E_2^*} - (1-E_2^*)}{2}\right\}$$

and

$$\Lambda_2 \neq \left(\frac{k_1k_2}{2} - 1\right)(1-E_2^*).$$

Let

$$\widetilde{\mathbf{V}}^k(\mathbf{x}) = \mathbf{V}^k\left(\mathbf{x}', x_3 + x_3^k\right), \quad \forall \mathbf{x} \in \mathbb{R}^3.$$

By Lemmas 2.2 and 2.3 and Corollary 1 we have that $\widetilde{\mathbf{V}}^k(\mathbf{x})$ satisfies:

(a) $\widetilde{V}_2^k(\mathbf{0}) = \theta_2$;
(b) $\frac{\partial}{\partial \nu}\widetilde{V}_i^k(\mathbf{x}) > 0$ for any $\mathbf{x} \in \mathbb{R}^3$, where $i = 1,2$; $k \in \mathbb{N}$; $\nu = \frac{1}{\sqrt{1+\nu_1^2+\nu_2^2}}(\nu_1, \nu_2, 1)$ satisfies $\sqrt{\nu_1^2+\nu_2^2} \leq \frac{1}{m_*}$;
(c) For any $(x_0, y_0, z_0) \in \mathbb{R}^3$ with $z_0 \geq h^k(x_0, y_0)$, there holds

$$\widetilde{\mathbf{V}}^k(x_1+x_0, x_2+y_0, x_3) \leq \widetilde{\mathbf{V}}^k(x_1, x_2, x_3+z_0), \quad \forall\ (x_1,x_2,x_3) \in \mathbb{R}^3;$$

(d) $\widetilde{\mathbf{V}}^k(\mathbf{x}', x_3) = \widetilde{\mathbf{V}}^k(\mathbf{B}_k\mathbf{x}', x_3)$, $\forall \mathbf{x} \in \mathbb{R}^3$;
(e) There are $\frac{\partial}{\partial x_1}\widetilde{V}_i^k(\mathbf{x}) > 0$ in $x_1 \in (0,\infty)$ and $\frac{\partial}{\partial x_2}\widetilde{V}_i^k(\mathbf{x}) > 0$ in $x_2 \in (0,\infty)$, where $k \in \mathbb{N}$, $i = 1,2$;
(f) There exists a constant $K > 0$ such that $\left\|\widetilde{\mathbf{V}}^k(\mathbf{x})\right\|_{C^3(\mathbb{R}^3)} \leq K$, $\forall\ k \in \mathbb{N}$.

Now we define a function $\mathbf{W}(\mathbf{x}) = (W_1(\mathbf{x}), W_2(\mathbf{x})) \in C^2\left(\mathbb{R}^3, \left[\mathbf{E}^0, \mathbf{E}^1\right]\right)$ by (up to an extraction of some subsequence)

$$\widetilde{\mathbf{V}}^k(\mathbf{x}) \to \mathbf{W}(\mathbf{x}) \text{ in } \|\cdot\|_{C^2_{loc}(\mathbb{R}^3)} \text{ as } k \to \infty.$$

Then we have the following theorem for the function $\mathbf{W}(\mathbf{x}) \in C^2\left(\mathbb{R}^3, \left[\mathbf{E}^0, \mathbf{E}^1\right]\right)$.

Theorem 3.1. *Assume that $k_1, k_2 > 1$ and $c > 0$. There exists a function $\mathbf{W}(\mathbf{x}) = (W_1(\mathbf{x}), W_2(\mathbf{x})) \in C^2\left(\mathbb{R}^3, \left[\mathbf{E}^0, \mathbf{E}^1\right]\right)$ satisfying*

$$\begin{cases} \Delta W_1(\mathbf{x}) - s\frac{\partial}{\partial x_3}W_1(\mathbf{x}) + W_1(\mathbf{x})\left[1 - k_1 - W_1(\mathbf{x}) + k_1W_2(\mathbf{x})\right] = 0, \\ d\Delta W_2(\mathbf{x}) - s\frac{\partial}{\partial x_3}W_2(\mathbf{x}) + r\left(1 - W_2(\mathbf{x})\right)\left[k_2W_1(\mathbf{x}) - W_2(\mathbf{x})\right] = 0, \end{cases} \mathbf{x} \in \mathbb{R}^3. \tag{14}$$

In addition, one has

(i) $W_2(\mathbf{0}) = \theta_2$;
(ii) $\mathbf{W}(\mathbf{x}_1', x_3) = \mathbf{W}(\mathbf{x}_2', x_3)$, $\forall\ \mathbf{x}_1', \mathbf{x}_2' \in \mathbb{R}^2$ *with* $|\mathbf{x}_1'| = |\mathbf{x}_2'|$, $x_3 \in \mathbb{R}$;
(iii) for any $(x_0, y_0, z_0) \in \mathbb{R}^3$ *with* $z_0 \geq m_*\sqrt{x_0^2 + y_0^2}$, *one has*

$$\mathbf{W}(x_1+x_0, x_2+y_0, x_3) \leq \mathbf{W}(x_1, x_2, x_3+z_0), \quad \forall\ (x_1,x_2,x_3) \in \mathbb{R}^3;$$

(iv) $\frac{\partial}{\partial x_3}\mathbf{W}(\mathbf{x}) \gg 0$ *for any* $\mathbf{x} \in \mathbb{R}^3$, $i = 1,2$;
(v) $\frac{\partial}{\partial x_1}\mathbf{W}(\mathbf{x}) \gg 0$ *for* $x_1 \in (0,\infty)$, $\frac{\partial}{\partial x_2}\mathbf{W}(\mathbf{x}) \gg 0$ *for* $x_2 \in (0,\infty)$;
(vi) $\lim\limits_{x_3 \to +\infty} \|\mathbf{W}(\cdot, x_3) - \mathbf{E}^1\|_{C(\mathbb{R}^2)} = 0$ *and* $\lim_{x_3 \to -\infty} \|\mathbf{W}(\cdot, x_3) - \mathbf{E}^0\|_{C_{loc}(\mathbb{R}^2)} = 0$.

(vii) $\frac{\partial}{\partial \nu}\mathbf{W}(\mathbf{x}) \gg 0$ *for any* $\mathbf{x} \in \mathbb{R}^3$, *where*

$$\nu = \frac{1}{\sqrt{1+\nu_1^2+\nu_2^2}}(\nu_1, \nu_2, 1) \quad \text{with} \quad \sqrt{\nu_1^2+\nu_2^2} \le \frac{1}{m_*}.$$

It can be shown that $\mathbf{W}(\mathbf{x})$ satisfies (14) and (i) and (ii) of Theorem 3.1. In view of $h^k(x_1,x_2) \le m_*\sqrt{x_1^2+x_2^2}$ for any $(x_1,x_2) \in \mathbb{R}^2$ and $h^k(x_1,x_2) \to m_*\sqrt{x_1^2+x_2^2}$ in $C_{loc}(\mathbb{R}^2)$ as $k \to +\infty$, we can prove (iii) of Theorem 3.1. In the following we prove (iv)-(vii) of Theorem 3.1 by a sequence of lemmas. Following the properties (a)-(f) of $\widetilde{\mathbf{V}}(\mathbf{x})^k$, we have: (I) $\frac{\partial}{\partial x_3}W_i(\mathbf{x}) \ge 0$ for $\mathbf{x} \in \mathbb{R}^3$, $i = 1, 2$; (II) $\frac{\partial}{\partial x_1}W_i(0, x_2, x_3) = 0$ for $(x_2, x_3) \in \mathbb{R}^2$ and $\frac{\partial}{\partial x_2}W_i(x_1, 0, x_3) = 0$ for $(x_1, x_3) \in \mathbb{R}^2$, $i = 1, 2$; (III) $\frac{\partial}{\partial x_1}W_i(\mathbf{x}) \ge 0$ for $\mathbf{x} \in (0,+\infty) \times \mathbb{R}^2$ and $\frac{\partial}{\partial x_2}W_i(\mathbf{x}) \ge 0$ for $\mathbf{x} \in \mathbb{R} \times (0, +\infty) \times \mathbb{R}$, $i = 1, 2$.

Lemma 3.2. $W_1(\mathbf{x}) \not\equiv \theta_1$ *and* $W_2(\mathbf{x}) \not\equiv \theta_2$ *for* $\mathbf{x} \in \mathbb{R}^3$, *where* $\theta_i = W_i(\mathbf{0})$.

Proof. Notice that $0 < W_2(\mathbf{0}) = \theta_2 < E_2^* = \frac{(k_1-1)k_2}{k_1k_2-1}$. If $W_2(\mathbf{x}) \equiv \theta_2$ for any $\mathbf{x} \in \mathbb{R}^3$, we have $k_2W_1(\mathbf{x}) - W_2(\mathbf{x}) \equiv 0$ for any $\mathbf{x} \in \mathbb{R}^3$. Thus, $0 < W_1(\mathbf{x}) \equiv \frac{\theta_2}{k_2} < E_1^* = \frac{k_1-1}{k_1k_2-1}$. It follows that system (5) admits an equilibrium $\left(\frac{\theta_2}{k_2}, \theta_2\right)$ with $\mathbf{E}^0 \ll \left(\frac{\theta_2}{k_2}, \theta_2\right) \ll \mathbf{E}^*$, which is impossible. Therefore, $W_2(\mathbf{x}) \not\equiv \theta_2$ for $\mathbf{x} \in \mathbb{R}^3$.

Now we show that $W_1(\mathbf{x}) \not\equiv \theta_1$ for $\mathbf{x} \in \mathbb{R}^3$. On the contrary we assume that $W_1(\mathbf{x}) \equiv \theta_1$ for $\mathbf{x} \in \mathbb{R}^3$. In view of $0 \le \theta_1 \le 1$, we consider two cases: (a) $\theta_1 = 0$ and (b) $0 < \theta_1 \le 1$.

(a) If $\theta_1 = 0$, it follows from the second equation of (14) that

$$d\Delta W_2(\mathbf{x}) - s\frac{\partial}{\partial x_3}W_2(\mathbf{x}) - rW_2(\mathbf{x})\left(1 - W_2(\mathbf{x})\right) = 0, \quad \forall\, \mathbf{x} \in \mathbb{R}^3.$$

Let $\widetilde{W}(\mathbf{x}) = 1 - W_2(\mathbf{x})$ for any $\mathbf{x} \in \mathbb{R}^3$. It follows that $\frac{\partial}{\partial x_1}\widetilde{W}(\mathbf{x}) \le 0$ for any $\mathbf{x} \in (0,+\infty) \times \mathbb{R}^2$, $\frac{\partial}{\partial x_2}\widetilde{W}(\mathbf{x}) \le 0$ for any $\mathbf{x} \in \mathbb{R} \times (0,+\infty) \times \mathbb{R}$ and $\frac{\partial}{\partial x_3}\widetilde{W}(\mathbf{x}) \le 0$ for any $\mathbf{x} \in \mathbb{R}^3$. In addition, we have

$$d\Delta\widetilde{W}(\mathbf{x}) - s\frac{\partial}{\partial x_3}\widetilde{W}(\mathbf{x}) + r\widetilde{W}(\mathbf{x})\left(1 - \widetilde{W}(\mathbf{x})\right) = 0, \quad \forall\, \mathbf{x} \in \mathbb{R}^3.$$

Furthermore, let $\widetilde{\widetilde{W}}(\mathbf{x}) = \widetilde{W}(x_1, x_2, -x_3)$ for any $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$. Then we have $\frac{\partial}{\partial x_3}\widetilde{\widetilde{W}}(\mathbf{x}) \ge 0$ for any $\mathbf{x} \in \mathbb{R}^3$ and

$$d\Delta\widetilde{\widetilde{W}}(\mathbf{x}) + s\frac{\partial}{\partial x_3}\widetilde{\widetilde{W}}(\mathbf{x}) + r\widetilde{\widetilde{W}}(\mathbf{x})\left(1 - \widetilde{\widetilde{W}}(\mathbf{x})\right) = 0, \quad \forall\, \mathbf{x} \in \mathbb{R}^3,$$

which implies that $\widetilde{\widetilde{W}}(x_1, x_2, x_3 - st)$ is a solution of the following Fisher-KPP equation

$$\frac{\partial}{\partial t}u(\mathbf{x},t) = d\Delta u(\mathbf{x},t) + ru(\mathbf{x},t)\left(1 - u(\mathbf{x},t)\right), \quad \mathbf{x} \in \mathbb{R}^3,\ t > 0. \tag{15}$$

Let

$$\begin{aligned} \overline{u}(\mathbf{x},t) &= \min\left\{\widetilde{\widetilde{W}}(x_1,x_2,x_3-st), \widetilde{\widetilde{W}}(x_1,x_2,-x_3-st)\right\} \\ &= \widetilde{\widetilde{W}}(x_1,x_2,-|x_3|-st), \quad \forall\, \mathbf{x} = (x_1,x_2,x_3) \in \mathbb{R}^3. \end{aligned}$$

Then $\overline{u}(\mathbf{x},t)$ is also a supersolution of (15). In particular, $\overline{u}(\mathbf{x},t) \leq \overline{u}(\mathbf{x},0) \leq \widetilde{\widetilde{W}}(\mathbf{0}) = 1-\theta_2 < 1$ for any $\mathbf{x} \in \mathbb{R}^3$. By the classical results of Aronson and Weinberger [2, Corollary 1] on the asymptotic speed of propagation for the Fisher-KPP equation, we know that the solution $u(\mathbf{x},t;\varphi)$ of (15) with initial value φ satisfies

$$\lim_{t\to+\infty} \inf_{|\mathbf{x}|\leq \tau t} u(\mathbf{x},t;\varphi) = 1 \quad \text{for any } \tau \in (0,\tau^*) \tag{16}$$

if $\varphi(\cdot) \in C\left(\mathbb{R}^3,[0,1]\right)$ satisfies that $\varphi(\cdot) \not\equiv 0$ and supp φ is compact, where $\tau^* = 2\sqrt{dr}$. However, the comparison principle implies that

$$u(\mathbf{x},t;\varphi) \leq \overline{u}(\mathbf{x},t) \leq \overline{u}(\mathbf{x},0) \leq \widetilde{\widetilde{W}}(\mathbf{0}) < 1 \quad \text{for any } \mathbf{x}\in\mathbb{R}^3,\ t>0, \tag{17}$$

if φ further satisfies $\varphi(\mathbf{x}) < \overline{u}(\mathbf{x},0)$ for any $\mathbf{x}\in\mathbb{R}^3$. It is obvious that (17) contradicts the fact (16). Therefore, $\theta_1 \neq 0$.

