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Existence and stability of traveling wave fronts in reaction advection diffusion equations with nonlocal delay

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Abstract

This paper is concerned with the existence, uniqueness and globally asymptotic stability of traveling wave fronts in the quasi-monotone reaction advection diffusion equations with nonlocal delay. Under bistable assumption, we construct various pairs of super- and subsolutions and employ the comparison principle and the squeezing technique to prove that the equation has a unique nondecreasing traveling wave front (up to translation), which is monotonically increasing and globally asymptotically stable with phase shift. The influence of advection on the propagation speed is also considered. Comparing with the previous results, our results recovers and/or improves a number of existing ones. In particular, these results can be applied to a reaction advection diffusion equation with nonlocal delayed effect and a diffusion population model with distributed maturation delay, some new results are obtained.
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1. Introduction

In this paper, we are concerned with an one space dimensional reaction advection diffusion equation with nonlocal delay of the form

$$\frac{\partial u}{\partial t} = d\Delta u + B\frac{\partial u}{\partial x} + g\big(u(x,t), \big(h * S(u)\big)(x,t)\big), \quad x \in \mathbb{R},\ t > 0, \tag{1.1}$$

where $d > 0$, $B \in \mathbb{R}$, Δ is the Laplacian operator on $\mathbb{R}$, h is a nonnegative kernel satisfying

$$\int_0^\tau \int_{-\infty}^{\infty} h(y,s)\,dy\,ds = 1, \qquad \int_0^\tau \int_{-\infty}^{\infty} |y|h(y,s)\,dy\,ds < \infty, \tag{1.2}$$

and the convolution is defined by

$$\big(h * S(u)\big)(x,t) = \int_{-\tau}^{0} \int_{-\infty}^{\infty} h(x-y,-s)S\big(u(y,t+s)\big)\,dy\,ds.$$

For $g(u,v)$ and $S(u)$, we impose the following conditions:

(H1) $g \in C^2([0,1] \times [S(0), S(1)], \mathbb{R})$ and $\partial_2 g(u,v) \geqslant 0$ for $(u,v) \in [0,1] \times [S(0), S(1)]$; $S \in C^2([0,1], \mathbb{R})$ and $S'(u) \geqslant 0$ for $u \in [0,1]$.

(H2) $g(0, S(0)) = g(1, S(1)) = 0$, $\partial_1 g(0, S(0)) + \partial_2 g(0, S(0))S'(0) < 0$, and $\partial_1 g(1, S(1)) + \partial_2 g(1, S(1))S'(1) < 0$.

Under condition (H2), it is obvious that 0 and 1 are stable equilibria of (1.1). We are interested in traveling wave solutions that connect the two stable equilibria 0 and 1. Throughout this paper, a *traveling wave solution* of (1.1) always refers to a pair (U, c), where $U = U(\xi)$ is a function on $\mathbb{R}$ and c is a constant, such that $u(x,t) := U(x - ct)$ is a solution of (1.1) and

$$\lim_{\xi \to -\infty} U(\xi) = 0, \qquad \lim_{\xi \to +\infty} U(\xi) = 1. \tag{1.3}$$

We call c the *traveling wave speed* and U the profile of the wave front. If $c = 0$, we say U is a *standing wave*. Moreover, we say a traveling wave $U(x - ct)$ is monotone if $U(\cdot) : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function.

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For some special cases of Eq. (1.1), many well-known results have been obtained under the bistable assumption. Some of them can be summarized as follows:

(i) If $B = 0$, $S(u) = u$ and $h(x,t) = \delta(t)\delta(x)$, $\delta(\cdot)$ is the Dirac delta function, then (1.1) reduces to the local equation without delay

$$\frac{\partial u}{\partial t} = d\Delta u + g(u,u), \quad x \in \mathbb{R},\ t > 0. \tag{1.4}$$

In [20], Fife and McLeod have proved the globally exponential stability of traveling wave solutions of (1.4), see also Volpert et al. [44].

(ii) If $B = 0$, $S(u) = u$ and $h(x,t) = \delta(t-\tau)\delta(x)$, then (1.1) reduces to the local equation with a discrete delay

$$\frac{\partial u}{\partial t} = d\Delta u + g\big(u(x,t), u(x,t-\tau)\big), \quad x \in \mathbb{R},\ t > 0,\ \tau > 0. \tag{1.5}$$

Schaaf [38] considered Eq. (1.5) for the so-called Huxley nonlinearity as well as Fisher nonlinearity. He studied the existence of traveling wave solutions in such equations by using the phase-plane technique, the maximum principle for parabolic functional differential equations, and the general theory for ordinary functional differential equations. Smith and Zhao [42] proved the global asymptotic stability, Lyapunov stability and uniqueness of traveling wave solutions of (1.5) with *bistable* nonlinear term, by first establishing the existence and comparison theorem of solutions for (1.5), where they appealed to the theory of abstract functional differential equations of Martin and Smith [35], and then using the elementary sub- and supersolutions comparison and the squeezing technique developed by Chen [13].

(iii) If $B = 0$, $h(x,t) = \delta(t)J(x)$, then (1.1) reduces to the nonlocal equation

$$\frac{\partial u}{\partial t} = d\Delta u + g\left(u(x,t), \int_{-\infty}^{\infty} J(x-y)S\big(u(y,t)\big)\,dy\right), \quad x \in \mathbb{R},\ t > 0, \tag{1.6}$$

which was considered by Chen [13]. He proved the existence, uniqueness and global asymptotic stability of traveling wave solutions by developing the so-called squeezing technique. See also Alikakos et al. [1], Berestycki and Nirenberg [6], Chen [12], Chen and Guo [14,15], Ermentrout and McLeod [18], Evans et al. [19], Fife and McLeod [20], Ma and Zou [29,30] and Shen [39,40] for similar results related to this technique.

(iv) If $B = 0$, $g(u,v) = -\alpha u + v$, $S(u) = b(u)$ and $h(x,t) = \delta(t-\tau)J(x)$, then (1.1) reduces to the nonlocal equation

$$\frac{\partial u}{\partial t} = d\Delta u - \alpha u(x,t) + \int_{-\infty}^{\infty} J(x-y)b\big(u(y,t-\tau)\big)\,dy, \quad x \in \mathbb{R},\ t > 0,\ \tau > 0, \tag{1.7}$$

which was studied by Ma and Wu [28]. Under the bistable assumption, by establishing the existence and comparison theorem of solutions for (1.7), which is similar to that of Smith and Zhao [42], they proved the uniqueness and global asymptotic stability of traveling wave solutions by using the moving plane technique and the squeezing technique. Moreover, they proved the existence of traveling wave solutions by considering a nonlocal equation without delay, which

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is similar to the approach of Chen [13], and then passing to Eq. (1.7). In fact, this method is also used by Chen [11] for a neural network model and Ou and Wu [36] for a delayed hyperbolic–parabolic model.

Observing the above equations (1.4)–(1.7), we can see that these equations are either *local* or *nonlocal*, either with a *discrete delay* or without delay. In some situations (for example, when it models a feedback signal transmitted as a nerve impulse as discussed in Wu [48]), discrete delay is a good approximation, but in other situations (for example, pollution of an environment by dead organisms is clearly a cumulative effect), discrete delay is not realistic. However, even when a discrete delay can be regarded as a good approximation, there is likely to be some spread of the delay around some mean value and the use of a *distributed delay* can be regarded as allowing for stochastic effects in what is otherwise a deterministic model (see [16,26,31]). In fact, Volterra [45] already used a logistic equation with distributed delay to examine a cumulative effect on mortality of a deteriorating environment due to the accumulation of waste products and dead organisms.