(b) Assume that $W_1(\mathbf{x}) \equiv \theta_1$ and $0<\theta_1\leq 1$. Then by the first equation of (14) we have

$$1-k_1-\theta_1+k_1W_2(\mathbf{x}) \equiv 0.$$

Then we have $W_2(\mathbf{x}) \equiv \frac{k_1+\theta_1-1}{k_1}$ in $\mathbb{R}^3$, which contradicts the fact that $W_2(0)=\theta_2$ and $W_2(\cdot)\not\equiv\theta_2$.

Following the above arguments, we conclude that $W_1(\mathbf{x}) \not\equiv \theta_1$ for $\mathbf{x}\in\mathbb{R}^3$. This completes the proof. □

In the following we prove (v) of Theorem 3.1. To reach the aim, we first present a lemma, which will be proved in Section 5. Consider the following reaction-diffusion system in $\mathbb{R}$

$$\begin{cases} \frac{\partial}{\partial t}u_1 = \frac{\partial^2}{\partial x^2}u_1 + (E_1^*-u_1)(k_1u_2-u_1), \\ \frac{\partial}{\partial t}u_2 = d\frac{\partial^2}{\partial x^2}u_2 + r(1-E_2^*+u_2)(k_2u_1-u_2), \end{cases} \quad x\in\mathbb{R},\ t>0. \tag{18}$$

Lemma 3.3. *There exists $\kappa^*>0$ such that for any $\kappa\geq\kappa^*$ system (18) admits an increasing traveling wave front $(\rho_1(x+\kappa t),\rho_2(x+\kappa t))$ connecting the equilibria $\mathbf{E}^0$ and $\mathbf{E}^*$, and for any $\kappa<\kappa^*$ system (18) does not admit an increasing traveling wave front $(\rho_1(x+\kappa t),\rho_2(x+\kappa t))$ connecting the equilibria $\mathbf{E}^0$ and $\mathbf{E}^*$.*

Now we prove (v) of Theorem 3.1, namely, we have the following lemma.

Lemma 3.4. *$\frac{\partial}{\partial x_1}W_i(\mathbf{x})>0$ for any $\mathbf{x}\in(0,+\infty)\times\mathbb{R}^2$ and $\frac{\partial}{\partial x_2}W_i(\mathbf{x})>0$ for any $\mathbf{x}\in\mathbb{R}\times(0,+\infty)\times\mathbb{R}$, $i=1,2$.*

Proof. We first show that it is impossible that $\frac{\partial}{\partial x_1}W_1(\mathbf{x})\not\equiv 0$ and $\frac{\partial}{\partial x_1}W_2(\mathbf{x})\equiv 0$ for $\mathbf{x}\in(0,+\infty)\times\mathbb{R}^2$. We prove it by contradiction. Assume on the contrary that $\frac{\partial}{\partial x_1}W_1(\mathbf{x})\not\equiv 0$ and $\frac{\partial}{\partial x_1}W_2(\mathbf{x})\equiv 0$ for $\mathbf{x}\in(0,+\infty)\times\mathbb{R}^2$. Since $\frac{\partial}{\partial x_1}W_1(0,x_2,x_3)\equiv 0$ for $(x_2,x_3)\in\mathbb{R}^2$ and $\frac{\partial}{\partial x_1}W_i(\mathbf{x})\geq 0$ for $\mathbf{x}\in(0,+\infty)\times\mathbb{R}^2$, by a similar argument to that in Corollary 1 we have $\frac{\partial}{\partial x_1}W_1(\mathbf{x})>0$ for any $\mathbf{x}\in(0,+\infty)\times\mathbb{R}^2$. Note that $\frac{\partial}{\partial x_1}W_2(\mathbf{x})\equiv 0$ for $\mathbf{x}\in(0,+\infty)\times\mathbb{R}^2$. Differentiating with respect to x_1 the second equation of (14), we obtain

$$rk_2(1-W_2(\mathbf{x}))\frac{\partial}{\partial x_1}W_1(\mathbf{x})\equiv 0, \quad \forall\ \mathbf{x}\in\mathbb{R}^3,$$

which yields that $W_2(\mathbf{x})\equiv 1$ for any $\mathbf{x}\in\mathbb{R}^3$. This contradicts (i) of Theorem 3.1. Similarly, we can show that it is also impossible that $\frac{\partial}{\partial x_1}W_1(\mathbf{x})\equiv 0$ and $\frac{\partial}{\partial x_1}W_2(\mathbf{x})\not\equiv 0$ for $\mathbf{x}\in(0,+\infty)\times\mathbb{R}^2$.

By (ii) of Theorem 3.1, we have $\frac{\partial}{\partial x_1}W_i(x,y,x_3)=\frac{\partial}{\partial x_2}W_i(y,x,x_3)$ for any $(x,y)\in\mathbb{R}^2$ and $x_3\in\mathbb{R}$, where $i=1,2$. Following the above arguments, we have that either $\frac{\partial}{\partial x_1}W_i(\mathbf{x})>0$ for any $\mathbf{x}\in(0,+\infty)\times\mathbb{R}^2$ and $\frac{\partial}{\partial x_2}W_i(\mathbf{x})>0$ for any $\mathbf{x}\in\mathbb{R}\times(0,+\infty)\times\mathbb{R}$, or $\frac{\partial}{\partial x_1}W_i(\mathbf{x})\equiv 0$ for $\mathbf{x}\in(0,+\infty)\times\mathbb{R}^2$ and $\frac{\partial}{\partial x_2}W_i(\mathbf{x})\equiv 0$ for $\mathbf{x}\in\mathbb{R}\times(0,+\infty)\times\mathbb{R}$, where $i=1,2$.

We then prove that it is not true that $\frac{\partial}{\partial x_1}W_i(\mathbf{x})\equiv 0$ for $\mathbf{x}\in(0,+\infty)\times\mathbb{R}^2$ and $\frac{\partial}{\partial x_2}W_i(\mathbf{x})\equiv 0$ for $\mathbf{x}\in\mathbb{R}\times(0,+\infty)\times\mathbb{R}$, $i=1,2$. For a contrary, we assume that

$$\frac{\partial}{\partial x_1}W_1(\mathbf{x})\equiv\frac{\partial}{\partial x_2}W_1(\mathbf{x})\equiv\frac{\partial}{\partial x_1}W_2(\mathbf{x})\equiv\frac{\partial}{\partial x_2}W_2(\mathbf{x})\equiv 0,\quad \mathbf{x}\in\mathbb{R}^3,$$

which implies that $W_i(\mathbf{x})$ only depend on $x_3\in\mathbb{R}$. We rewrite $W_i(\mathbf{x})$ as $W_i(x_3)$ and denote $W_i(x_3)$ by $W_i(z)$ with $z=x_3$, $i=1,2$. By Lemma 3.2 we have $W_i(z)\not\equiv\theta_i$ and $W_i(0)=\theta_i$, which implies that $\frac{d}{dz}W_i(z)\not\equiv 0$ on $z\in\mathbb{R}$. It follows from the maximum principle that $\frac{d}{dz}W_i(z)>0$ for any $z\in\mathbb{R}$, $i=1,2$. Let

$$\begin{aligned}&W_1(-\infty)=\alpha^-,\quad W_1(+\infty)=\alpha^+,\quad 0\le\alpha^-<\alpha^+\le 1,\\&W_2(-\infty)=\beta^-,\quad W_2(+\infty)=\beta^+,\quad 0\le\beta^-<\beta^+\le 1.\end{aligned}$$

In particular, we have $\beta^-<E_2^*$. In this case we rewrite the system (14) as

$$\begin{cases}W_1''-sW_1'+W_1(1-k_1-W_1+k_1W_2)=0,\\ dW_2''-sW_2'+r(1-W_2)(k_2W_1-W_2)=0.\end{cases}\tag{19}$$

Obviously, (α^-,β^-) and (α^+,β^+) are the roots of the following algebra equations

$$\begin{cases}u(1-k_1-u+k_1v)=0,\\(1-v)(k_2u-v)=0.\end{cases}$$

Therefore, we have either $(\alpha^-,\beta^-)=(0,0)$ and $(\alpha^+,\beta^+)=(1,1)$ or $(\alpha^-,\beta^-)=(0,0)$ and $(\alpha^+,\beta^+)=(E_1^*,E_2^*)$. We argue that it is impossible to have $(\alpha^+,\beta^+)=(1,1)$. Otherwise, system (5) admits a planar traveling front $(W_1(x+st),W_2(x+st))$ with wave speed $s>c$ connecting two stable equilibria $(0,0)$ and $(1,1)$, which contradicts the uniqueness of the planar traveling wave front $(\mathbf{U},c)$ of (5).

Assume that $(\alpha^-,\beta^-)=(0,0)$ and $(\alpha^+,\beta^+)=(E_1^*,E_2^*)$. Let $\psi_1(z)=E_1^*-W_1(z)$ and $\psi_2(z)=E_1^*-W_2(z)$ for any $z\in\mathbb{R}$. We have

$$\begin{cases}\psi_1''-s\psi_1'+(E_1^*-\psi_1)(k_1\psi_2-\psi_1)=0,\\ d\psi_2''-s\psi_2'+r(1-E_2^*+\psi_2)(k_2\psi_1-\psi_2)=0.\end{cases}$$

Let $\rho_i(z)=\psi_i(-z)$ for any $z\in\mathbb{R}$, $i=1,2$. Then $(\rho_1(z),\rho_2(z))$ satisfies

$$\begin{cases}\rho_1''+s\rho_1'+(E_1^*-\rho_1)(k_1\rho_2-\rho_1)=0,\\ d\rho_2''+s\rho_2'+r(1-E_2^*+\rho_2)(k_2\rho_1-\rho_2)=0\end{cases}\tag{20}$$

and

$$\rho_i(-\infty)=E_i^*-W_i(+\infty)=0,\ \rho_i(+\infty)=E_i^*-W_i(-\infty)=E_i^*,\ i=1,2.$$

System (20) implies that the system (18) admits an increasing traveling wave front $(\rho_1(x-st),\rho_2(x-st))$ connecting $\mathbf{E}^0$ and $\mathbf{E}^*$. Thus, $(\rho_1(x+s't),\rho_2(x+s't))$ is an increasing traveling wave front connecting $\mathbf{E}^0$ and $\mathbf{E}^*$ with wave speed $s'=-s<0$, but this is impossible due to the fact showed by Lemma 3.3. The proof is completed. □

Following Lemma 3.4, we have $\mathbf{W}(x_1, x_2, 0) \gg \mathbf{W}(0, 0, 0)$ for any $(x_1, x_2) \in \mathbb{R}^2$ with $x_1^2+x_2^2 > 0$. By (iii) of Theorem 3.1 we have that $\mathbf{W}(0, 0, x_3) \geq \mathbf{W}(x_1, x_2, 0) \gg \mathbf{W}(0, 0, 0)$ for any $x_3 > m_* \sqrt{x_1^2 + x_2^2} > 0$, which implies that $\frac{\partial}{\partial x_3}\mathbf{W}(\mathbf{x}) \geq \mathbf{0}$ and $\frac{\partial}{\partial x_3}\mathbf{W}(\mathbf{x}) \not\equiv \mathbf{0}$ on $\mathbf{x} \in \mathbb{R}^3$. Furthermore, the maximum principle yields that $\frac{\partial}{\partial x_3}\mathbf{W}(\mathbf{x}) \gg \mathbf{0}$ for $\mathbf{x} \in \mathbb{R}^3$. Thus, we have proved (iv) of Theorem 3.1.

In the following we prove (vi) of Theorem 3.1. Since $\frac{\partial}{\partial x_3} W_i(\mathbf{x}) > 0$ $(i = 1, 2)$ for $\mathbf{x} \in \mathbb{R}^3$, we define

$$(\alpha_1, \alpha_2) := \lim_{x_3 \to +\infty} \mathbf{W}(0, 0, x_3) \quad \text{and } (\beta_1, \beta_2) := \lim_{x_3 \to -\infty} \mathbf{W}(0, 0, x_3).$$

Lemma 3.5. *One has $(\beta_1, \beta_2) = \mathbf{E}^0$, and either $(\alpha_1, \alpha_2) = \mathbf{E}^*$ or $(\alpha_1, \alpha_2) = \mathbf{E}^1$. In particular, one has*

$$\lim_{x_3 \to -\infty} \left\| \mathbf{W}(\cdot, \cdot, x_3) - \mathbf{E}^0 \right\|_{C_{loc}(\mathbb{R}^2)} = 0$$

and

$$\lim_{x_3 \to +\infty} \|\mathbf{W}(\cdot, \cdot, x_3) - (\alpha_1, \alpha_2)\|_{C(\mathbb{R}^2)} = 0.$$

Proof. By (iii) of Theorem 3.1, for any $z_0 \geq m_* \sqrt{x_0^2 + y_0^2}$ we have

$$\mathbf{W}(0, 0, x_3) \leq \mathbf{W}(x_0, y_0, x_3) \leq \mathbf{W}(0, 0, x_3 + z_0) \quad \forall x_3 \in \mathbb{R}.$$

It follows that

$$\lim_{x_3 \to +\infty} \|\mathbf{W}(\cdot, \cdot, x_3) - (\alpha_1, \alpha_2)\|_{C(\mathbb{R}^2)} = 0$$

and

$$\lim_{n \to +\infty} \|\mathbf{W}(\cdot, \cdot, z_n + \cdot) - (\alpha_1, \alpha_2)\|_{C^2_{loc}(\mathbb{R}^3)} = 0$$

for some sequence $\{z_n\}$ satisfying $z_n \to +\infty$ as $n \to +\infty$. Thus, the vector $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ satisfies

$$f_1(\alpha_1, \alpha_2) = 0 \quad \text{and} \quad f_2(\alpha_1, \alpha_2) = 0.$$

In view of $\mathbf{E}^0 \ll (\alpha_1, \alpha_2) \leq \mathbf{E}^*$, we conclude that either $(\alpha_1, \alpha_2) = \mathbf{E}^*$ or $(\alpha_1, \alpha_2) = \mathbf{E}^1$. Similarly, we have $(\beta_1, \beta_2) = \mathbf{E}^0$,

$$\lim_{x_3 \to -\infty} \|\mathbf{W}(\cdot, \cdot, x_3) - (\beta_1, \beta_2)\|_{C_{loc}(\mathbb{R}^2)} = 0$$

and

$$\lim_{n \to +\infty} \left\|\mathbf{W}(\cdot, \cdot, -z_n + \cdot) - \mathbf{E}^0\right\|_{C^2_{loc}(\mathbb{R}^3)} = 0,$$

where $z_n \to +\infty$ as $n \to +\infty$. This completes the proof. □

In order to complete the proof of Theorem 3.1 (vi), we need further to show that

$$\lim_{x_3 \to +\infty} \|\mathbf{W}(\cdot, \cdot, x_3) - \mathbf{E}^1\|_{C(\mathbb{R}^2)} = 0. \tag{21}$$

To do this, it is sufficient to show that there must be $(\alpha_1, \alpha_2) = \mathbf{E}^1$. Our method is to assume $(\alpha_1, \alpha_2) = \mathbf{E}^*$ and derive a contradiction, which is similar to the argument in Lemma 3.4. To obtain the contradiction, we first consider the spreading speed for a cooperation reaction-diffusion system, namely, the below (23). Since we are working in *three-dimensional spatial space*, the results of Liang and Zhao [33] on the spreading speed of the monotone semiflow are not applicable, so we use the theory of Thieme and Zhao [49]. Before presenting the lemma, we introduce some notations, which come from Thieme and Zhao [49]. Let $\sigma > 0$ and $\mathbf{u}(\mathbf{x},t) : \mathbb{R}^3 \times [0,+\infty) \to \mathbb{R}^2$. Define

$$\liminf_{t\to\infty,|\mathbf{x}|\le\sigma t} \mathbf{u}(\mathbf{x},t) = \sup_{t\ge 0} \inf \{\mathbf{u}(\mathbf{x},s) : s \ge t, |\mathbf{x}| \le \sigma s\}$$

and

$$\limsup_{t\to\infty,|\mathbf{x}|\le\sigma t} \mathbf{u}(\mathbf{x},t) = \inf_{t\ge 0} \sup \{\mathbf{u}(\mathbf{x},s) : s \ge t, |\mathbf{x}| \le \sigma s\}.$$

We say that $\lim_{t\to\infty,|\mathbf{x}|\le\sigma t} \mathbf{u}(\mathbf{x},t) = \mathbf{u}^*$ if and only if

$$\liminf_{t\to\infty,|\mathbf{x}|\le\sigma t} \mathbf{u}(\mathbf{x},t) = \limsup_{t\to\infty,|\mathbf{x}|\le\sigma t} \mathbf{u}(\mathbf{x},t) = \mathbf{u}^*.$$

This is equivalent to the statement that for any $\varepsilon > 0$, there exists some $t > 0$ such that $|\mathbf{u}(\mathbf{x},s) - \mathbf{u}^*| < \varepsilon$ whenever $s > t$ and $|x| \le \sigma s$.

Let

$$\begin{aligned} v_1^* &= \left(\frac{k_1k_2\left(1-E_2^*\right)}{1-E_2^*+\Lambda_2} - 1\right)\frac{4\left(1-E_2^*+\Lambda_2\right)^2}{k_1^2k_2^2\left(1-E_2^*\right)^2}E_1^*, \\ v_2^* &= \left(\frac{k_1k_2\left(1-E_2^*\right)}{1-E_2^*+\Lambda_2} - 1\right)\frac{4\left(1-E_2^*+\Lambda_2\right)}{k_1^2k_2\left(1-E_2^*\right)}E_1^* \end{aligned}$$

and $\mathbf{E}^{**} = (v_1^*, v_2^*)$. It is clear that $\mathbf{E}^{**} = (v_1^*, v_2^*)$ is the unique positive root of the equation

$$\begin{cases} k_1E_1^*v_2\left(1-\frac{k_1}{4E_1^*}\right) - E_1^*v_1 = 0, \\ rk_2\left(1-E_2^*\right)v_1 - r\left(1-E_2^*+\Lambda_2\right)v_2 = 0. \end{cases}$$

Recall that $\Lambda_2 = E_2^* - \theta_2$, where θ_2 satisfies the assumption (H). Define

$$g(u) = \begin{cases} \inf_{v\in\left(u,v_2^*\right]}\left\{v - \frac{k_1}{4E_1^*}v^2\right\}, & u \in [0, v_2^*], \\ \inf_{v\in\left[v_2^*,u\right)}\left\{v - \frac{k_1}{4E_1^*}v^2\right\}, & u > v_2^*. \end{cases} \tag{22}$$

Consider the following system

$$\begin{cases} \frac{\partial}{\partial t}\widetilde{\widetilde{u}}_1 = \Delta\widetilde{\widetilde{u}}_1(\mathbf{x},t) + k_1E_1^*g(\widetilde{\widetilde{u}}_2) - E_1^*\widetilde{\widetilde{u}}_1, \\ \frac{\partial}{\partial t}\widetilde{\widetilde{u}}_2 = d\Delta\widetilde{\widetilde{u}}_2(\mathbf{x},t) + rk_2\left(1-E_2^*\right)\widetilde{\widetilde{u}}_1 - r\left(1-E_2^*+\Lambda_2\right)\widetilde{\widetilde{u}}_2, \end{cases} \quad \mathbf{x}\in\mathbb{R}^3,\ t>0. \tag{23}$$

Due to the strict convexity of the function $v - \frac{k_1}{4E_1^*}v^2$, we have that $\mathbf{E}^{**} = (v_1^*, v_2^*)$ is the unique positive equilibrium of system (23). In particular, $g(\cdot)$ is nondecreasing on $[0, v_2^*]$. Here we emphasize that the reason that we can establish the spreading speed for solutions of system (23) is that the theory of Thieme and Zhao [49] only works for *scalar* nonlinear integral equations and system (23) exactly enables us to reduce it to a scalar integral equation for the second component $\widetilde{\widetilde{u}}_2$. But it is difficult to reduce any component of solutions of the original Lotka-Volterra competition-diffusion system to a scalar integral equation.

Lemma 3.6. *Assume that $\phi = (\phi_1, \phi_2) \in C\left(\mathbb{R}^3, \left[\mathbf{E}^0, \mathbf{E}^{**}\right]\right)$ is compactly supported with $\phi_1(\cdot) + \phi_2(\cdot) \not\equiv 0$. Let $\widetilde{\widetilde{\mathbf{u}}}(\mathbf{x},t;\phi) = \left(\widetilde{\widetilde{u}}_1(\mathbf{x},t;\phi), \widetilde{\widetilde{u}}_2(\mathbf{x},t;\phi)\right)$ be the solution of system (23) with the initial value ϕ. Then there exists $\sigma^* > 0$ such that*

(i) *$\widetilde{\widetilde{\mathbf{u}}}(\mathbf{x},t;\phi) \in \left[\mathbf{E}^0, \mathbf{E}^{**}\right]$ for any $\mathbf{x}\in\mathbb{R}^3$ and $t > 0$;*

(ii) *$\lim_{t\to\infty,|\mathbf{x}|\ge\sigma t} \widetilde{\widetilde{\mathbf{u}}}(\mathbf{x},t) = \mathbf{E}^0$ for any $\sigma > \sigma^*$;*

(iii) *$\lim_{t\to\infty,|\mathbf{x}|\le\sigma t} \widetilde{\widetilde{\mathbf{u}}}(\mathbf{x},t) = \mathbf{E}^{**}$ for any $\sigma < \sigma^*$.*

We postpone to prove the lemma in subsection 5.2. Now we are in the position to prove (21).

Lemma 3.7. *One has*

$$\lim_{x_3\to+\infty}\|\mathbf{W}(\cdot,\cdot,x_3)-\mathbf{E}^1\|_{C(\mathbb{R}^2)}=0.$$

Proof. As mentioned above, we need only to show $\lim_{x_3\to+\infty}\mathbf{W}(0,0,x_3)=\mathbf{E}^1$. By Lemma 3.5, we have that

$$\text{either}\quad \lim_{x_3\to+\infty}\mathbf{W}(0,0,x_3)=\mathbf{E}^1\quad\text{or}\quad \lim_{x_3\to+\infty}\mathbf{W}(0,0,x_3)=\mathbf{E}^*.$$

Therefore, it is sufficient to show that the latter is impossible.

On the contrary we assume that

$$\lim_{x_3\to+\infty}\mathbf{W}(0,0,x_3)=\mathbf{E}^*. \tag{24}$$

Consequently, we have $\mathbf{E}^0\le\mathbf{W}(\mathbf{x})\le\mathbf{E}^*$ for all $\mathbf{x}\in\mathbb{R}^3$. Set $\check{\Phi}_i(x_1,x_2,x_3)=E_i^*-W_i(x_1,x_2,-x_3)$, $i=1,2$. We have

$$\begin{cases}\Delta\check{\Phi}_1+s\frac{\partial}{\partial x_3}\check{\Phi}_1+\left(E_1^*-\check{\Phi}_1\right)\left(k_1\check{\Phi}_2-\check{\Phi}_1\right)=0,\\ d\Delta\check{\Phi}_2+s\frac{\partial}{\partial x_3}\check{\Phi}_2+r\left(1-E_2^*+\check{\Phi}_2\right)\left(k_2\check{\Phi}_1-\check{\Phi}_2\right)=0,\end{cases}\quad\forall\,\mathbf{x}\in\mathbb{R}^3.$$

By virtue of $W_i(-x_1,x_2,x_3)=W_i(x_1,x_2,x_3)$ and $W_i(x_1,-x_2,x_3)=W_i(x_1,x_2,x_3)$ for all $(x_1,x_2,x_3)\in\mathbb{R}^3$, and $\frac{\partial}{\partial x_1}W_i>0$ for $x_1>0$, $\frac{\partial}{\partial x_2}W_i>0$ for $x_2>0$ and $\frac{\partial}{\partial x_3}W_i>0$ for any $x_3\in\mathbb{R}$, we have that $\check{\Phi}_i(-x_1,x_2,x_3)=\check{\Phi}_i(x_1,x_2,x_3)$ and $\check{\Phi}_i(x_1,-x_2,x_3)=\check{\Phi}_i(x_1,x_2,x_3)$ for all $(x_1,x_2,x_3)\in\mathbb{R}^3$, and $\frac{\partial}{\partial x_1}\check{\Phi}_i<0$ for $x_1>0$, $\frac{\partial}{\partial x_2}\check{\Phi}_i<0$ for $x_2>0$ and $\frac{\partial}{\partial x_3}\check{\Phi}_i>0$ for any $x_3\in\mathbb{R}$, $i=1,2$. It is obvious that $\left(\check{\Phi}_1(x_1,x_2,x_3-st),\check{\Phi}_2(x_1,x_2,x_3-st)\right)$ is a solution of the following system

$$\begin{cases}\frac{\partial}{\partial t}\widehat{u}_1=\Delta\widehat{u}_1+\left(E_1^*-\widehat{u}_1\right)\left(k_1\widehat{u}_2-\widehat{u}_1\right),\\ \frac{\partial}{\partial t}\widehat{u}_2=d\Delta\widehat{u}_2+r\left(1-E_2^*+\widehat{u}_2\right)\left(k_2\widehat{u}_1-\widehat{u}_2\right),\end{cases}\quad\forall\,\mathbf{x}\in\mathbb{R}^3,\ t>0. \tag{25}$$

Note that $\left[\mathbf{E}^0,\mathbf{E}^*\right]$ is an invariant interval of solutions of system (25). Define

$$\widehat{\Phi}_i(\mathbf{x},t):=\min\left\{\check{\Phi}_i(x_1,x_2,x_3-st),\check{\Phi}_i(x_1,x_2,-x_3-st)\right\}$$

for $\mathbf{x}=(x_1,x_2,x_3)\in\mathbb{R}^3$ and $t\ge0$, where $i=1,2$. Then

$$\widehat{\mathbf{\Phi}}(\mathbf{x},t)=\left(\widehat{\Phi}_1(\mathbf{x},t),\widehat{\Phi}_2(\mathbf{x},t)\right)$$

is a supersolution of (25). In particular, we have that

$$0\le\widehat{\Phi}_i(\mathbf{x},t)=\check{\Phi}_i(x_1,x_2,-|x_3|-st)\le\check{\Phi}_i(0,0,-st)\le E_i^*-\theta_i.$$

Following from (24), we have

$$\lim_{t\to+\infty}\check{\Phi}_i(0,0,-st)=\lim_{t\to+\infty}E_i^*-W_i(0,0,-st)=0,\quad i=1,2.$$

Therefore,

$$\lim_{t\to\infty}\sup_{\mathbf{x}\in\mathbb{R}^3}\widehat{\mathbf{\Phi}}(\mathbf{x},t)=\mathbf{0}. \tag{26}$$