In view of individuals taking time to move, spatial dispersal/diffusion was dealt with by simply adding a diffusion term to corresponding delayed ODE model in previous literatures, namely, adding a Laplacian term to the ODE model. But in recent years it has become recognized that there are modelling difficulties with this approach. The difficulty is that diffusion and time delay are independent of each other, since individuals have not been at the same point in space at previous times. Britton [9] made the first comprehensive attempt to address this difficulty by introducing a *nonlocal delay*, that is, the delay term involves a weighted spatial–temporal average over the whole of the infinite domain and the whole of the previous times. Another approach to overcome this difficulty was developed by Smith and Thieme [41], where the technique of integration along characteristics was used to derive a system of (ordinary) delay differential equations for the matured population of single species with two age classes (immature and mature) and with spatial dispersal among discrete patches. Since then, great progress has been made on the existence of travelling wave fronts in reaction–diffusion equations with nonlocal delays, see Ashwin et al. [2], Al-Omari and Gourley [3], Billingham [8], Gourley [22,23], Gourley and Kuang [24], Gourley and Ruan [25], Ruan and Xiao [37], So et al. [43], Wang et al. [46] and Zou [49]. Notice that all these equations are *monostable*.

Though there have been many results for reaction–diffusion equations with bistable nonlinearity and nonlocal delays, some new problems have arisen recently. In [27], Liang and Wu derived a reaction advection diffusion equation with nonlocal delayed effects of the form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(D_m \frac{\partial u}{\partial x} + Bu\right) - d_m u(x,t) + \varepsilon \int_{-\infty}^{\infty} J_\alpha(x + B\tau - y) b\big(u(y, t-\tau)\big)\, dy, \tag{1.8}$$

where $J_\alpha(x) = \frac{1}{\sqrt{4\pi\alpha}} e^{\frac{-x^2}{4\alpha}}$, $\tau > 0$, is the time delay, ε reflects the impact of the death rate of the immature, α represents the effect of the dispersal rate of the immature on the growth rate of the matured population, and B is the velocity of the spatial transport field. By choosing three different birth functions $b(u)$, they established the existence of traveling wave fronts of Eq. (1.8). We note that they only considered (1.8) with monostable nonlinearity. The bistable case remains open.

In fact, reaction advection diffusion equations are widely used to model some reaction–diffusion processes taking place in moving media such as fluids, for example, combustion,

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atmospheric chemistry, and plankton distributions in the sea, see Berestycki [7], Cencini et al. [10], Gilding and Kersner [21] and the references therein. Of particular interest is the influence of advection terms on the propagation of traveling wave fronts, which were studied by many researchers, see Berestycki [7], Gilding and Kersner [21], Malaguti and Marcelli [32,33], Malaguti et al. [34]. However, in their works, the delay and nonlocal effect were not considered.

Recently, Al-Omari and Gourley [4] rigorously derived a nonlocal reaction–diffusion model for a single population with stage structure and distributed maturation delay, namely,

$$\begin{cases} \dfrac{\partial u_i}{\partial t} = D_i \Delta u_i + b\big(u_m(x,t)\big) - \gamma u_i(x,t) - \displaystyle\int_0^\tau \int_\Omega G(x,y,s) f(s) e^{-\gamma s} b\big(u_m(y,t-s)\big)\,dy\,ds, \\ \dfrac{\partial u_m}{\partial t} = D_m \Delta u_m - d\big(u_m(x,t)\big) + \displaystyle\int_0^\tau \int_\Omega G(x,y,s) f(s) e^{-\gamma s} b\big(u_m(y,t-s)\big)\,dy\,ds, \end{cases} \tag{1.9}$$

where $\Omega \subset \mathbb{R}^N$ is open and bounded, $G(x,y,t)$ is the solution subject to homogeneous Neumann boundary condition of

$$\frac{\partial G}{\partial t} = D_i \Delta_x G, \qquad G(x,y,0) = \delta(x-y).$$

If the bounded domain Ω is replaced by the whole real line $(-\infty,\infty)$, then the second equation of (1.9) reduces to

$$\frac{\partial u_m}{\partial t} = D_m \Delta u_m - d\big(u_m(x,t)\big) + \int_0^\tau \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi D_i s}} e^{\frac{-(x-y)^2}{4D_i s}} f(s) e^{-\gamma s} b\big(u_m(y,t-s)\big)\,dy\,ds. \tag{1.10}$$

For some special cases of (1.10), the existence of traveling wave fronts has been studied by many authors. For example, Gourley and Kuang [24] and Al-Omari and Gourley [5] considered the case where $d(u_m(x,t)) = \alpha u_m^2(x,t)$ and $b(u_m(x,t)) = \beta u_m(x,t)$; So et al. [43] studied the case where $d(u_m(x,t)) = \alpha u_m(x,t)$, $b(u_m(x,t)) = \beta u_m(x,t) e^{-a u_m(x,t)}$. We also note that, under the bistable assumptions, system (1.7) considered by Ma [28] is also a special case of (1.10) with discrete delay. However, to the best of our knowledge, there is no result for Eq. (1.10) with bistable nonlinearity and distributed delay.

Motivated by the above discussion, in this paper we treat the existence, uniqueness and global asymptotic stability of traveling wave fronts of (1.1) under the bistable assumptions, that is, (H2). The assumption (H1) is necessary to establish the comparison theorem for the Cauchy problem of (1.1). Contrasting to [13,42], we only require that the quasi-monotone condition holds on $[0,1]^2$, namely, (H1) holds. Under (H1), as showed in Section 2, if the Cauchy-type initial value lies between 0 and 1, then the solutions of (1.1) also lie between 0 and 1. Though some special models can be modified so that the quasi-monotone condition holds on a larger domain by extending the nonlinearity, for example, Eq. (1.5) with the Huxley nonlinearity and Eq. (1.7), in which nonlinearities were extended to $[-\delta, 1+\delta]$ and $\mathbb{R}^2$, respectively, see [28,42]. However, we do not know whether each g satisfying (H1) can be extended to $\tilde{g} \in C^2([-\delta, 1+\delta] \times$

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$[S(0)-\varepsilon, S(1)+\varepsilon]$) with $\partial_2\tilde{g}(u,v) \geqslant 0$ for $(u,v) \in [-\delta, 1+\delta] \times [S(0)-\varepsilon, S(1)+\varepsilon]$. Therefore, in this paper we use different super- and subsolutions from that in [13,28,42], which was used in our paper [47]. Thus, we can discuss Eq. (1.1) only under condition (H1).

In order to take advantage of our estimates for super- and subsolutions, we make the following extensions for g and S. Define a function $\hat{S}:[-9,10] \to \mathbb{R}$ by

$$\hat{S}(u) = \begin{cases} S(0) + S'(0)u - u^2, & u \in [-9,0], \\ S(u), & u \in (0,1), \\ S(1) + S'(1)(u-1) + (u-1)^2, & u \in [1,10]. \end{cases}$$

Then define $\hat{g}:[0,1] \times [S(-9), S(10)] \to \mathbb{R}$ by

$$\hat{g}(u,v) = \begin{cases} g(u,S(0)) + \partial_2 g(u,S(0))(v - S(0)), & (u,v) \in [0,1] \times [S(-9), S(0)], \\ g(u,v), & (u,v) \in [0,1] \times [S(0), S(1)], \\ g(u,S(1)) + \partial_2 g(u,S(1))(v - S(1)), & (u,v) \in [0,1] \times [S(1), S(10)]. \end{cases}$$

Obviously, $\hat{S}'(u)$ is continuous and nonnegative on $[-9,10]$, $\partial_1\hat{g}(u,v)$ is continuous on $[0,1] \times [S(0), S(1)]$, and $\partial_2\hat{g}(u,v)$ is continuous and nonnegative on $[0,1] \times [S(-9), S(10)]$. For the sake of convenience, we still denote $\hat{S}$ and $\hat{g}$ by S and g in the remainder of this paper.

The rest of the paper is organized as follows. In Section 2, we establish the existence and comparison principle of solutions for the initial value problem of (1.1). In Sections 3 and 4, by constructing some super- and subsolutions of (1.1) and using the comparison result established in Section 2 and the squeezing technique of Chen [13] and Smith and Zhao [42], we consider the uniqueness and asymptotic stability of traveling wave fronts, respectively. In Section 5, we first consider the existence of traveling wave fronts for a class of reaction–diffusion equation without delay, where the method of Chen [13] can be applied, and then obtain the existence of traveling wave fronts of (1.1). In particular, in Lemma 5.4, we use a uniformly continuous function $H(x)$, which is different from the Heaviside function used by Chen [13] and Ma and Wu [28], so that the existence and comparison principle can still be used. Thus, we exactly and rigorously show the existence of traveling wave fronts. In Section 6, we apply our results to the above equations (1.8) and (1.10) and obtain some new results.