Following the definition of $\widehat{\mathbf{\Phi}}(\mathbf{x},t)$, we have $\widehat{\mathbf{\Phi}}(\mathbf{x},0)\gg\mathbf{0}$ for any $\mathbf{x}\in\mathbb{R}^3$. Fix $\phi\in C\left(\mathbb{R}^3,\left[\mathbf{E}^0,\mathbf{E}^{**}\right]\right)$ such that it is compactly supported with $\phi_1(\cdot)+\phi_2(\cdot)\not\equiv0$ and $\phi(\mathbf{x})\le\widehat{\mathbf{\Phi}}(\mathbf{x},0)$ for any $\mathbf{x}\in\mathbb{R}^3$. Let $\widehat{\mathbf{u}}(\mathbf{x},t;\phi)$ be the solution of (25) with initial value $\widehat{\mathbf{u}}(\mathbf{x},0;\phi)=\phi(\mathbf{x})$. Because $\widehat{\mathbf{\Phi}}(\mathbf{x},t)$ is a supersolution of (25), we have $\widehat{\mathbf{u}}(\mathbf{x},t;\phi)\leq\widehat{\mathbf{\Phi}}(\mathbf{x},t)$ for all $x\in\mathbb{R}^3$ and $t>0$. It follows from (26) that

$$\lim_{t\to\infty}\sup_{\mathbf{x}\in\mathbb{R}^3}\widehat{\mathbf{u}}(\mathbf{x},t;\phi)=\mathbf{0}. \tag{27}$$

Consider the following system

$$\begin{cases}\frac{\partial}{\partial t}\widetilde{u}_1=\Delta\widetilde{u}_1+\left(E_1^*-\widetilde{u}_1\right)\left(k_1\widetilde{u}_2-\widetilde{u}_1\right),\\ \frac{\partial}{\partial t}\widetilde{u}_2=d\Delta\widetilde{u}_2+rk_2\left(1-E_2^*\right)\widetilde{u}_1-r\left(1-E_2^*+\Lambda_2\right)\widetilde{u}_2.\end{cases} \tag{28}$$

This system has two equilibria $\mathbf{E}^0=(0,0)$ and $\mathbf{E}_+^*=\left(E_1^*,\frac{k_2(1-E_2^*)}{1-E_2^*+\Lambda_2}E_1^*\right)$. In particular, system (28) satisfies the comparison principle on $\left[\mathbf{E}^0,\mathbf{E}_+^*\right]$. Notice that $\widehat{u}_2(\mathbf{x},t;\phi)\leq\widehat{\Phi}_2(\mathbf{x},t)\leq\widehat{\Phi}_2(\mathbf{0},0)\leq E_2^*-\theta_2=\Lambda_2$ for all $\mathbf{x}\in\mathbb{R}^3$ and $t>0$. By virtue of

$$\begin{aligned}&r\left(1-E_2^*+\widehat{u}_2(\mathbf{x},t;\phi)\right)\left(k_2\widehat{u}_1(\mathbf{x},t;\phi)-\widehat{u}_2(\mathbf{x},t;\phi)\right)\\ &\qquad\geq rk_2\left(1-E_2^*\right)\widehat{u}_1(\mathbf{x},t;\phi)-r\left(1-E_2^*+\Lambda_2\right)\widehat{u}_2(\mathbf{x},t;\phi),\end{aligned}$$

we have

$$\begin{cases}\frac{\partial}{\partial t}\widehat{u}_1(\mathbf{x},t;\phi)=\Delta\widehat{u}_1(\mathbf{x},t;\phi)+\left(E_1^*-\widehat{u}_1(\mathbf{x},t;\phi)\right)\left(k_1\widetilde{u}_2(\mathbf{x},t;\phi)-\widehat{u}_1(\mathbf{x},t;\phi)\right),\\ \frac{\partial}{\partial t}\widehat{u}_2(\mathbf{x},t;\phi)\geq d\Delta\widehat{u}_2(\mathbf{x},t;\phi)+rk_2\left(1-E_2^*\right)\widehat{u}_1(\mathbf{x},t;\phi)\\ \qquad\qquad -r\left(1-E_2^*+\Lambda_2\right)\widehat{u}_2(\mathbf{x},t;\phi),\end{cases}$$

which implies that $\widehat{\mathbf{u}}(\mathbf{x},t;\phi)$ is a supersolution of (28). Let $\widetilde{\mathbf{u}}(\mathbf{x},t;\phi)$ be the solution of (28) with initial value $\phi(\cdot)$. By the condition (H), we have $\Lambda_2<E_+^*$. Then the comparison principle implies that $\widehat{\mathbf{u}}(\mathbf{x},t;\phi)\geq\widetilde{\mathbf{u}}(\mathbf{x},t;\phi)\geq\mathbf{0}$ for any $\mathbf{x}\in\mathbb{R}^3$ and $t>0$. Therefore, it follows from (27) that

$$\lim_{t\to\infty}\sup_{\mathbf{x}\in\mathbb{R}^3}\widetilde{\mathbf{u}}(\mathbf{x},t;\phi)=\mathbf{0}. \tag{29}$$

Using the inequality

$$uv\leq\frac{\eta}{4}u^2+\frac{1}{\eta}v^2,$$

where $\eta>0$ is a constant, we have

$$\begin{aligned}\left(E_1^*-u\right)\left(k_1v-u\right)&=k_1E_1^*v-k_1uv-E_1^*u+u^2\\ &\geq k_1E_1^*v-\frac{\eta k_1}{4}v^2-\frac{k_1}{\eta}u^2-E_1^*u+u^2.\end{aligned}$$

Setting $\eta=k_1$, we get

$$\left(E_1^*-u\right)\left(k_1v-u\right)\geq k_1E_1^*v-\frac{k_1^2}{4}v^2-E_1^*u,\quad\forall\ u,v\geq0.$$

By the definition of $g(\cdot)$, we have

$$\begin{aligned}\left(E_1^*-u\right)\left(k_1v-u\right)&\geq k_1E_1^*v-\frac{k_1^2}{4}v^2-E_1^*u\\ &\geq k_1E_1^*g(v)-E_1^*u\end{aligned}$$

for any $(u,v)\in\left[\mathbf{E}^0,\mathbf{E}^{**}\right]$. In particular, when (H) holds, one has $\mathbf{E}^{**}\leq\mathbf{E}_+^*$. Let $\widetilde{\widetilde{\mathbf{u}}}(\mathbf{x},t;\phi)$ be the solution of (23) with initial value $\phi(\cdot)$. Then by the above inequality and Lemma 3.6 (i) we obtain

$$\begin{cases} \frac{\partial}{\partial t}\widetilde{\widetilde{u}}_1(\mathbf{x},t;\phi) & = \Delta\widetilde{\widetilde{u}}_1(\mathbf{x},t;\phi) + k_1E_1^*g\left(\widetilde{\widetilde{u}}_2(\mathbf{x},t;\phi)\right) - E_1^*\widetilde{\widetilde{u}}_1(\mathbf{x},t;\phi) \\ & \leq \Delta\widetilde{\widetilde{u}}_1(\mathbf{x},t;\phi) + \left(E_1^* - \widetilde{\widetilde{u}}_1(\mathbf{x},t;\phi)\right)\left(k_1\widetilde{\widetilde{u}}_2(\mathbf{x},t;\phi) - \widetilde{\widetilde{u}}_1(\mathbf{x},t;\phi)\right), \\ \frac{\partial}{\partial t}\widetilde{\widetilde{u}}_2(\mathbf{x},t;\phi) & = d\Delta\widetilde{\widetilde{u}}_2(\mathbf{x},t;\phi) + rk_2(1-E_2^*)\widetilde{\widetilde{u}}_1(\mathbf{x},t;\phi) \\ & \quad -r(1-E_2^*+\Lambda_2)\widetilde{\widetilde{u}}_2(\mathbf{x},t;\phi), \end{cases}$$

which implies that $\widetilde{\widetilde{\mathbf{u}}}(\mathbf{x},t;\phi)$ is a subsolution of (28). Using the comparison principle for systems (28), we have

$$\widetilde{\widetilde{\mathbf{u}}}(\mathbf{x},t;\phi) \leq \widetilde{\mathbf{u}}(\mathbf{x},t;\phi), \quad \forall\, \mathbf{x}\in\mathbb{R}^3,\ t>0. \tag{30}$$

Following Lemma 3.6 (iii), we have

$$\lim_{t\to\infty,|\mathbf{x}|\leq\sigma t}\widetilde{\widetilde{\mathbf{u}}}(\mathbf{x},t;\phi) = \mathbf{E}^{**} \quad \text{for any } \sigma\in(0,\sigma^*),$$

which implies

$$\widetilde{\widetilde{\mathbf{u}}}(\mathbf{x},t;\phi)\to\mathbf{E}^{**} \text{ in the sense of } C_{loc}\left(\mathbb{R}^3\right) \text{ as } t\to+\infty. \tag{31}$$

Combining (30) and (31), we get

$$\lim_{t\to\infty}\inf_{|\mathbf{x}|<R}\widetilde{\mathbf{u}}(\mathbf{x},t;\phi)\geq\mathbf{E}^{**} \quad \text{for any fixed } R>0. \tag{32}$$

Finally, there exists a contradiction between (29) and (32). This contradiction shows that (24) is impossible. Therefore,

$$\lim_{z\to+\infty}\mathbf{W}(0,0,z) = \mathbf{E}^1.$$

This completes the proof. □

Up to now, we have completed the proof of Theorem 3.1 (vi). Theorem 3.1 (vii) can be easily proved by using the results of Theorem 3.1 (ii)-(v) and the maximum principle. Here we omit the details of the proof. Thus, we have completed the proof of Theorem 3.1.

Theorem 3.8. *Let $s>c>0$ and denote the axisymmetric traveling front $\mathbf{W}(\mathbf{x})$ defined in Theorem 3.1 by $\mathbf{W}^s(\mathbf{x})$. Let $U_2(0)=W_2^s(\mathbf{0})=\theta_2$. Then one has*

$$\lim_{s\to c+0}\|\mathbf{W}^s(\mathbf{x})-\mathbf{U}(x_3)\|_{C^2_{loc}(\mathbb{R}^3)}=0,$$

where $(\mathbf{U},c)$ is the planar traveling wave front of (5) connecting $\mathbf{E}^0$ and $\mathbf{E}^1$.

Proof. Note that there exists $K>0$ such that $\|\mathbf{W}^s(\cdot)\|_{C^3(\mathbb{R}^3)}<K$ for any $s\in(c,c+1)$. Let $\{s_n\}$ satisfy $s_n<s_{n+1}<c+1$ and $s_n\to c$ as $n\to\infty$. Then there exists a function $\widehat{\mathbf{U}}(\cdot)\in C^2\left(\mathbb{R},\left[\mathbf{E}^0,\mathbf{E}^1\right]\right)$ such that

$$\mathbf{W}^{s_n}(0,0,\cdot)\to\widehat{\mathbf{U}}(\cdot) \quad \text{under the norm } \|\cdot\|_{C^2_{loc}(\mathbb{R})} \quad \text{as } n\to\infty.$$

By (iii) of Theorem 3.1, we have

$$\mathbf{W}^{s_n}(0,0,x_3)\leq\mathbf{W}^{s_n}(x_1,x_2,x_3)\leq\mathbf{W}^{s_n}\left(0,0,x_3+m_*^n\sqrt{x_1^2+x_2^2}\right)$$

for any $(x_1,x_2,x_3)\in\mathbb{R}^3$, where $m_*^n=\sqrt{\frac{s_n^2-c^2}{c}}$. Due to $m_*^n\to 0$ as $n\to\infty$, we have that $\mathbf{W}^{s_n}(x_1,x_2,x_3)$ converges to $\widehat{\mathbf{U}}(x_3)$ uniformly in any compact set $\Omega\subset\mathbb{R}^3$ as $n\to\infty$. Consequently, we have that $\mathbf{W}^{s_n}(x_1,x_2,x_3)$ converges to $\widehat{\mathbf{U}}(x_3)$ in the sense of $\|\cdot\|_{C^2_{loc}(\mathbb{R}^3)}$ as $n\to\infty$. Thus, we have that $\widehat{\mathbf{U}}(\cdot)=\left(\widehat{U}_1(\cdot),\widehat{U}_1(\cdot)\right)\in C^2\left(\mathbb{R},\left[\mathbf{E}^0,\mathbf{E}^1\right]\right)$ satisfies

$$\begin{cases} \widehat{U}_1''(x)-c\widehat{U}_1'(x)+\widehat{U}_1(x)\left(1-\widehat{U}_1(x)-k_1\widehat{U}_2(x)\right)=0, \\ d\widehat{U}_2''(x)-c\widehat{U}_2'(x)+r\left(1-\widehat{U}_2(x)\right)\left(k_2\widehat{U}_1(x)-\widehat{U}_2(x)\right)=0, \end{cases} \quad x\in\mathbb{R}.$$

In view of $\widehat{U}_2(0)=\theta_2$ and $\widehat{U}_i'(x)\geq 0$ for any $x\in\mathbb{R}$, similar to the proof of Theorem 3.1 we can show that $\widehat{\mathbf{U}}(+\infty)=\mathbf{E}^1$, $\widehat{\mathbf{U}}(-\infty)=\mathbf{E}^0$ and $\widehat{U}_i'(x)>0$ for any $x\in\mathbb{R}$, where $i=1,2$. It then follows from the uniqueness of planar traveling wave fronts of (5) connecting two equilibria $\mathbf{E}^0$ and $\mathbf{E}^1$ that $\widehat{\mathbf{U}}(x)\equiv\mathbf{U}(x)$ in $x\in\mathbb{R}$. This completes the proof. □

4. **Nonexistence of axisymmetric traveling fronts.** In this section we prove Theorems 1.3 and 1.4, which imply the nonexistence of axisymmetric traveling fronts. Here we give only the proof of Theorem 1.4. Theorem 1.3 can be similarly proved.

Proof of Theorem 1.4. We prove it by a contradiction argument. On the contrary, we assume that for $s<c$, there exists an axisymmetric traveling front $\mathbf{\Psi}(\mathbf{x})$ satisfying (4), $\lim_{x_3\to+\infty}\mathbf{\Psi}(0,0,x_3)=\mathbf{E}_u$, $\lim_{x_3\to-\infty}\mathbf{\Psi}(0,0,x_3)=\mathbf{E}_v$ and

$$\frac{\partial^2}{\partial x_i^2}\Psi_1(\mathbf{x})\Big|_{x_1=x_2=0}\geq 0,\ \ \frac{\partial^2}{\partial x_i^2}\Psi_2(\mathbf{x})\Big|_{x_1=x_2=0}\leq 0,\ \ \frac{\partial}{\partial x_3}\Psi_1(\mathbf{x})>0,\ \ \frac{\partial}{\partial x_3}\Psi_2(\mathbf{x})<0.$$

Let $W_1(\mathbf{x})=\Psi_1(\mathbf{x})$ and $W_2(\mathbf{x})=1-\Psi_2(\mathbf{x})$ for any $\mathbf{x}\in\mathbb{R}^3$. Then $\mathbf{W}(\mathbf{x})=(W_1(\mathbf{x}),W_2(\mathbf{x}))$ satisfies (14), $\lim_{x_3\to+\infty}\mathbf{W}(0,0,x_3)=\mathbf{E}^1$, $\lim_{x_3\to-\infty}\mathbf{W}(0,0,x_3)=\mathbf{E}^0$ and

$$\frac{\partial^2}{\partial x_i^2}W_1(\mathbf{x})\Big|_{x_1=x_2=0}\geq 0,\ \ \frac{\partial^2}{\partial x_i^2}W_2(\mathbf{x})\Big|_{x_1=x_2=0}\geq 0,\ \ \frac{\partial}{\partial x_3}W_1(\mathbf{x})>0,\ \ \frac{\partial}{\partial x_3}W_2(\mathbf{x})>0.$$

Let $\widetilde{\mathbf{U}}(x_3)=\mathbf{W}(0,0,x_3)$ for any $x_3\in\mathbb{R}$. Then we have

$$\begin{aligned}&\widetilde{U}_1''(x_3)-s\widetilde{U}_1'(x_3)+\widetilde{U}_1(x_3)\left(1-k_1-\widetilde{U}_1(x_3)+k_1\widetilde{U}_2(x_3)\right)\\ &\quad=-\frac{\partial^2}{\partial x_1^2}W_1(\mathbf{x})\Big|_{x_1=x_2=0}-\frac{\partial^2}{\partial x_2^2}W_1(\mathbf{x})\Big|_{x_1=x_2=0}\leq 0\end{aligned}$$

and

$$\begin{aligned}&d\widetilde{U}_2''(x_3)-s\widetilde{U}_2'(x_3)+r\left(1-\widetilde{U}_2(x_3)\right)\left(k_2\widetilde{U}_1(x_3)-\widetilde{U}_2(x_3)\right)\\ &\quad=-d\,\frac{\partial^2}{\partial x_1^2}W_2(\mathbf{x})\Big|_{x_1=x_2=0}-d\,\frac{\partial^2}{\partial x_2^2}W_2(\mathbf{x})\Big|_{x_1=x_2=0}\leq 0,\end{aligned}$$

which imply that $\mathbf{u}^+(x,t)=\left(u_1^+(x,t),u_2^+(x,t)\right)$ with $u_i^+(x,t)=\widetilde{U}_i(x+st)$ $(i=1,2)$ is a supersolution of the following system

$$\begin{cases} \frac{\partial}{\partial t}\check{u}_1(x,t)=\frac{\partial^2}{\partial x^2}\check{u}_1(x,t)+\check{u}_1(x,t)\left(1-\check{u}_1(x,t)-k_1\check{u}_2(x,t)\right), \\ \frac{\partial}{\partial t}\check{u}_2(x,t)=d\frac{\partial^2}{\partial x^2}\check{u}_2(x,t)+r\left(1-\check{u}_2(x,t)\right)\left(k_2\check{u}_1(x,t)-\check{u}_2(x,t)\right), \end{cases} \quad x\in\mathbb{R},t>0. \tag{33}$$

On the other hand, following Wang [51, Lemma 4.2] (see also Lin and Li [34]), we obtain that the function

$$\begin{aligned}\mathbf{u}^-(x,t) &= \mathbf{U}\left(x+ct-\xi^- - \rho\delta\left(1-e^{-\beta t}\right)\right)\\ &\quad -\delta e^{-\beta t}\mathbf{Q}\left(x+ct-\xi^- - \rho\delta\left(1-e^{-\beta t}\right)\right)\end{aligned}$$

is a subsolution of (33), where ρ, δ, β are appropriate positive constants, $\mathbf{Q}(\cdot) \in C^2\left(\mathbb{R}, \mathbb{R}_+^2\right)$ is a monotone vector-valued function and satisfies $\mathbf{Q}(\pm\infty) \gg \mathbf{0}$, $\xi^- \in \mathbb{R}$ is an arbitrary number. In view of

$$\mathbf{u}^-(x,0) = \mathbf{U}(x-\xi^-) - \delta\mathbf{Q}(x-\xi^-), \quad \mathbf{u}^+(-\infty,0) = \mathbf{E}^0 \text{ and } \mathbf{u}^+(+\infty,0) = \mathbf{E}^1,$$

there exists a sufficiently large $\xi^- > 0$ so that $\mathbf{u}^-(x,0) \le \mathbf{u}^+(x,0)$ for all $x \in \mathbb{R}$. Applying the comparison principle (see also Wang [51, Sections 2 and 5]), we have

$$\mathbf{u}^-(x,t) \le \mathbf{u}^+(x,t) = \widetilde{\mathbf{U}}(x+st), \quad \forall\, x \in \mathbb{R},\ t > 0.$$

It follows that

$$\begin{aligned}\mathbf{E}^1 &\gg \widetilde{\mathbf{U}}(0) = \mathbf{u}^+(-st,t)\\ &\ge \mathbf{u}^-(-st,t)\\ &= \mathbf{U}\left((c-s)t-\xi^- - \rho\delta\left(1-e^{-\beta t}\right)\right)\\ &\quad -\delta e^{-\beta t}\mathbf{Q}\left((c-s)t-\xi^- - \rho\delta\left(1-e^{-\beta t}\right)\right)\\ &\to \mathbf{E}^1 \qquad \text{as } t \to +\infty,\end{aligned}$$

which is a contradiction, this completes the proof of Theorem 1.4. □

5. **Proofs of Lemmas 3.3 and 3.6.**

In this section we prove Lemmas 3.3 and 3.6.

5.1. **Proof of Lemma 3.3.**

Consider system (18), namely, the following reaction-diffusion system

$$\left\{\begin{array}{l}\frac{\partial}{\partial t}u_1 = \frac{\partial^2}{\partial x^2}u_1 + \left(E_1^* - u_1\right)\left(k_1u_2 - u_1\right),\\ \frac{\partial}{\partial t}u_2 = d\frac{\partial^2}{\partial x^2}u_2 + r\left(1 - E_2^* + u_2\right)\left(k_2u_1 - u_2\right).\end{array}\right. \tag{34}$$

For the corresponding ODE system

$$\left\{\begin{array}{l}\frac{d}{dt}u_1 = \left(E_1^* - u_1\right)\left(k_1u_2 - u_1\right),\\ \frac{d}{dt}u_2 = r\left(1 - E_2^* + u_2\right)\left(k_2u_1 - u_2\right),\end{array}\right. \tag{35}$$

the equilibrium $\mathbf{E}^0$ is unstable and the equilibrium $\mathbf{E}^*$ is stable. Following Volpert et al. [50, Theorem 4.2] and Liang and Zhao [33, Theorems 4.3 and 4.4] we know that there exists $\kappa^* \in \mathbb{R}$ such that for any $\kappa \ge \kappa^*$, system (18) admits an increasing traveling wave front $\left(\rho_1\left(x+\kappa t\right), \rho_2\left(x+\kappa t\right)\right)$ connecting the equilibria $\mathbf{E}^0$ and $\mathbf{E}^*$ and for any $\kappa < \kappa^*$, system (18) does not admit an increasing traveling wave front $\left(\rho_1\left(x+\kappa t\right), \rho_2\left(x+\kappa t\right)\right)$ connecting the equilibria $\mathbf{E}^0$ and $\mathbf{E}^*$. To complete the proof, it is sufficient to show that $\kappa^* > 0$.

By Volpert et al. [50] we know that $\kappa^* := \inf_{\phi\in\mathcal{K}} \psi^*(\phi)$, where

$$\psi^*(\phi) := \max\left\{\sup_{x\in\mathbb{R}} \frac{\phi_1''(x) + f_1(\phi(x))}{\phi_1'(x)}, \sup_{x\in R} \frac{d\phi_2''(x) + f_2(\phi(x))}{\phi_2'(x)}\right\}$$

and

$$\mathcal{K} = \left\{\phi(x) = \left(\phi_1(x), \phi_2(x)\right) \in C^2\left(\mathbb{R}, \left[\mathbf{E}^0, \mathbf{E}^*\right]\right) \,\middle|\, \phi_i'(x) > 0, i = 1, 2\right\}.$$

But it is difficult to show $\kappa^* > 0$ via $\kappa^* := \inf_{\phi \in \mathcal{K}} \psi^* (\phi)$. Therefore, we use the setting of Liang and Zhao [33] to show $\kappa^* > 0$.

It is not difficult to show that the hypothesis (A1)-(A6) and (C1)-(C6) of Liang and Zhao [33] hold in $\left[\mathbf{E}^0, \mathbf{E}^*\right]$ for system (18). Linearizing system (18) at $\mathbf{E}^0 = (0,0)$, we obtain the linear system

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t} u_1 = \frac{\partial^2}{\partial x^2} u_1 - E_1^* u_1 + k_1 E_1^* u_2, \\ \frac{\partial}{\partial t} u_2 = d \frac{\partial^2}{\partial x^2} u_2 + r k_2 \left(1 - E_2^*\right) u_1 - r \left(1 - E_2^*\right) u_2. \end{array} \right. \tag{36}$$

For any $\alpha \in \mathbb{R}$, let $u_i (x,t) = e^{\alpha x} \eta_i (t)$, $i = 1, 2$. Then we have

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t} \eta_1 (t) = \left(\alpha^2 - E_1^*\right) \eta_1 (t) + k_1 E_1^* \eta_2 (t), \\ \frac{\partial}{\partial t} \eta_2 (t) = r k_2 \left(1 - E_2^*\right) \eta_1 (t) + \left(d\alpha^2 - r\left(1 - E_2^*\right)\right) \eta_2 (t). \end{array} \right. \tag{37}$$

Namely,

$$\frac{d}{dt} \eta (t) = \mathbf{A} (\alpha) \eta (t),$$

where

$$\eta (t) = \begin{pmatrix} \eta_1 (t) \\ \eta_2 (t) \end{pmatrix} \quad \text{and} \quad \mathbf{A} (\alpha) = \begin{pmatrix} \alpha^2 - E_1^* & k_1 E_1^* \\ r k_2 \left(1 - E_2^*\right) & d\alpha^2 - r\left(1 - E_2^*\right) \end{pmatrix}.$$

It is obvious that the matrix $\mathbf{A}(\alpha)$ is cooperative and irreducible (see Smith [45, page 56]). Let $\lambda (\alpha) := s(\mathbf{A}(\alpha))$ be a simple eigenvalue of $\mathbf{A}(\alpha)$ with a strongly positive eigenvector $\mathbf{p}(\alpha) = (p_1(\alpha), p_2(\alpha)) \gg (0,0)$. A direct calculation yields

$$\lambda (\alpha) = \frac{-\left[r\left(1 - E_2^*\right) + E_1^* - (1+d)\alpha^2\right] + \sqrt{\vartheta(\alpha)}}{2},$$

where

$$\begin{aligned} \vartheta(\alpha) \;\; = \;\; & \left[r\left(1 - E_2^*\right) + E_1^* - (1+d)\alpha^2\right]^2 \\ & -4\left(E_1^* - \alpha^2\right)\left(r\left(1 - E_2^*\right) - d\alpha^2\right) + 4 r k_1 k_2 E_1^* \left(1 - E_2^*\right). \end{aligned}$$

Define

$$\mathbf{B}_\alpha^t \left(\eta^0\right) := M_t \left(\eta^0 e^{-\alpha x}\right)(0) = \eta\left(t; \eta^0\right) = \exp\left(\mathbf{A}(\alpha) t\right) \eta^0,$$

where M_t is the linear solution map defined by (36) and $\eta\left(t, \eta^0\right)$ is the solution of (37) with $\eta (0) = \eta^0$. Therefore, $\mathbf{B}_\alpha^t$ is the solution map associated with (37) on $\mathbb{R}^2$, and hence, $\mathbf{B}_\alpha^t$ is the strongly positive matrix for each $t > 0$. Since $\eta(t; \mathbf{p}(\alpha)) = \exp(\mathbf{A}(\alpha)) \mathbf{p}(\alpha)$, we have

$$\mathbf{B}_\alpha^t (\mathbf{p}(\alpha)) = \eta(t; \mathbf{p}(\alpha)) = \exp(\mathbf{A}(\alpha) t) \mathbf{p}(\alpha) = \exp(\lambda(\alpha) t) \mathbf{p}(\alpha),$$

which implies that $\exp(\lambda(\alpha) t)$ is the principle eigenvalue of $\mathbf{B}_\alpha^t$, and $\mathbf{p}(\alpha)$ is the associated strongly positive eigenvalue. Let $t = 1$. Then $\gamma(\alpha) := \exp(\lambda(\alpha))$ is the principle eigenvalue of $\mathbf{B}_\alpha := \mathbf{B}_\alpha^1$. Define a function

$$\Phi(\alpha) := \frac{1}{\alpha} \gamma(\alpha) = \frac{\lambda(\alpha)}{\alpha}.$$

A direct verification shows that:

(1) $\Phi(\alpha) \to +\infty$ as $\alpha \to 0^+$;
(2) $\Phi(\alpha)$ is decreasing near 0;
(3) $\Phi'(\alpha)$ changes sign once on $(0, +\infty)$;
(4) $\lim_{\alpha \to +\infty} \Phi(\alpha) = +\infty$.