2. Existence and comparison of solutions

Let $X = BUC(\mathbb{R},\mathbb{R})$ be the Banach space of all bounded and uniformly continuous functions from $\mathbb{R}$ into $\mathbb{R}$ with the usual supremum norm. Let

$$X^+ = \{\varphi \in X\colon\ \varphi(x) \geqslant 0,\ x \in \mathbb{R}\}.$$

It is easy to see that X^+ is a closed cone of X and X is a Banach lattice under the partial ordering induced by X^+. By [17, Theorem 1.5], it then follows that the X-realization $d\Delta_X$ of $d\Delta$ generates a strongly continuous analytic semigroup $T(t)$ on X and $T(t)X^+ \subset X^+$, $t \geqslant 0$. Moreover, by the explicit expression of solutions of the heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = d\Delta u, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = \varphi(x), & x \in \mathbb{R}, \end{cases} \tag{2.1}$$

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we have

$$T(t)\varphi(x)=\frac{1}{\sqrt{4\pi dt}}\int_{-\infty}^{\infty}\exp\left(-\frac{(x-y)^2}{4dt}\right)\varphi(y)\,dy,\quad x\in\mathbb{R},\ t>0,\ \varphi(\cdot)\in X.$$

Consider the following equation

$$\begin{cases}\dfrac{\partial v}{\partial t}=d\Delta v+B\dfrac{\partial v}{\partial x}, & x\in\mathbb{R},\ t>0,\\ v(x,0)=\varphi(x), & x\in\mathbb{R}.\end{cases}\tag{2.2}$$

In fact, if $u(x,t)$ is the solution of (2.1), then $v(x,t)=u(x+Bt,t)$ is a solution of (2.2). Inversely, if $v(x,t)$ is a solution of (2.2), then $u(x,t)=v(x-Bt,t)$ is a solution of (2.1). Thus, the existence and uniqueness of solutions of (2.2) follow from the existence and uniqueness of solutions of (2.1). In particular,

$$v(x,t)=u(x+Bt,t)=\frac{1}{\sqrt{4\pi dt}}\int_{-\infty}^{\infty}\exp\left(-\frac{(x+Bt-y)^2}{4dt}\right)\varphi(y)\,dy.$$

Define bounded linear operators $U(t):X\to X$, $t\geqslant 0$, by

$$\begin{aligned}&U(0)\varphi(x)=\varphi(x),\\&U(t)\varphi(x)=\frac{1}{\sqrt{4\pi dt}}\int_{-\infty}^{\infty}\exp\left(-\frac{(x+Bt-y)^2}{4dt}\right)\varphi(y)\,dy,\quad x\in\mathbb{R},\ t>0,\ \varphi(\cdot)\in X.\end{aligned}\tag{2.3}$$

It is easy to prove that $U(t)$ is a strongly continuous semigroup on X. Obviously, $U(t)X^+\subset X^+$, $t\geqslant 0$. In particular, when $B=0$, $U(t)=T(t)$.

Let $f_0(\cdot):[0,1]\to\mathbb{R}$ be defined by $f_0(u)=g(u,S(u))$, $u\in[0,1]$. By the continuity of f_0 and condition (H2), it then easily follows that there exist $a^-,a^+\in(0,1)$ with $a^-\leqslant a^+$ such that $f_0(\cdot):[0,1]\to\mathbb{R}$ satisfies

$$\begin{cases}f_0(0)=f_0(a^-)=f_0(a^+)=f_0(1)=0,\\ f_0(u)>0\quad\text{for } u\in(a^+,1),\quad\text{and}\quad f_0(u)<0\quad\text{for } u\in(0,a^-).\end{cases}$$

Let $L_1=\max\{|\partial_1 g(u,v)|:\ 0\leqslant u\leqslant 1,\ S(0)\leqslant v\leqslant S(1)\}$ and define

$$\Theta(J,t)=\frac{1}{\sqrt{4\pi dt}}\exp\left(-\frac{|B|J}{2d}-\left(L_1+\frac{|B|}{4d}\right)t-\frac{(J+1)^2}{4dt}\right),\quad J\geqslant 0,\ t>0.$$

Clearly, $\Theta\in C([0,\infty)\times(0,\infty),\mathbb{R})$.

Let $C=C([-\tau,0],X)$ be the Banach space of continuous functions from $[-\tau,0]$ into X with the supremum norm, and let $C^+=\{\varphi\in C:\ \varphi(s)\in X^+,\ s\in[-\tau,0]\}$. Then C^+ is a positive cone of C. As usual, we identify an element $\varphi\in C$ as a function from $\mathbb{R}\times[-\tau,0]$ into $\mathbb{R}$ defined by $\varphi(x,s)=\varphi(s)(x)$. For any continuous function $w:[-\tau,b)\to X$, $b>0$, we define $w_t\in C$,

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$t \in [0, b)$, by $w_t(s) = w(t + s)$, $s \in [-\tau, 0]$. Then $t \mapsto w_t$ is a continuous function from $[0, b)$ to C. For any $\varphi \in [0, 1]_C = \{\varphi \in C\colon \varphi(x, s) \in [0, 1],\ x \in \mathbb{R},\ s \in [-\tau, 0]\}$, define

$$F(\varphi)(x) = g\left(\varphi(x, 0), \int_{-\tau}^{0} \int_{-\infty}^{\infty} h(x - y, -s) S\big(\varphi(y, s)\big)\, dy\, ds\right).$$

By the global Lipschitz continuity of $g(\cdot,\cdot)$ on $[0, 1] \times [S(0), S(1)]$, we can verify that $F(\varphi) \in X$ and $F : [0, 1]_C \to X$ is globally Lipschitz continuous.

Definition 2.1. A continuous function $v : [-\tau, b) \to X$, $b > 0$, is called a supersolution (subsolution) of (1.1) on $[0, b)$ if

$$v(t) \geqslant (\leqslant)\, U(t - s)v(s) + \int_{s}^{t} U(t - r)F(v_r)\, dr, \tag{2.4}$$

for all $0 \leqslant s < t < b$. If v is both a supersolution and a subsolution on $[0, b)$, then it is said to be a mild solution of (1.1).

Remark 2.2. Assume that there is a $v \in BUC(\mathbb{R} \times [-\tau, b), \mathbb{R})$, $b > 0$, such that v is C^2 in $x \in \mathbb{R}$, C^1 in $t \in (0, b)$ and for $x \in \mathbb{R}$, $t \in (0, b)$

$$\frac{\partial v}{\partial t} \geqslant (\leqslant)\, d\Delta v + B\frac{\partial v}{\partial x} + g\left(v(x, t), \int_{-\tau}^{0} \int_{-\infty}^{\infty} h(x - y, -s) S\big(v(y, t + s)\big)\, dy\, ds\right). \tag{2.5}$$

Then, the positivity of the linear semigroup $U(t) : X^+ \to X^+$ implies that (2.4) holds. Hence v is a supersolution (subsolution) of (1.1) on $[0, b)$.

We now establish the following existence and comparison result.

Theorem 2.3. *Assume that* (H1) *and* (H2) *hold. Then for any* $\varphi \in [0, 1]_C$, (1.1) *has a unique mild solution* $u(x, t, \varphi)$ *on* $[0, \infty)$ *which is a classical solution to* (1.1) *for* $(x, t) \in \mathbb{R} \times (\tau, \infty)$. *Furthermore, for any pair of supersolution* $\varphi^+(x, t)$ *and subsolution* $\varphi^-(x, t)$ *of* (1.1) *on* $[0, b)$ *with* $0 \leqslant \varphi^+(x, t), \varphi^-(x, t) \leqslant 1$ *for* $x \in \mathbb{R}$, $t \in [-\tau, b)$, *and* $\varphi^+(x, s) \geqslant \varphi^-(x, s)$ *for* $x \in \mathbb{R}$, $s \in [-\tau, 0]$, $0 < b \leqslant \infty$, *we have* $\varphi^+(x, t) \geqslant \varphi^-(x, t)$ *for* $x \in \mathbb{R}$, $0 \leqslant t < b$, *and*

$$\varphi^+(x, t) - \varphi^-(x, t) \geqslant \Theta(J, t - t_0) \int_{z}^{z+1} \big(\varphi^+(y, t_0) - \varphi^-(y, t_0)\big)\, dy$$

for any $J \geqslant 0$, x *and* $z \in \mathbb{R}$ *with* $|x - z| \leqslant J$, *and* $b > t > t_0 \geqslant 0$.

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Proof. Under an abstract setting as in [35], a mild solution of (1.1) is a solution to its associated integral equation

$$\begin{cases} u(t) = U(t)\varphi(0) + \int_0^t U(t-s)F(u_s)\,ds, \quad t>0, \\ u_0 = \varphi \in [0,1]_C. \end{cases}$$

We firstly prove the conclusions except the last inequality in the theorem when $B=0$. Assume that $B=0$. In this case, $U(t)=T(t)$. Clearly, $v^+=1$ and $v^-=0$ are an ordered pair of super- and subsolutions of (1.1) on $[0,\infty)$. Notice that $F:[0,1]_C \to X$ is globally Lipschitz continuous. We further claim that F is quasi-monotone on $[0,1]_C$ in the sense that

$$\lim_{h\to 0^+} \frac{1}{h} \operatorname{dist}\big(\psi(0)-\phi(0)+h\big[F(\psi)-F(\phi)\big]; X^+\big) = 0$$

for all $\psi,\phi \in [0,1]_C$ with $\psi \geqslant \phi$. In fact, it follows from (H1) that

$$\begin{aligned} F(\psi)-F(\phi) &= g\left(\psi(\cdot,0), \int_{-\tau}^{0}\int_{-\infty}^{\infty} h(\cdot-y,-s)S\big(\psi(y,s)\big)\,dy\,ds\right) \\ &\quad - g\left(\phi(\cdot,0), \int_{-\tau}^{0}\int_{-\infty}^{\infty} h(\cdot-y,-s)S\big(\phi(y,s)\big)\,dy\,ds\right) \\ &\geqslant g\left(\psi(\cdot,0), \int_{-\tau}^{0}\int_{-\infty}^{\infty} h(\cdot-y,-s)S\big(\phi(y,s)\big)\,dy\,ds\right) \\ &\quad - g\left(\phi(\cdot,0), \int_{-\tau}^{0}\int_{-\infty}^{\infty} h(\cdot-y,-s)S\big(\phi(y,s)\big)\,dy\,ds\right) \\ &\geqslant -L_1\big(\psi(0)-\phi(0)\big) \quad \text{in } X. \end{aligned} \tag{2.6}$$

Hence, for any $\mu>0$ such that $L_1\mu<1$,

$$\psi(0)-\phi(0)+\mu\big[F(\psi)-F(\phi)\big] \geqslant (1-L_1\mu)\big(\psi(0)-\phi(0)\big) \geqslant 0 \quad \text{in } X.$$

Then when $B=0$, the existence and uniqueness of $u(x,t,\varphi)$ follows from [35, Corollary 5] with $S(t,s)=T(t,s)$, $t\geqslant s\geqslant 0$, and $\mathcal{B}(t,\varphi)=F(\varphi)$. Moreover, by a semigroup theory argument given in the proof of [35, Theorem 1], it follows that $u(x,t,\varphi)$ is a classical solution for $t>\tau$.

Assume $B=0$ and $\varphi^+(x,t)$ and $\varphi^-(x,t)$ are a pair of super- and subsolutions of (1.1). For simplicity, let $\psi(x,s)=\varphi^+(x,s)$, $\phi(x,s)=\varphi^-(x,s)$, $x\in\mathbb{R}$, $s\in[-\tau,0]$. Then $\psi,\phi\in[0,1]_C$ and $\psi\geqslant\phi$ in C. Again by [35, Corollary 5], we have

$$0 \leqslant u(x,t,\phi) \leqslant u(x,t,\psi) \leqslant 1, \quad x\in\mathbb{R},\ b>t\geqslant 0. \tag{2.7}$$

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Applying [35, Corollary 5] with $v^+(x,t)=1$ and $v^-(x,t)=w(x,t)$, $v^+(x,t)=u(x,t)$ and $v^-(x,t)=0$, respectively, we have

$$\varphi^-(x,t)\leqslant u(x,t,\phi)\leqslant 1,\quad x\in\mathbb{R},\ b>t\geqslant 0, \tag{2.8}$$

and

$$0\leqslant u(x,t,\psi)\leqslant \varphi^+(x,t),\quad x\in\mathbb{R},\ b>t\geqslant 0. \tag{2.9}$$

Combining (2.7), (2.8) and (2.9), we obtain $\varphi^+(x,t)\geqslant\varphi^-(x,t)$ for all $x\in\mathbb{R}$ and $b>t\geqslant 0$ when $B=0$.

Now we consider the case $B\neq 0$. Define $\bar h(y,s)=h(y-Bs,s)$ for $y\in\mathbb{R}$ and $s\in[0,\tau]$. Consider the initial value problem

$$\begin{cases} \dfrac{\partial v}{\partial t}=d\Delta v+g\big(v(x,t),\big(\bar h * S(v)\big)(x,t)\big), & x\in R,\ t>0,\\ v_0=\varphi\in[0,1]_C. \end{cases} \tag{2.10}$$

By a direct verification, we can show that if $u(x,t)$ is a (wild) solution of (1.1) with initial value $\varphi\in[0,1]_C$, then $v(x,t)=u(x-Bt,t)$ is a (wild) solution of (2.10). Inversely, if $v(x,t)$ is a (wild) solution of (2.10), then $u(x,t)=v(x+Bt,t)$ is a solution of (1.1) with initial value $\varphi\in[0,1]_C$. Moreover, if $u(x,t)$ is a supersolution (subsolution) of (1.1), then $v(x,t)=u(x-Bt,t)$ is a supersolution (subsolution) of (2.10). Inversely, if $v(x,t)$ is a supersolution (subsolution) of (2.10), then $u(x,t)=v(x+Bt,t)$ is a supersolution (subsolution) of (1.1). Applying the results for the case $B=0$ to (2.10), we can show that the conclusions hold for $B\neq 0$ except the last inequality in the theorem.

It remains to prove the last inequality in the theorem. Let $v(x,t)=\varphi^+(x,t)-\varphi^-(x,t)$, $x\in\mathbb{R}$, $t\in[-\tau,\infty)$. Then $v(x,t)\geqslant 0$, $x\in\mathbb{R}$, $t\in[-\tau,b)$. Clearly, $\varphi_t^+,\varphi_t^-\in[0,1]_C$ and $\varphi_t^+\geqslant\varphi_t^-$ in C for all $b>t\geqslant 0$. For any given $t_0\geqslant 0$, by Definition 2.1 and (2.6), for any $b>t\geqslant t_0$, it follows that

$$\begin{aligned} v(t)&\geqslant U(t-t_0)v(t_0)+\int_{t_0}^{t}U(t-\theta)\big(F\big(\varphi_\theta^+\big)-F\big(\varphi_\theta^-\big)\big)\,d\theta\\ &\geqslant U(t-t_0)v(t_0)-L_1\int_{t_0}^{t}U(t-\theta)v(\theta)\,d\theta. \end{aligned}$$

Let

$$z(t)=e^{-L_1(t-t_0)}U(t-t_0)v(t_0),\quad b>t\geqslant t_0.$$

Then $z(t)$ satisfies

$$z(t)=U(t-t_0)z(t_0)-L_1\int_{t_0}^{t}U(t-\theta)z(\theta)\,d\theta,\quad b>t\geqslant t_0.$$

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Using [35, Proposition 3] with $v^- \equiv z(t)$, $v^+ = +\infty$, $S(t,s) = S^-(t,s) = U(t,s) = U(t-s)$, $b > t \geqslant s \geqslant 0$, and $B(t,\varphi) \equiv B^-(t,\varphi) \equiv -L_1\varphi(0)$, we have $v(t) \geqslant z(t)$ for all $b > t \geqslant t_0$. Thus it follows that

$$\varphi^+(t) - \varphi^-(t) \geqslant e^{-L_1(t-t_0)} U(t-t_0)\big(\varphi^+(t_0) - \varphi^-(t_0)\big), \quad b > t \geqslant t_0. \tag{2.11}$$