In the following we prove that $\kappa^* \geq \inf_{\alpha > 0} \Phi(\alpha) = \inf_{\alpha > 0} \frac{\lambda(\alpha)}{\alpha}$. In view of $\lambda(0) > 0$, we have $\gamma(0) = \exp(\lambda(0)) > 1$. Therefore, the condition (C7) in Liang and Zhao [33] is satisfied.

For $\delta > 0$, denote $\mathcal{C}_\delta = \left\{ \varphi = (\varphi_1, \varphi_2) \in BC(\mathbb{R}, \mathbb{R}^2) \middle| 0 \le \varphi_i(\cdot) \le \delta,\ i = 1, 2 \right\}$. For any $\epsilon \in (0, 1)$, there exists a $\delta > 0$ such that $0 \le u_i(x, t; \varphi) \le u_i(t; \delta) \le \epsilon$ for any $x \in \mathbb{R}$, $t \in [0, 1]$ and $\varphi \in \mathcal{C}_\delta$, where $i = 1, 2$, $\mathbf{u}(x, t; \varphi) = (u_1(x, t; \varphi), u_2(x, t; \varphi))$ is the solution of (18) with initial value $\varphi \in \mathcal{C}_\delta$ and $\mathbf{u}(t; \delta) = (u_1(t; \delta), u_2(t; \delta))$ is the solution of (35) with initial value $\mathbf{u}(0) = (\delta, \delta)$. Thus, $\mathbf{u}(x, t) = \mathbf{u}(x, t; \varphi)$ satisfies

$$\begin{aligned}
&\tfrac{\partial}{\partial t} u_1 \ge \tfrac{\partial^2}{\partial x^2} u_1 - E_1^* u_1 + k_1 (E_1^* - \epsilon) u_2, \\
&\tfrac{\partial}{\partial t} u_2 \ge d \tfrac{\partial^2}{\partial x^2} u_2 + r k_2 (1 - E_2^*) u_1 - r (1 - E_2^* + \epsilon) u_2
\end{aligned}$$

for any $x \in \mathbb{R}$ and $t \in [0, 1]$.

Let $M_t^\epsilon, t \ge 0$, be the solution map associated with the linear system

$$\begin{aligned}
&\tfrac{\partial}{\partial t} v_1 = \tfrac{\partial^2}{\partial x^2} v_1 - E_1^* v_1 + k_1 (E_1^* - \epsilon) v_2, \\
&\tfrac{\partial}{\partial t} v_2 = d \tfrac{\partial^2}{\partial x^2} v_2 + r k_2 (1 - E_2^*) v_1 - r (1 - E_2^* + \epsilon) v_2.
\end{aligned}$$

The comparison principle implies that $M_t^\epsilon(\varphi) \le Q_t(\varphi)$ for any $\varphi \in \mathcal{C}_\delta$ and $t \in [0, 1]$, where Q_t is the solution map associated with system (18), namely, $Q_t(\varphi)(\cdot) = \mathbf{u}(\cdot, t; \varphi)$. In particular, $M_1^\epsilon(\varphi) \le Q(\varphi) := Q_1(\varphi)$ for any $\varphi \in \mathcal{C}_\delta$. As we did for M_t, a similar analysis can be made for M_t^ϵ. It then follows from Liang and Zhao [33, Theorem 3.10] that

$$\inf_{\alpha > 0} \Phi_\epsilon(\alpha) \le \kappa^*,\ \forall\ \epsilon \in (0, 1).$$

Letting $\epsilon \to 0$, we obtain $\kappa^* \ge \inf_{\alpha > 0} \Phi(\alpha)$.

Now we show that $\inf_{\alpha > 0} \Phi(\alpha) > 0$. First, we have $\Phi(0 + 0) = +\infty$ and $\Phi(+\infty) = +\infty$. Therefore, it suffices to show that $\lambda(\alpha) > 0$ for any $\alpha > 0$. We consider three cases in the proof.

Case (i). $\left(E_1^* - \alpha^2\right)\left(r(1 - E_2^*) - d\alpha^2\right) > 0$ and $E_1^* - \alpha^2 < 0$. In this case we have $-\left[r(1 - E_2^*) + E_1^* - (1 + d)\alpha^2\right] > 0$, which implies that $\lambda(\alpha) > 0$.

Case (ii). $\left(E_1^* - \alpha^2\right)\left(r(1 - E_2^*) - d\alpha^2\right) < 0$. By the fact that

$$\begin{aligned}
\vartheta(\alpha) &= \left[r(1 - E_2^*) + E_1^* - (1 + d)\alpha^2\right]^2 \\
&\quad -4\left(E_1^* - \alpha^2\right)\left(r(1 - E_2^*) - d\alpha^2\right) + 4 r k_1 k_2 E_1^* (1 - E_2^*) \\
&= \left[\left(r(1 - E_2^*) - d\alpha^2\right) - \left(E_1^* - \alpha^2\right)\right]^2 + 4 r k_1 k_2 E_1^* (1 - E_2^*),
\end{aligned}$$

we obtain $\lambda(\alpha) > 0$.

Case (iii). $\left(E_1^* - \alpha^2\right)\left(r(1 - E_2^*) - d\alpha^2\right) > 0$ and $E_1^* - \alpha^2 > 0$. In this case we have

$$4 r k_1 k_2 E_1^* (1 - E_2^*) > 4 r E_1^* (1 - E_2^*) > 4\left(E_1^* - \alpha^2\right)\left(r(1 - E_2^*) - d\alpha^2\right).$$

Therefore, we have

$$\lambda(\alpha) > \frac{-\left[r(1 - E_2^*) + E_1^* - (1 + d)\alpha^2\right] + \left|r(1 - E_2^*) + E_1^* - (1 + d)\alpha^2\right|}{2} = 0.$$

Combining the above cases (i)-(iii), we obtain $\kappa^* \ge \inf_{\alpha > 0} \Phi(\alpha) = \inf_{\alpha > 0} \frac{\lambda(\alpha)}{\alpha} > 0$. This completes the proof of Lemma 3.3. □

5.2. **Proof of Lemma 3.6.** In this subsection we prove Lemma 3.6 by using the theory of Thieme and Zhao [49]. For the reader's convenience, in subsection 5.2.1 we state some results of Thieme and Zhao [49] on the spreading speed for scalar nonlinear integral equations, which are needed in the next subsection. In subsection 5.2.2, we use these results to prove Lemma 3.6 in details.

5.2.1. *Spreading speed for scalar nonlinear integral equations.* Consider the following nonlinear integral equation

$$u(\mathbf{x},t) = u_0(\mathbf{x},t) + \int_0^t \int_{\mathbb{R}^n} F(u(\mathbf{x}-\mathbf{y},t-r),\mathbf{y},r)\, d\mathbf{y}dr, \tag{38}$$

where $F : \mathbb{R}_+^2 \times \mathbb{R}^n \to \mathbb{R}$ is continuous in u and Borel measurable in $(\mathbf{y},r)$, and $u_0 : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ is Borel measurable and bounded. We further impose the following assumptions on F:

(A) There exists a function $k : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

(A1) $k^* = \int_0^\infty \int_{\mathbb{R}^n} k(\mathbf{x},t)\, d\mathbf{x}dt < \infty$;

(A2) $0 \le F(u,\mathbf{x},t) \le uk(\mathbf{x},t)$, $\forall u,t \ge 0$, $\mathbf{x} \in \mathbb{R}^n$;

(A3) For every compact interval I in $(0,\infty)$, there exists some $\varepsilon > 0$ such that

$$F(u,\mathbf{x},t) \ge \varepsilon k(\mathbf{x},t),\ \forall u \in I,\ t \ge 0,\ \mathbf{x} \in \mathbb{R}^n;$$

(A4) For every $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$F(u,\mathbf{x},t) \ge (1-\varepsilon)\, uk(\mathbf{x},t),\ \forall u \in [0,\delta],\ t \ge 0,\ \mathbf{x} \in \mathbb{R}^n;$$

(A5) For every $w > 0$, there exists some $\Lambda > 0$ such that

$$|F(u,\mathbf{x},t) - F(v,\mathbf{x},t)| \ge \Lambda |u-v|\, k(\mathbf{x},t),\ \forall u,v \in [0,w],\ t \ge 0,\ \mathbf{x} \in \mathbb{R}^n.$$

Consequently, we make a couple of assumptions concerning the function k:

(B) $k : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ is a Borel measurable function such that

(B1) $k^* = \int_0^\infty \int_{\mathbb{R}^n} k(\mathbf{x},t)\, d\mathbf{x}dt\ \in (1,\infty)$;

(B2) There exists some $\lambda > 0$ such that $\int_0^\infty \int_{\mathbb{R}^n} e^{\lambda x_1} k(\mathbf{x},t)\, d\mathbf{x}dt < \infty$;

(B3) There exists numbers $\varrho_2 > \varrho_1 > 0$, $\rho > 0$ such that

$$k(\mathbf{x},t) > 0, \quad \forall t \in (\varrho_1,\varrho_2), |\mathbf{x}| \in [0,\rho);$$

(B4) $k(\mathbf{x},t)$ is isotropic on $\mathbf{x}$ for almost all $t > 0$.

To consider the special case $F(u,\mathbf{x},t) = f(u)\, k(\mathbf{x},t)$, the function f needs to satisfy the following assumptions:

(C) $f(u) : \mathbb{R}_+ \to \mathbb{R}_+$ is a Lipschitz continuous function such that

(C1) $f(0) = 0$ and $f(u) > 0$ for any $u > 0$;

(C2) f is differentiable at $u = 0$, $f'(0) = 1$ and $f(u) \le u$, $\forall u > 0$;

(C3) $\lim_{u\to\infty} \frac{f(u)}{u} = 0$;

(C4) There exists a positive solution u^* of $u = k^* f(u)$ such that $k^* f(u) > u$ for $u \in (0,u^*)$, and $k^* f(u) < u$ for $u > u^*$.

It is clear that if (B) and (C) hold, then (A) holds. Define

$$\mathcal{K}(\sigma,\lambda) = \int_0^\infty \int_{\mathbb{R}^n} e^{-\lambda(\sigma r - z_1)} k(\mathbf{z},t)\, d\mathbf{z}dr, \quad \forall \sigma \ge 0,\ \lambda \ge 0$$

and

$$\sigma^* := \inf\{\sigma \ge 0 : \mathcal{K}(\sigma,\lambda) < 1 \text{ for some } \lambda > 0\}.$$

We say that u_0 is admissible if for every $\sigma,\lambda > 0$ with $\mathcal{K}(\sigma,\lambda) < 1$, there exists some $\gamma > 0$ such that

$$u_0(\mathbf{x},t) \le \gamma e^{\lambda(\sigma t - |\mathbf{x}|)},\ \forall t \ge 0,\ \mathbf{x} \in \mathbb{R}^n.$$

Now we state two theorems of Thieme and Zhao [49].

Theorem 5.1. (Thieme and Zhao [49, Theorem 2.1]) *Let* (A) *and* (B) *hold. Then for every admissible* u_0*, the unique solution* $u(\mathbf{x},t)$ *of* (38) *satisfies*

$$\lim_{t\to\infty,|\mathbf{x}|\geq\sigma t} u(\mathbf{x},t)=0$$

for each $\sigma>\sigma^*$.

Theorem 5.2. (Thieme and Zhao [49, Theorem 2.2]) *Let* $F(u,\mathbf{x},r)=f(u)k(\mathbf{x},r)$. *Assume that* (B) *and* (C) *hold, and* f *is monotone increasing. Then for any Borel measurable function* $u_0:\mathbb{R}^n\times\mathbb{R}_+\to\mathbb{R}_+$ *with the property that* $u_0(x,t)\geq\eta>0$, $\forall t\in(t_1,t_2)$, $|\mathbf{x}|\leq\eta$, *for appropriate* $t_2>t_1\geq 0,\eta>0$, *there holds* $\liminf_{t\to\infty,|x|\leq\sigma t} u(\mathbf{x},t)=u^*$, $\forall\sigma\in(0,\sigma^*)$.

5.2.2. *Concrete proof of Lemma 3.6.* Let

$$f(u)=\begin{cases} g(u), & u\in[0,v_2^*],\\ g(v_2^*), & u>v_2^*,\end{cases}$$

where $g(\cdot)$ is defined by (22). Then $f(u)$ is increasing and satisfies the assumption (C). In particular, v_2^* is the positive solution of $u=k^*f(u)$ and satisfies $k^*f(u)>u$ for $u\in(0,v_2^*)$ and $k^*f(u)<u$ for $u>v_2^*$, where $k^*=\frac{k_1k_2(1-E_2^*)}{1-E_2^*+\Lambda_2}$.

Consider the following system

$$\begin{cases} \frac{\partial}{\partial t}u_1(\mathbf{x},t)=\Delta u_1(\mathbf{x},t)+k_1E_1^*f(u_2(\mathbf{x},t))-E_1^*u_1(\mathbf{x},t),\\ \frac{\partial}{\partial t}u_2(\mathbf{x},t)=d\Delta u_2(\mathbf{x},t)+rk_2(1-E_2^*)u_1(\mathbf{x},t)-r(1-E_2^*+\Lambda_2)u_2(\mathbf{x},t),\end{cases} \tag{39}$$

where $\mathbf{x}\in\mathbb{R}^3$ and $t>0$. System (39) has two equilibria $\mathbf{E}^0=(0,0)$ and $\mathbf{E}^{**}=(v_1^*,v_2^*)$. Therefore, the interval $\left[\mathbf{E}^0,\mathbf{E}^{**}\right]$ is invariant for the solution semiflows of system (39). System (39) also satisfies the comparison principle on $(u_1,u_2)\in[0,+\infty)^2$.