Combining (2.3), (2.11) and the definition of $\Theta \in C([0,\infty) \times (0,b), \mathbb{R})$, we then have

$$\varphi^+(x,t) - \varphi^-(x,t) \geqslant \Theta(J, t-t_0) \int_z^{z+1} \big(\varphi^+(y,t_0) - \varphi^-(y,t_0)\big)\, dy$$

for all $x \in \mathbb{R}$ with $|x - z| \leqslant J$ and $b > t > t_0 \geqslant 0$. The proof is complete. $\square$

Remark 2.4. By Theorem 2.3, it follows that if $\varphi^+(x,t)$ and $\varphi^-(x,t)$ are the pair of supersolution and subsolution of (1.1) given in Theorem 2.3 and $\varphi^+(x,0) \not\equiv \varphi^-(x,0)$, then for any $b > t > 0$,

$$\varphi^+(x,t) - \varphi^-(x,t) \geqslant \Theta(J,t) \int_z^{z+1} \big(\varphi^+(y,0) - \varphi^-(y,0)\big)\, dy > 0.$$

In particular, if $u(x,t,\varphi)$ is a solution of (1.1) with the initial value $\varphi \in [0,1]_C$ and $\varphi(x,0)$ ($\not\equiv$ constant) is nondecreasing on $\mathbb{R}$, then for any $t > \tau$ and $x \in \mathbb{R}$, $\frac{\partial}{\partial x} u(x,t) > 0$.

Lemma 2.5. *Assume that* (H1) *and* (H2) *hold. Let $U(x - ct)$ be a nondecreasing traveling wave front of* (1.1). *Then $U'(\xi) > 0$ for $\xi \in \mathbb{R}$.*

Remark 2.6. For $\tau = 0$, that is, for the equation without delay, Theorem 2.3, Remark 2.4 and Lemma 2.5 still hold.

3. Uniqueness of traveling wave fronts

In this section, we will consider the uniqueness of traveling wave fronts of (1.1). To prove our results, we need the following two lemmas.

Lemma 3.1. *Assume that* (H1) *and* (H2) *hold. For any travelling wave front $U(x - ct)$ of* (1.1) *with $0 \leqslant U(\xi) \leqslant 1$, $\xi \in \mathbb{R}$, we have $\lim_{\xi \to \pm\infty} U'(\xi) = 0$.*

In fact, noting that

$$\lim_{\xi \to -\infty} \int_{-\tau}^{0} \int_{-\infty}^{+\infty} h(y,-s) U(\xi - y + cs)\, dy\, ds = 0$$

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and

$$\lim_{\xi\to+\infty}\int_{-\tau}^{0}\int_{-\infty}^{+\infty} h(y,-s)U(\xi-y+cs)\,dy\,ds=1,$$

we can prove Lemma 3.1 by an argument similar to that of [42].

Lemma 3.2. *Assume that* (H1) *and* (H2) *hold and let $U(x-ct)$ be an increasing traveling wave front of* (1.1). *Then there exist three positive numbers β_0 (which is independent of U), σ_0 and $\bar{\delta}$ such that for any $\delta\in(0,\bar{\delta}]$ and every $\xi_0\in\mathbb{R}$, the functions w^+ and w^- defined by*

$$\begin{aligned}
w^+(x,t) &:= \min\bigl\{U\bigl(x-ct+\xi_0+\sigma_0\delta\bigl(1-e^{-\beta_0 t}\bigr)\bigr)+\delta e^{-\beta_0 t},1\bigr\},\\
w^-(x,t) &:= \max\bigl\{U\bigl(x-ct+\xi_0-\sigma_0\delta\bigl(1-e^{-\beta_0 t}\bigr)\bigr)-\delta e^{-\beta_0 t},0\bigr\}
\end{aligned}$$

are a supersolution and a subsolution of (1.1) *on* $[0,\infty)$*, respectively.*

Proof. Clearly, $0<U(\xi)<1$. Hence, $0<U(x-ct)<1$, $x\in\mathbb{R}$, $t\in\mathbb{R}$. By Theorem 2.3 and the monotonicity of $U(\cdot)$, it follows that $U(\cdot)\in C^1(\mathbb{R})$ and $U'(\xi)>0$, $\xi\in\mathbb{R}$. Since

$$\begin{aligned}
&\lim_{(u,v,r,s,\varpi,\beta)\to(0+,S(0),0+,S(0),S'(0),0)}\bigl[\partial_1 g(u,v)+\varpi e^{\beta\tau}\partial_2 g(r,s)+\beta\bigr]\\
&\quad=\partial_1 g\bigl(0,S(0)\bigr)+S'(0)\partial_2 g\bigl(0,S(0)\bigr)<0
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{(u,v,r,s,\varpi,\beta)\to(1-,S(1),1-,S(1),S'(1),0)}\bigl[\partial_1 g(u,v)+\varpi e^{\beta\tau}\partial_2 g(r,s)+\beta\bigr]\\
&\quad=\partial_1 g\bigl(1,S(1)\bigr)+S'(1)\partial_2 g\bigl(1,S(1)\bigr)<0,
\end{aligned}$$

we can fix $\beta_0>0$ and $\delta^*>0$ such that

$$\partial_1 g(u,v)+\varpi e^{\beta_0\tau}\partial_2 g(r,s)<-\beta_0 \tag{3.1}$$

for all

$$\begin{aligned}
(u,v,r,s,\varpi)\in{}&[0,\delta^*]\times\bigl[S(0)-\delta^*,S(0)+\delta^*\bigr]\times[0,\delta^*]\\
&\times\bigl[S(0)-\delta^*,S(0)+\delta^*\bigr]\times\bigl[S'(0)-\delta^*,S'(0)+\delta^*\bigr]
\end{aligned}$$

and

$$\begin{aligned}
(u,v,r,s,\varpi)\in{}&[1-\delta^*,1]\times\bigl[S(1)-\delta^*,S(1)+\delta^*\bigr]\times[1-\delta^*,1]\\
&\times\bigl[S(1)-\delta^*,S(1)+\delta^*\bigr]\times\bigl[S'(1)-\delta^*,S'(1)+\delta^*\bigr].
\end{aligned}$$

By (1.2) and the following results:

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$$\lim_{(\xi,\delta)\to(\infty,0)} \int_0^\tau \int_{-\infty}^\infty h(y,s) S\big(U(\xi - y + cs) + \delta\big)\, dy\, ds = S(1),$$

$$\lim_{(\xi,\delta)\to(-\infty,0)} \int_0^\tau \int_{-\infty}^\infty h(y,s) S\big(U(\xi - y + cs) + \delta\big)\, dy\, ds = S(0),$$

$$\lim_{(\xi,\delta)\to(\infty,0)} \int_0^\tau \int_{-\infty}^\infty h(y,s) S'\big(U(\xi - y + cs) + \delta\big)\, dy\, ds = S'(1),$$

$$\lim_{(\xi,\delta)\to(-\infty,0)} \int_0^\tau \int_{-\infty}^\infty h(y,s) S'\big(U(\xi - y + cs) + \delta\big)\, dy\, ds = S'(0),$$

there exist $M_0 = M_0(U, \beta_0, \delta^*) > 0$ and $\hat\delta = \hat\delta(U, \beta_0, \delta^*) \in (0, \delta^*)$ such that for all $\xi \geqslant M_0$ and $\delta \in [0, \hat\delta]$,

$$U(\xi) \geqslant 1 - \delta^*, \qquad S(1) + \delta^* \geqslant \int_0^\tau \int_{-\infty}^\infty h(y,s) S\big(U(\xi - y + cs) + \delta\big)\, dy\, ds \geqslant S(1) - \delta^*,$$

$$S'(1) + \delta^* \geqslant \int_0^\tau \int_{-\infty}^\infty h(y,s) S'\big(U(\xi - y + cs) + \delta\big)\, dy\, ds \geqslant S'(1) - \delta^* \tag{3.2}$$

and for all $\xi \leqslant -M_0$ and $\delta \in [0, \hat\delta]$,

$$U(\xi) \leqslant \delta^*, \qquad S(0) - \delta^* \leqslant \int_0^\tau \int_{-\infty}^\infty h(y,s) S\big(U(\xi - y + cs) - \delta\big)\, dy\, ds \leqslant S(0) + \delta^*,$$

$$S'(0) - \delta^* \leqslant \int_0^\tau \int_{-\infty}^\infty h(y,s) S'\big(U(\xi - y + cs) - \delta\big)\, dy\, ds \leqslant S'(0) + \delta^*. \tag{3.3}$$

Set

$$\begin{aligned} c_1 &= c_1\big(\beta_0, \delta^*\big) \\ &= \max\big\{\big|\partial_1 g(u,v)\big| + \kappa e^{\beta_0 \tau}\big|\partial_2 g(r,s)\big|\colon\ u, r \in [0,1],\ v, s \in \big[S(0), S(1+\delta^*)\big]\big\} \end{aligned}$$

and

$$m_0 = m_0\big(U, \beta_0, \delta^*\big) = \min\big\{U'(\xi)\colon\ |\xi| \leqslant M_0\big\} > 0,$$

where $\kappa = \max\{S'(u)\colon\ u \in [0, 1+\delta^*]\} > 0$, and define

$$\sigma_0 = \sigma_0\big(U, \beta_0, \delta^*\big) = \frac{\beta_0 + c_1}{m_0 \beta_0}, \qquad \bar\delta = \hat\delta e^{-\beta_0 \tau}.$$

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We only prove that $w^+(x,t)$ is a supersolution of (1.1) since a similar argument can be used for $w^-(x,t)$. For any given $\delta \in (0,\bar{\delta})$, let $\xi(x,t) = x - ct + \xi_0 + \sigma_0\delta[1-e^{-\beta_0 t}]$. Define

$$A^+ = \{(x,t)\colon U(\xi(x,t)) + \delta e^{-\beta_0 t} > 1\} \quad \text{and} \quad A^- = \{(x,t)\colon U(\xi(x,t)) + \delta e^{-\beta_0 t} < 1\}.$$

Then it is easy to show that for $(x,t) \in A^+$,

$$\frac{\partial w^+(x,t)}{\partial t} - d\Delta w^+(x,t) - B\frac{\partial w^+(x,t)}{\partial x} - g\big(w^+(x,t), \big(h * S(w^+)\big)(x,t)\big) \geqslant 0. \tag{3.4}$$

In the following, we show that (3.4) holds for $(x,t) \in A^-$. Let $(x,t) \in A^-$. Then $w^+(x,t) = U(\xi(x,t)) + \delta e^{-\beta_0 t} < 1$. It follows that, for any $t \geqslant 0$,

$$\begin{aligned}
&\frac{\partial w^+(x,t)}{\partial t} - d\Delta w^+(x,t) - B\frac{\partial w^+(x,t)}{\partial x} - g\big(w^+(x,t), \big(h * S(w^+)\big)(x,t)\big) \\
&\quad = U'\big(\xi(x,t)\big)\big(-c + \sigma_0\delta\beta_0 e^{-\beta_0 t}\big) - \beta_0\delta e^{-\beta_0 t} - dU''\big(\xi(x,t)\big) - BU'\big(\xi(x,t)\big) \\
&\qquad - g\big(U\big(\xi(x,t)\big) + \delta e^{-\beta_0 t}, \big(h * S(w^+)\big)(x,t)\big) \\
&\quad = \big(\sigma_0 U'\big(\xi(x,t)\big) - 1\big)\beta_0\delta e^{-\beta_0 t} + g\left(U\big(\xi(x,t)\big), \int_0^\tau \int_{-\infty}^{+\infty} h(y,s) S\big(U\big(\eta(y,s)\big)\big)\,dy\,ds\right) \\
&\qquad - g\big(U\big(\xi(x,t)\big) + \delta e^{-\beta_0 t}, \big(h * S(w^+)\big)(x,t)\big) \\
&\quad = \big(\sigma_0 U'\big(\xi(x,t)\big) - 1\big)\beta_0\delta e^{-\beta_0 t} + g\left(U\big(\xi(x,t)\big), \int_0^\tau \int_{-\infty}^{+\infty} h(y,s) S\big(U\big(\eta(y,s)\big)\big)\,dy\,ds\right) \\
&\qquad - g\left(U\big(\xi(x,t)\big) + \delta e^{-\beta_0 t},\right. \\
&\qquad \left.\int_0^\tau \int_{-\infty}^{+\infty} h(y,s) S\big(U\big(\eta(y,s) - \sigma_0\delta\big[e^{\beta_0 s} - 1\big]e^{-\beta_0 t}\big) + \delta e^{-\beta_0(t-s)}\big)\,dy\,ds\right) \\
&\quad \geqslant \big(\sigma_0 U'\big(\xi(x,t)\big) - 1\big)\beta_0\delta e^{-\beta_0 t} + g\left(U\big(\xi(x,t)\big), \int_0^\tau \int_{-\infty}^{+\infty} h(y,s) S\big(U\big(\eta(y,s)\big)\big)\,dy\,ds\right) \\
&\qquad - g\left(U\big(\xi(x,t)\big) + \delta e^{-\beta_0 t}, \int_0^\tau \int_{-\infty}^{+\infty} h(y,s) S\big(U\big(\eta(y,s)\big) + \delta e^{-\beta_0(t-s)}\big)\,dy\,ds\right) \\
&\quad \geqslant \delta e^{-\beta_0 t}\left\{\sigma_0\beta_0 U'\big(\xi(x,t)\big) - \beta_0\right. \\
&\qquad - \int_0^1 \partial_1 g\left(U\big(\xi(x,t)\big) + \theta\delta e^{-\beta_0 t}, \int_0^\tau \int_{-\infty}^{+\infty} h(y,s) S\big(U\big(\eta(y,s)\big)\big)\,dy\,ds\right) d\theta
\end{aligned}$$

$$-\int_0^1 \Bigg[\partial_2\hat{g}\Bigg(U\big(\xi(x,t)\big)+\delta e^{-\beta_0 t},\int_0^\tau\int_{-\infty}^{+\infty}h(y,s)S\big(U\big(\eta(y,s)\big)+\theta\delta e^{-\beta_0(t-s)}\big)\,dy\,ds\Bigg)$$

$$\times e^{\beta_0\tau}\int_0^\tau\int_{-\infty}^{+\infty}h(y,s)S'\big(U\big(\eta(y,s)\big)+\theta\delta e^{-\beta_0(t-s)}\big)\,dy\,ds\Bigg]\,d\theta\Bigg\}, \tag{3.5}$$

where $\eta(y,s)=\xi(x,t)-y+cs$, $s\in[0,\tau]$. We need to consider three cases.