Lemma 5.3. *Assume that* $\phi=(\phi_1,\phi_2)\in C\left(\mathbb{R}^3,\left[\mathbf{E}^0,\mathbf{E}^{**}\right]\right)$ *satisfies that* $\phi_1(\cdot)+\phi_2(\cdot)\not\equiv 0$ *and for every* $\mu_1>0$*, there exists some* $\mu_2>0$ *such that* $\phi_1(\mathbf{x})+\phi_2(\mathbf{x})\leq\mu_2e^{-\mu_1|\mathbf{x}|}$ *for any* $\mathbf{x}\in\mathbb{R}^3$. *Then there exists* $\sigma^*>0$ *such that the solution* $\mathbf{u}(\mathbf{x},t;\phi)=(u_1(\mathbf{x},t;\phi),u_2(\mathbf{x},t;\phi))$ *of* (39) *with initial value* ϕ *satisfies:*

(i) $\mathbf{u}(\mathbf{x},t;\phi)\in\left[\mathbf{E}^0,\mathbf{E}^{**}\right]$ *for any* $\mathbf{x}\in\mathbb{R}^3$ *and* $t>0$;

(ii) $\lim_{t\to\infty,|\mathbf{x}|\geq\sigma t}\mathbf{u}(\mathbf{x},t)=\mathbf{E}^0$ *for any* $\sigma>\sigma^*$;

(iii) $\lim_{t\to\infty,|\mathbf{x}|\leq\sigma t}\mathbf{u}(\mathbf{x},t)=\mathbf{E}^{**}$ *for any* $\sigma<\sigma^*$.

Proof. By (39), we have

$$u_1(\mathbf{x},t)=e^{-E_1^*t}\Gamma(t)\phi_1(\mathbf{x})+k_1E_1^*\int_0^t e^{-E_1^*(t-s)}\Gamma(t-s)f(u_2(\mathbf{x},s))\,ds \tag{40}$$

and

$$\begin{aligned} u_2(\mathbf{x},t) &= e^{-r(1-E_2^*+\Lambda_2)t}\Gamma(dt)\phi_2(\mathbf{x})\\ &\quad+rk_2(1-E_2^*)\int_0^t e^{-r(1-E_2^*+\Lambda_2)(t-s)}\Gamma(d(t-s))u_1(\mathbf{x},s)\,ds,\end{aligned}$$

where

$$\Gamma(t)\varphi(\mathbf{x})=\int_{\mathbb{R}^3}\frac{1}{\left(\sqrt{4\pi t}\right)^3}e^{-\frac{|\mathbf{y}|^2}{4t}}\varphi(\mathbf{x}-\mathbf{y})\,d\mathbf{y}.$$

Consequently, we have

$$
\begin{aligned}
&u_2(\mathbf{x},t)\\
&= u_2^0(\mathbf{x},t) + rk_1k_2E_1^*(1-E_2^*)\int_0^t e^{-r(1-E_2^*+\Lambda_2)(t-s)}\int_{\mathbb{R}^3}(4\pi dt)^{-\frac{3}{2}}e^{-\frac{|\mathbf{y}|^2}{4d(t-s)}}\\
&\quad\times\int_0^s e^{-E_1^*(s-\tau)}\Gamma(s-\tau)f(u_2(\mathbf{x}-\mathbf{y},\tau))\,d\tau d\mathbf{y}ds\\
&= u_2^0(\mathbf{x},t) + rk_1k_2E_1^*(1-E_2^*)\int_0^t e^{-r(1-E_2^*+\Lambda_2)(t-s)}\int_{\mathbb{R}^3}(4\pi dt)^{-\frac{3}{2}}e^{-\frac{|\mathbf{y}|^2}{4d(t-s)}}\\
&\quad\times\int_0^s e^{-E_1^*(s-\tau)}\int_{\mathbb{R}^3}(4\pi t)^{-\frac{3}{2}}e^{-\frac{|\mathbf{z}|^2}{4t}}f(u_2(\mathbf{x}-\mathbf{y}-\mathbf{z},\tau))\,d\mathbf{z}d\tau d\mathbf{y}ds\\
&= u_2^0(\mathbf{x},t) + rk_1k_2E_1^*(1-E_2^*)\int_0^t e^{-r(1-E_2^*+\Lambda_2)(t-s)}\int_{\mathbb{R}^3}(4\pi dt)^{-\frac{3}{2}}e^{-\frac{|\mathbf{y}|^2}{4d(t-s)}}\\
&\quad\times\int_0^s e^{-E_1^*(s-\tau)}\int_{\mathbb{R}^3}(4\pi(s-\tau))^{-\frac{3}{2}}e^{-\frac{|\mathbf{z}-\mathbf{y}|^2}{4(s-\tau)}}f(u_2(\mathbf{x}-\mathbf{z},\tau))\,d\mathbf{z}d\tau d\mathbf{y}ds, \quad (41)
\end{aligned}
$$

where

$$
\begin{aligned}
u_2^0(\mathbf{x},t) &= e^{-r(1-E_2^*+\Lambda_2)t}\Gamma(dt)\phi_2(\mathbf{x}) + rk_2(1-E_2^*)\\
&\quad\times\int_0^t e^{-r(1-E_2^*+\Lambda_2)(t-s)}\Gamma(d(t-s))\left(e^{-E_1^*s}\Gamma(s)\phi_1(\mathbf{x})\right)ds\\
&= e^{-r(1-E_2^*+\Lambda_2)t}\int_{\mathbb{R}^3}(4\pi dt)^{-\frac{3}{2}}e^{-\frac{|\mathbf{y}|^2}{4dt}}\phi_2(\mathbf{x}-\mathbf{y})\,d\mathbf{y}\\
&\quad+rk_2(1-E_2^*)\int_0^t e^{-r(1-E_2^*+\Lambda_2)(t-s)}\\
&\quad\times\int_{\mathbb{R}^3}(4\pi d(t-s))^{-\frac{3}{2}}e^{-\frac{|\mathbf{y}|^2}{4(t-s)}}\left(e^{-E_1^*s}\Gamma(s)\phi_1(\mathbf{x}-\mathbf{y})\right)d\mathbf{y}ds\\
&= e^{-r(1-E_2^*+\Lambda_2)t}\int_{\mathbb{R}^3}(4\pi dt)^{-\frac{3}{2}}e^{-\frac{|\mathbf{y}|^2}{4dt}}\phi_2(\mathbf{x}-\mathbf{y})\,d\mathbf{y}\\
&\quad+rk_2(1-E_2^*)\int_0^t e^{-r(1-E_2^*+\Lambda_2)(t-s)}\int_{\mathbb{R}^3}(4\pi d(t-s))^{-\frac{3}{2}}e^{-\frac{|\mathbf{y}|^2}{4(t-s)}}e^{-E_1^*s}\\
&\quad\times\int_{\mathbb{R}^3}(4\pi s)^{-\frac{3}{2}}e^{-\frac{|\mathbf{z}|^2}{4s}}\phi_1(\mathbf{x}-\mathbf{y}-\mathbf{z})\,d\mathbf{z}d\mathbf{y}ds\\
&= e^{-r(1-E_2^*+\Lambda_2)t}\int_{\mathbb{R}^3}(4\pi dt)^{-\frac{3}{2}}e^{-\frac{|\mathbf{y}|^2}{4dt}}\phi_2(\mathbf{x}-\mathbf{y})\,d\mathbf{y}\\
&\quad+rk_2(1-E_2^*)\int_0^t e^{-r(1-E_2^*+\Lambda_2)(t-s)}\int_{\mathbb{R}^3}(4\pi d(t-s))^{-\frac{3}{2}}e^{-\frac{|\mathbf{y}|^2}{4(t-s)}}e^{-E_1^*s}\\
&\quad\times\int_{\mathbb{R}^3}(4\pi s)^{-\frac{3}{2}}e^{-\frac{|\mathbf{z}-\mathbf{y}|^2}{4s}}\phi_1(\mathbf{x}-\mathbf{z})\,d\mathbf{z}\;d\mathbf{y}ds.
\end{aligned}
$$

In the following we define appropriate $k(\mathbf{x},t)$ so that we can rewrite $u_2(\mathbf{x},t)$ in the form of (38) with the special case $F(u,\mathbf{x},t) = f(u)k(\mathbf{x},t)$. In particular, we show that assumption (B) holds in this case and $u_2^0(\mathbf{x},t)$ is admissible. Following

(41), we have

$$\begin{aligned}
& rk_1k_2E_1^*\left(1-E_2^*\right)\int_0^t e^{-r(1-E_2^*+\Lambda_2)(t-s)}\int_{\mathbb{R}^3}(4\pi d(t-s))^{-\frac{3}{2}}e^{-\frac{|\mathbf{y}|^2}{4d(t-s)}} \\
& \quad\times\int_0^s e^{-E_1^*(s-\tau)}\int_{\mathbb{R}^3}(4\pi(s-\tau))^{-\frac{3}{2}}e^{-\frac{|\mathbf{z}-\mathbf{y}|^2}{4(s-\tau)}}f\left(u_2\left(\mathbf{x}-\mathbf{z},\tau\right)\right)d\mathbf{z}d\tau d\mathbf{y}ds \\
= \; & rk_1k_2E_1^*\left(1-E_2^*\right)\int_0^t e^{-r(1-E_2^*+\Lambda_2)(t-s)}\int_0^s e^{-E_1^*(s-\tau)}\int_{\mathbb{R}^3}(4\pi(s-\tau))^{-\frac{3}{2}} \\
& \quad\times\int_{\mathbb{R}^3}(4\pi d(t-s))^{-\frac{3}{2}}e^{-\frac{|\mathbf{y}|^2}{4d(t-s)}}e^{-\frac{|\mathbf{z}-\mathbf{y}|^2}{4(s-\tau)}}d\mathbf{y}f\left(u_2\left(\mathbf{x}-\mathbf{z},\tau\right)\right)d\mathbf{z}d\tau ds \\
= \; & rk_2k_1E_1^*\left(1-E_2^*\right)\int_0^t\int_\tau^t e^{-r(1-E_2^*+\Lambda_2)(t-s)}e^{-E_1^*(s-\tau)}\int_{\mathbb{R}^3}(4\pi(s-\tau))^{-\frac{3}{2}} \\
& \quad\times\int_{\mathbb{R}^3}(4\pi d(t-s))^{-\frac{3}{2}}e^{-\frac{|\mathbf{y}|^2}{4d(t-s)}}e^{-\frac{|\mathbf{z}-\mathbf{y}|^2}{4(s-\tau)}}d\mathbf{y}f\left(u_2\left(\mathbf{x}-\mathbf{z},\tau\right)\right)d\mathbf{z}dsd\tau \\
= \; & rk_1k_2E_1^*\left(1-E_2^*\right)\int_0^t\int_0^{t-\tau} e^{-r(1-E_2^*+\Lambda_2)(t-s-\tau)}e^{-E_1^*s}\int_{\mathbb{R}^3}(4\pi s)^{-\frac{3}{2}} \\
& \quad\times\int_{\mathbb{R}^3}(4\pi d(t-s-\tau))^{-\frac{3}{2}}e^{-\frac{|\mathbf{y}|^2}{4d(t-s-\tau)}}e^{-\frac{|\mathbf{z}-\mathbf{y}|^2}{4s}}d\mathbf{y}f\left(u_2\left(\mathbf{x}-\mathbf{z},\tau\right)\right)d\mathbf{z}dsd\tau \\
= \; & rk_1k_2E_1^*\left(1-E_2^*\right)\int_0^t\int_{\mathbb{R}^3}\left[\int_0^{t-\tau}e^{-r(1-E_2^*+\Lambda_2)(t-s-\tau)}e^{-E_1^*s}\right. \\
& \quad\left.\times\int_{\mathbb{R}^3}\frac{e^{-\frac{|\mathbf{z}-\mathbf{y}|^2}{4s}}}{\left(\sqrt{4\pi s}\right)^3}\frac{e^{-\frac{|\mathbf{y}|^2}{4d(t-s-\tau)}}}{\left(\sqrt{4\pi d\left(t-s-\tau\right)}\right)^3}d\mathbf{y}ds\right]f\left(u_2\left(\mathbf{x}-\mathbf{z},\tau\right)\right)d\mathbf{z}d\tau.
\end{aligned}$$

Let

$$\widetilde{k}\left(\mathbf{z},t\right)=\int_0^t e^{-r(1-E_2^*+\Lambda_2)(t-s)}e^{-E_1^*s}\int_{\mathbb{R}^3}\frac{e^{-\frac{|\mathbf{z}-\mathbf{y}|^2}{4s}}}{\left(\sqrt{4\pi s}\right)^3}\frac{e^{-\frac{|\mathbf{y}|^2}{4d(t-s)}}}{\left(\sqrt{4\pi d\left(t-s\right)}\right)^3}d\mathbf{y}ds.$$

A direct calculation yields

$$\int_0^\infty\int_{\mathbb{R}^3}\widetilde{k}\left(\mathbf{z},t\right)d\mathbf{z}dt=\frac{1}{rE_1^*\left(1-E_2^*+\Lambda_2\right)}.$$

Set

$$k\left(\mathbf{z},t\right)=rk_1k_2E_1^*\left(1-E_2^*\right)\widetilde{k}\left(\mathbf{z},t\right).$$

It is easy to show that $k\left(\mathbf{z},t\right)$ satisfies the assumption (B). In particular, we have $k^*=\int_0^\infty\int_{\mathbb{R}^3}k\left(\mathbf{z},t\right)d\mathbf{z}dt=\frac{k_1k_2(1-E_2^*)}{1-E_2^*+\Lambda_2}>1$ due to the assumption (H).