Case (i): $|\xi(x,t)|\leqslant M_0$. Since

$$0\leqslant U\big(\eta(y,s)\big)+\theta\delta e^{-\beta_0(t-s)}\leqslant 1+\delta^*,$$

$$S(0)\leqslant\int_0^\tau\int_{-\infty}^{\infty}h(y,s)S\big(U\big(\eta(y,s)\big)+\theta\delta e^{-\beta_0(t-s)}\big)\,dy\,ds\leqslant S(1+\delta^*),$$

and

$$\left|\int_0^\tau\int_{-\infty}^{\infty}h(y,s)S'\big(U\big(\eta(y,s)\big)+\delta e^{-\beta_0(t-s)}\big)\,dy\,ds\right|\leqslant\kappa,$$

by the choice of c_1, we have

$$\begin{aligned}
&\int_0^1\left|\partial_1 g\Bigg(U\big(\xi(x,t)\big)+\theta\delta e^{-\beta_0 t},\int_0^\tau\int_{-\infty}^{+\infty}h(y,s)S\big(U\big(\eta(y,s)\big)\big)\,dy\,ds\Bigg)\right|d\theta\\
&\quad+\int_0^1\Bigg\{\left|\partial_2 g\Bigg(U\big(\xi(x,t)\big)+\delta e^{-\beta_0 t},\int_0^\tau\int_{-\infty}^{+\infty}h(y,s)S\big(U\big(\eta(y,s)\big)+\theta\delta e^{-\beta_0(t-s)}\big)\,dy\,ds\Bigg)\right|\\
&\quad\times e^{\beta_0\tau}\left|\int_{-\infty}^{+\infty}h(y,s)S'\big(U\big(\eta(y,s)\big)+\theta\delta e^{-\beta_0(t-s)}\big)\,dy\,ds\right|\Bigg\}\,d\theta\\
&\leqslant c_1.
\end{aligned}$$

Then, by the choice of m_0 and σ_0, we have

$$\begin{aligned}
&\frac{\partial w^+(x,t)}{\partial t}-d\Delta w^+(x,t)-B\frac{\partial w^+(x,t)}{\partial x}-g\big(w^+(x,t),\big(h*S\big(w^+\big)\big)(x,t)\big)\\
&\quad\geqslant[\sigma_0\beta_0 m_0-\beta_0-c_1]\delta e^{-\beta_0 t}=0.
\end{aligned}$$

Case (ii): $\xi(x,t)\geqslant M_0$. By (3.2) we have

$$S(1)-\delta^* \leqslant \int_0^\tau \int_{-\infty}^{+\infty} h(y,s) S\big(U\big(\eta(y,s)\big)\big)\,dy\,ds \leqslant S(1),$$

$$S(1)-\delta^* \leqslant \int_0^\tau \int_{-\infty}^{+\infty} h(y,s) S\big(U\big(\eta(y,s)\big)+\theta\delta e^{-\beta_0(t-s)}\big)\,dy\,ds \leqslant S(1)+\delta^*,$$

and

$$S'(1)-\delta^* \leqslant \int_0^\tau \int_{-\infty}^{+\infty} h(y,s) S'\big(U\big(\eta(y,s)\big)+\theta\delta e^{-\beta_0(t-s)}\big)\,dy\,ds \leqslant S'(1)+\delta^*.$$

Therefore, by (3.1) and (3.5), it follows that

$$\begin{aligned}&\frac{\partial w^+(x,t)}{\partial t} - d\Delta w^+(x,t) - B\frac{\partial w^+(x,t)}{\partial x} - g\big(w^+(x,t), \big(h * S\big(w^+\big)\big)(x,t)\big)\\ &\quad \geqslant \delta e^{-\beta_0 t}\big[\sigma_0\beta_0 U'\big(\xi(x,t)\big) - \beta_0 + \beta_0\big] \geqslant 0.\end{aligned}$$

Case (iii): $\xi(x,t) \leqslant -M_0$. The proof is similar to that for the case (ii) and is omitted.

Thus, we have shown that (3.4) holds for any $(x,t) \in A^+ \cup A^-$. Now we further show that (2.4) holds for w^+. Define $\tilde{w}^+(x,t) = w^+(x-Bt,t)$. To complete the proof, we only need to show that $\tilde{w}^+(x,t)$ is a supersolution of (2.10), namely, the following inequality

$$\tilde{w}^+(t) \geqslant T(t-s)\tilde{w}^+(s) + \int_s^t T(t-r)\tilde{F}\big(\tilde{w}_r^+\big)\,dr \tag{3.6}$$

holds, where

$$\tilde{F}(\varphi)(x) = g\left(\varphi(x,0), \int_{-\tau}^{0}\int_{-\infty}^{\infty} \bar{h}(x-y,-s) S\big(\varphi(y,s)\big)\,dy\,ds\right).$$

Define $\tilde{A}^+ = \{(x,t)\colon (x-Bt,t) \in A^+\}$, $\tilde{A}^- = \{(x,t)\colon (x-Bt,t) \in A^-\}$. By (3.3), then for any $(x,t) \in \tilde{A}^+ \cup \tilde{A}^-$,

$$\frac{\partial \tilde{w}^+(x,t)}{\partial t} - d\Delta\tilde{w}^+(x,t) - g\big(\tilde{w}^+(x,t), \big(\bar{h} * S\big(\tilde{w}^+\big)\big)(x,t)\big) \geqslant 0.$$

Since $\frac{\partial}{\partial x}\{U(\xi(x-Bt,t)) + \delta e^{-\beta_0 t}\} = U'(\xi(x-Bt,t)) > 0$, then for every $t_0 \in [0,\infty)$, there exists a unique $x^+(t_0) \in \mathbb{R}$ such that $U(\xi(x^+(t_0)-Bt_0,t_0)) + \delta e^{-\beta_0 t_0} = 1$, $(x,t_0) \in \tilde{A}^+$ for $x > x^+(t_0)$, $(x,t_0) \in \tilde{A}^-$ for $x < x^+(t_0)$ and

$$\frac{\partial \tilde{w}^+(x^+(t_0)-0,t)}{\partial x} = \lim_{x \to x^+(t_0)-0} U'\big(\xi(x-Bt_0,t)\big) > 0.$$

Define

$$\Phi\big(\tilde{w}^+\big)(x,t,r)=\frac{1}{\sqrt{4\pi d(t-r)}}\int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4d(t-r)}}\,\tilde{w}^+(y,r)\,dy,\quad t>r\geqslant 0,$$

and

$$H\big(\tilde{w}^+\big)(x,t)=-\frac{\partial\tilde{w}^+(x,t)}{\partial t}+d\Delta\tilde{w}^+(x,t)+g\big(\tilde{w}^+(x,t),\big(\bar{h}*S\big(\tilde{w}^+\big)\big)(x,t)\big)\leqslant 0.$$

Set $\tilde{F}(\tilde{w}_t^+)(x)=g(\tilde{w}^+(x,t),(\bar{h}*S(\tilde{w}^+))(x,t))$, then a direct calculation implies

$$\begin{aligned}\frac{\partial}{\partial r}\Phi\big(\tilde{w}^+\big)(x,t,r)&=\frac{1}{2(t-r)\sqrt{4\pi d(t-r)}}\int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4d(t-r)}}\,\tilde{w}^+(y,r)\,dy\\&\quad-\frac{1}{\sqrt{4\pi d(t-r)}}\int_{-\infty}^{\infty}\frac{(x-y)^2}{4d(t-r)^2}e^{\frac{-(x-y)^2}{4d(t-r)}}\,\tilde{w}^+(y,r)\,dy\\&\quad+\frac{d}{\sqrt{4\pi d(t-r)}}\int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4d(t-r)}}\,\frac{\partial^2\tilde{w}^+(y,r)}{\partial y^2}\,dy\\&\quad+\frac{1}{\sqrt{4\pi d(t-r)}}\int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4d(t-r)}}\big[F\big(\tilde{w}_r^+\big)(y)-H\big(\tilde{w}^+\big)(y,r)\big]\,dy.\end{aligned}$$

Furthermore, integration by parts gives

$$\begin{aligned}&\frac{d}{\sqrt{4\pi d(t-r)}}\int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4d(t-r)}}\,\frac{\partial^2\tilde{w}^+(y,r)}{\partial y^2}\,dy\\&\quad=\frac{d}{\sqrt{4\pi d(t-r)}}\int_{-\infty}^{x^+(r)} e^{\frac{-(x-y)^2}{4d(t-r)}}\,\frac{\partial^2\tilde{w}^+(y,r)}{\partial y^2}\,dy\\&\quad=\frac{d}{\sqrt{4\pi d(t-r)}}e^{\frac{-(x-x^+(r))^2}{4d(t-r)}}\,\frac{\partial\tilde{w}^+(x^+(r)-0,r)}{\partial x}\\&\qquad-\frac{1}{\sqrt{4\pi d(t-r)}}\,\frac{x-x^+(r)}{2(t-r)}e^{\frac{-(x-x^+(r))^2}{4d(t-r)}}\\&\qquad-\frac{1}{\sqrt{4\pi d(t-r)}}\int_{-\infty}^{x^+(r)}\frac{1}{2(t-r)}e^{\frac{-(x-y)^2}{4d(t-r)}}\,\tilde{w}^+(y,r)\,dy\end{aligned}$$