Let

$$\begin{aligned}
\mathcal{K}\left(\sigma,\lambda\right) &= \int_0^\infty\int_{\mathbb{R}^3}e^{-\lambda(\sigma t-z_1)}k\left(\mathbf{z},t\right)d\mathbf{z}dt \\
&= \frac{rk_1k_2E_1^*\left(1-E_2^*\right)}{\left(E_1^*+\lambda\sigma-\lambda^2\right)\left(r\left(1-E_2^*+\Lambda_2\right)+\lambda\sigma-d\lambda^2\right)}
\end{aligned}$$

and

$$\lambda^\sharp(\sigma)=\min\left\{\frac{\sigma+\sqrt{\sigma^2+4E_1^*}}{2},\frac{\sigma+\sqrt{\sigma^2+4dr\left(1-E_2^*+\Lambda_2\right)}}{2d}\right\}\quad\text{for }\sigma>0.$$

Then we have $\mathcal{K}(\sigma,\lambda) < +\infty$ for $\lambda \in \left(0,\lambda^\sharp(\sigma)\right)$ and $\mathcal{K}(\sigma,\lambda) \to +\infty$ as $\lambda \to \lambda^\sharp(\sigma) - 0$. In particular, we have $\mathcal{K}(\sigma,0) = k^* > 1$ for any $\sigma \ge 0$ and

$$\mathcal{K}(0,\lambda) = \frac{rk_1k_2E_1^*\left(1-E_2^*\right)}{\left(E_1^*-\lambda^2\right)\left(r\left(1-E_2^*+\Lambda_2\right)-d\lambda^2\right)} < \infty$$

for any $0 < \lambda < \min\left\{\sqrt{E_1^*}, \sqrt{r\left(1-E_2^*+\Lambda_2\right)/\,d}\right\}$. By Thieme and Zhao [49, Proposition 2.3], there exist a unique $\sigma^* > 0$ and $\lambda^* > 0$ such that

$$\mathcal{K}(\sigma,\lambda) = 1 \quad \text{and} \quad \frac{d}{d\lambda}\mathcal{K}(\sigma,\lambda) = 0.$$

Now we show that $u_2^0(\mathbf{x},t)$ is admissible. Note that

$$\begin{aligned}
&u_2^0(\mathbf{x},t)\\
&\quad= e^{-r(1-E_2^*+\Lambda_2)t}\int_{\mathbb{R}^3}(4\pi dt)^{-\frac{3}{2}}\,e^{-\frac{|\mathbf{y}|^2}{4dt}}\phi_2(\mathbf{x}-\mathbf{y})\,d\mathbf{y}\\
&\qquad+rk_2\left(1-E_2^*\right)\int_0^t e^{-r(1-E_2^*+\Lambda_2)(t-s)}\int_{\mathbb{R}^3}(4\pi d(t-s))^{-\frac{3}{2}}\,e^{-\frac{|\mathbf{y}|^2}{4d(t-s)}}e^{-E_1^*s}\\
&\qquad\times\int_{\mathbb{R}^3}(4\pi s)^{-\frac{3}{2}}\,e^{-\frac{|\mathbf{z}-\mathbf{y}|^2}{4s}}\phi_1(\mathbf{x}-\mathbf{z})\,d\mathbf{z}\;d\mathbf{y}ds.
\end{aligned}$$

It is easy to see that

$$u_2^0(\mathbf{x},t) \le v_2^* e^{-r(1-E_2^*+\Lambda_2)t} + \frac{rk_2\left(1-E_2^*\right)v_1^*}{r\left(1-E_2^*+\Lambda_2\right)-E_1^*}\left[e^{-E_1^*t}-e^{-r(1-E_2^*+\Lambda_2)t}\right]$$

for any $\mathbf{x}\in\mathbb{R}^3$ and $t>0$, which implies that $u_2^0(\mathbf{x},t)$ converges to 0 uniformly in $\mathbf{x}\in\mathbb{R}^3$ as $t\to+\infty$. Give $\sigma>0$ and $\lambda\in\left(0,\lambda^\sharp(\sigma)\right)$ with $\mathcal{K}(\sigma,\lambda)<1$. In this case we have $d\lambda^2-\lambda\sigma-r\left(1-E_2^*+\Lambda_2\right)<0$ and $\lambda^2-\lambda\sigma-E_1^*<0$. It follows from the assumption of the lemma that there exists $\gamma>0$ such that $\phi_1(\mathbf{x})+\phi_2(\mathbf{x})\le\gamma e^{-\lambda|\mathbf{x}|}$ for any $\mathbf{x}\in\mathbb{R}^3$. For any $\mathbf{e}\in\mathbb{R}^3$ with $|\mathbf{e}|=1$, we have $\phi_1(\mathbf{x})+\phi_2(\mathbf{x})\le\gamma e^{\lambda\mathbf{e}\cdot\mathbf{x}}$ for any $\mathbf{x}\in\mathbb{R}^3$ due to the inequality $-|\mathbf{x}|\le\mathbf{e}\cdot\mathbf{x}\le|\mathbf{x}|$. Consequently, we obtain

$$\begin{aligned}
&e^{-r(1-E_2^*+\Lambda_2)t}\int_{\mathbb{R}^3}(4\pi dt)^{-\frac{3}{2}}\,e^{-\frac{|\mathbf{y}|^2}{4dt}}\phi_2(\mathbf{x}-\mathbf{y})\,d\mathbf{y}\\
&\quad\le e^{-r(1-E_2^*+\Lambda_2)t}\int_{\mathbb{R}^3}(4\pi dt)^{-\frac{3}{2}}\,e^{-\frac{|\mathbf{y}|^2}{4dt}}\gamma e^{\lambda\mathbf{e}\cdot\mathbf{x}}e^{-\lambda\mathbf{e}\cdot\mathbf{y}}d\mathbf{y}\\
&\quad=\gamma e^{\lambda\mathbf{e}\cdot\mathbf{x}}e^{-r(1-E_2^*+\Lambda_2)t}\int_{\mathbb{R}^3}(4\pi dt)^{-\frac{3}{2}}\,e^{-\frac{|\mathbf{y}|^2}{4dt}}e^{-\lambda y_1}d\mathbf{y}\\
&\quad=\gamma e^{\lambda\mathbf{e}\cdot\mathbf{x}}e^{\left(d\lambda^2-r(1-E_2^*+\Lambda_2)\right)t}\\
&\quad\le\gamma e^{\lambda(\sigma t+\mathbf{e}\cdot\mathbf{x})}
\end{aligned}$$

and

$$
\begin{aligned}
& rk_2\left(1-E_2^*\right)\int_0^t e^{-r\left(1-E_2^*+\Lambda_2\right)(t-s)}\int_{\mathbb{R}^3}\left(4\pi d(t-s)\right)^{-\frac{3}{2}}e^{-\frac{|\mathbf{y}|^2}{4(t-s)}}e^{-E_1^* s} \\
& \qquad \times\int_{\mathbb{R}^3}\left(4\pi s\right)^{-\frac{3}{2}}e^{-\frac{|\mathbf{z}-\mathbf{y}|^2}{4s}}\phi_1\left(\mathbf{x}-\mathbf{z}\right)d\mathbf{z}\,d\mathbf{y}ds \\
\leq\ & rk_2\left(1-E_2^*\right)\int_0^t e^{-r\left(1-E_2^*+\Lambda_2\right)(t-s)}\int_{\mathbb{R}^3}\left(4\pi d(t-s)\right)^{-\frac{3}{2}}e^{-\frac{|\mathbf{y}|^2}{4(t-s)}}e^{-E_1^* s} \\
& \times\int_{\mathbb{R}^3}\left(4\pi s\right)^{-\frac{3}{2}}e^{-\frac{|\mathbf{z}-\mathbf{y}|^2}{4s}}\gamma e^{\lambda\mathbf{e}\cdot\mathbf{x}}e^{-\lambda\mathbf{e}\cdot\mathbf{z}}d\mathbf{z}\,d\mathbf{y}ds \\
=\ & rk_2\left(1-E_2^*\right)\gamma e^{\lambda\mathbf{e}\cdot\mathbf{x}}\int_0^t e^{-r\left(1-E_2^*+\Lambda_2\right)(t-s)}\int_{\mathbb{R}^3}\left(4\pi d(t-s)\right)^{-\frac{3}{2}}e^{-\frac{|\mathbf{y}|^2}{4(t-s)}}e^{-E_1^* s} \\
& \times\int_{\mathbb{R}^3}\left(4\pi s\right)^{-\frac{3}{2}}e^{-\frac{|\mathbf{z}-\mathbf{y}|^2}{4s}}e^{-\lambda z_1}d\mathbf{z}\,d\mathbf{y}ds \\
=\ & rk_2\left(1-E_2^*\right)\gamma e^{\lambda\mathbf{e}\cdot\mathbf{x}}\int_0^t e^{-r\left(1-E_2^*+\Lambda_2\right)(t-s)}e^{d\lambda^2(t-s)}e^{-E_1^* s}e^{\lambda^2 s}ds \\
\leq\ & rk_2\left(1-E_2^*\right)\gamma e^{\lambda\mathbf{e}\cdot\mathbf{x}}\int_0^t e^{\lambda\sigma(t-s)}e^{-E_1^* s}e^{\lambda^2 s}ds \\
\leq\ & \frac{r\gamma k_2\left(1-E_2^*\right)}{E_1^*+\lambda\sigma-\lambda^2}e^{\lambda(\sigma t+\mathbf{e}\cdot\mathbf{x})}
\end{aligned}
$$

for any $\mathbf{x}\in\mathbb{R}^3$ and $t\geq 0$. Letting $\mathbf{e}=-\frac{\mathbf{x}}{|\mathbf{x}|}$ yields

$$
u_2^0\left(\mathbf{x},t\right)\leq\left(\gamma+\frac{r\gamma k_2\left(1-E_2^*\right)}{E_1^*+\lambda\sigma-\lambda^2}\right)e^{\lambda(\sigma t-|\mathbf{x}|)},\quad\forall\ \mathbf{x}\in\mathbb{R}^3,\ t\geq 0,
$$

which implies that $u_2^0\left(\mathbf{x},t\right)$ is admissible.

Let $F(u,\mathbf{x},s)=f(u)k(\mathbf{x},s)$. It is easy to show that the assumption (A) holds. In addition, we have the integral equation

$$
u_2\left(\mathbf{x},t\right)=u_2^0\left(\mathbf{x},t\right)+\int_0^t\int_{\mathbb{R}^3}k(\mathbf{y},t-\tau)f\left(u_2\left(\mathbf{x}-\mathbf{y},\tau\right)\right)d\mathbf{y}d\tau. \tag{42}
$$

It follows from Thieme and Zhao [49, Proposition 2.1] that for any bounded $u_2^0(\mathbf{x},t)$, the integral equation (42) has a unique solution which is bounded on $(\mathbf{x},t)\in\mathbb{R}^3\times[0+\infty)$. It is obvious that the second component $u_2(\mathbf{x},t)$ of the solution of system (39) is a solution of (42). Thus, the result of the lemma for $u_2(\mathbf{x},t)$ follows from Theorems 5.1 and 5.2. Consequently, the result of the lemma for $u_1(\mathbf{x},t)$ follows from an argument on (40), see [49, Theorem 4.4]. This completes the proof. □

Now we prove Lemma 3.6.

Proof of Lemma 3.6: Assume that $\phi=(\phi_1,\phi_2)\in C\left(\mathbb{R}^3,\left[\mathbf{E}^0,\mathbf{E}^{**}\right]\right)$ is compactly supported with $\phi_1(\cdot)+\phi_2(\cdot)\not\equiv 0$. It is easy to show that ϕ satisfies the condition of Lemma 5.3. By the comparison principle, we have $\widetilde{\widetilde{\mathbf{u}}}(\mathbf{x},t;\phi)\in\left[\mathbf{E}^0,\mathbf{E}^{**}\right]$ for any $\mathbf{x}\in\mathbb{R}^3$ and $t>0$. By the definition of f, we have that the solution $\widetilde{\widetilde{\mathbf{u}}}(\mathbf{x},t;\phi)$ of (23) is also a solution of (39). Applying Lemma 5.3, we know that the conclusions of Lemma 3.6 hold. This completes the proof of Lemma 3.6. □

6. **Discussion.** Under the assumptions that $k_1,k_2>1$ and $c>0$, in this paper we have established the existence of axisymmetric traveling fronts of a two-species Lotka-Volterra competition-diffusion system in $\mathbb{R}^3$ for any $s>c$ and demonstrated

some important qualitative properties, such as monotonicity, of the axisymmetric traveling fronts. When s tends to c, we showed that the axisymmetric traveling fronts converge locally uniformly to the planar traveling wave fronts in $\mathbb{R}^3$. Furthermore, we showed the nonexistence of axisymmetric traveling fronts with convex level set. Note that the nonexistence results of Theorems 1.3 and 1.4 remain valid for two-dimensional V-shaped traveling fronts and high-dimensional pyramidal traveling fronts.

Due to the effect of the coupled nonlinearity, in this paper we did not consider the behavior of level sets of the axisymmetric traveling fronts at infinity. We conjecture that the level set admits an asymptotic behavior similar to that for the scalar equations, see Hamel et al. [19, 20] and Taniguchi [48]. Another natural problem is the uniqueness and stability of the axisymmetric traveling fronts. We leave these for our future studies. In addition, in the current paper we only considered the case $c > 0$. For the case $c = 0$, it should be expected that there exist more complex dynamics, such as those obtained for the balanced Allen-Cahn equation. This case is very interesting and remains open.

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