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$$+\frac{1}{\sqrt{4\pi d(t-r)}}\int_{-\infty}^{x^+(r)}\frac{(x-y)^2}{4d(t-r)^2}e^{\frac{-(x-y)^2}{4d(t-r)}}\tilde{w}^+(y,r)\,dy$$

and

$$\frac{1}{\sqrt{4\pi d(t-r)}}\int_{x^+(r)}^{\infty}\frac{(x-y)^2}{4d(t-r)^2}e^{\frac{-(x-y)^2}{4d(t-r)}}\,dy$$
$$=-\frac{x-x^+(r)}{2(t-r)\sqrt{4\pi d(t-r)}}e^{\frac{-(x-x^+(r))^2}{4d(t-r)}}+\frac{1}{2(t-r)\sqrt{4\pi d(t-r)}}\int_{x^+(r)}^{\infty}e^{\frac{-(x-y)^2}{4d(t-r)}}\,dy.$$

Hence, it follows that

$$\frac{\partial}{\partial r}\Phi\big(\tilde{w}^+\big)(x,t,r)$$
$$=\frac{d}{\sqrt{4\pi d(t-r)}}e^{\frac{-(x-x^+(r))^2}{4d(t-r)}}\frac{\partial\tilde{w}^+(x^+(r)-0,r)}{\partial x}$$
$$+\frac{1}{\sqrt{4\pi d(t-r)}}\int_{-\infty}^{\infty}e^{\frac{-(x-y)^2}{4d(t-r)}}\big[\tilde{F}\big(\tilde{w}_r^+\big)(y)-H\big(\tilde{w}^+\big)(y,r)\big]\,dy.$$

Since

$$\frac{d}{\sqrt{4\pi d(t-r)}}\exp\left\{\frac{-(x-x^+(r))^2}{4d(t-r)}\right\}\frac{\partial\tilde{w}^+(x^+(r)-0,r)}{\partial x}$$

is integrable in $r\in[0,t)$, $\frac{\partial}{\partial r}\Phi(\tilde{w}^+)(x,t,r)$ is continuous in $r\in[0,t)$, and

$$\lim_{r\to t-0}\frac{1}{\sqrt{4\pi d(t-r)}}\int_{-\infty}^{\infty}e^{\frac{-(x-y)^2}{4d(t-r)}}\tilde{w}^+(y,r)\,dy=\tilde{w}^+(x,t),$$

it follows that for $0\leqslant s<t$,

$$\tilde{w}^+(x,t)=\lim_{\eta\to 0+0}\Phi\big(\tilde{w}^+\big)(x,t,t-\eta)$$
$$=\Phi\big(\tilde{w}^+\big)(x,t,s)+\lim_{\eta\to 0+0}\int_{s}^{t-\eta}\frac{\partial}{\partial r}\Phi\big(\tilde{w}^+\big)(x,t,r)\,dr$$
$$=\frac{1}{\sqrt{4\pi d(t-r)}}\int_{-\infty}^{\infty}e^{\frac{-(x-y)^2}{4d(t-r)}}\tilde{w}^+(y,s)\,dy$$

$$+\int_s^t \frac{d}{\sqrt{4\pi d(t-r)}} e^{\frac{-(x-x^+(r))^2}{4d(t-r)}} \frac{\partial \tilde{w}^+(x^+(r)-0,r)}{\partial x}\,dr$$

$$+\int_s^t \frac{1}{\sqrt{4\pi d(t-r)}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4d(t-r)}} \big[\tilde{F}\big(\tilde{w}_r^+\big)(y) - H\big(\tilde{w}^+\big)(y,r)\big]\,dy\,dr.$$

In view of $\partial\tilde{w}^+(x^+(r)-0,r)/\partial x > 0$ and $H(\tilde{w}^+)(y,r) \leqslant 0$, we see that (3.6) holds, which implies that $w^+(x,t)$ is a supersolution of (1.1). This completes the proof. □

Theorem 3.3. *Assume that* (H1) *and* (H2) *hold. Assume further that* (1.1) *has a monotone traveling wave solution $U(x-ct)$. Then the traveling wave solutions of* (1.1) *are unique up to a translation in the sense that for any traveling wave solution $\bar{U}(x-\bar{c}t)$ with $0 \leqslant \bar{U}(\xi) \leqslant 1$, $\xi \in \mathbb{R}$, we have $\bar{c} = c$ and $\bar{U}(\cdot) = U(\xi_0 + \cdot)$ for some $\xi_0 = \xi_0(\bar{U}) \in \mathbb{R}$.*

Proof. Since $\bar{U}(\xi)$ and $U(\xi)$ have the same limits as $\xi \to \pm\infty$, there exist $\xi_1 \in \mathbb{R}$ and a sufficiently large number $p > 0$ such that for every $s \in [-\tau, 0]$ and $x \in \mathbb{R}$,

$$U(x-cs+\xi_1) - \bar{\delta} < \bar{U}(x-\bar{c}s) < U(x-cs+\xi_1+p) + \bar{\delta}.$$

Hence,

$$\begin{aligned}&\min\big\{U\big(x-cs+\xi_1+p+\sigma_0\bar{\delta}\big(e^{\beta_0\tau}-1\big)+\sigma_0\bar{\delta}\big(1-e^{-\beta_0 s}\big)\big)+\bar{\delta}e^{-\beta_0 s},1\big\}\\&\quad > \bar{U}(x-\bar{c}s) > \max\big\{U\big(x-cs+\xi_1-\sigma_0\bar{\delta}\big(e^{\beta_0\tau}-1\big)-\sigma_0\bar{\delta}\big(1-e^{-\beta_0 s}\big)\big)-\bar{\delta}e^{-\beta_0 s},0\big\},\end{aligned}$$

where β_0, σ_0 and $\bar{\delta}$ are given in Lemma 3.2. By comparison, we obtain that for all $t \geqslant 0$ and $x \in \mathbb{R}$,

$$\begin{aligned}&\min\big\{U\big(x-ct+\xi_1+p+\sigma_0\bar{\delta}\big(e^{\beta_0\tau}-1\big)+\sigma_0\bar{\delta}\big(1-e^{-\beta_0 t}\big)\big)+\bar{\delta}e^{-\beta_0 t},1\big\}\\&\quad > \bar{U}(x-\bar{c}t) > \max\big\{U\big(x-ct+\xi_1-\sigma_0\bar{\delta}\big(e^{\beta_0\tau}-1\big)-\sigma_0\bar{\delta}\big(1-e^{-\beta_0 t}\big)\big)-\bar{\delta}e^{-\beta_0 t},0\big\}.\end{aligned}$$

Keeping $\xi = x - ct$ fixed and letting $t \to \infty$, from the first and second inequalities we see that $c = \bar{c}$. In addition,

$$U\big(\xi+\xi_1-\sigma_0\bar{\delta}e^{\beta_0\tau}\big) < \bar{U}(\xi) < U\big(\xi+\xi_1+p+\sigma_0\bar{\delta}e^{\beta_0\tau}\big) \quad \text{for } \xi \in \mathbb{R}. \tag{3.7}$$

Define

$$\xi^* := \inf\big\{\xi\colon\ \bar{U}(\cdot) \leqslant U(\cdot+\xi)\big\} \quad \text{and} \quad \xi_* := \sup\big\{\xi\colon\ \bar{U}(\cdot) \geqslant U(\cdot+\xi)\big\}.$$

Then from (3.7), both ξ^* and ξ_* are well defined. Since $U(\cdot+\xi_*) \leqslant \bar{U}(\cdot) \leqslant U(\cdot+\xi^*)$, we have $\xi_* \leqslant \xi^*$.

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To complete the proof, it suffices to show that $\xi_* = \xi^*$. For the sake of contradiction, assume that $\xi_* < \xi^*$ and $\bar{U}(\cdot) \not\equiv U(\cdot + \xi^*)$. Since $\lim_{|\xi|\to\infty} U'(\xi) = 0$, there exists a large positive constant $M_1 = M_1(U) > 0$ such that

$$2\sigma_0 e^{\beta_0\tau} U'(\xi) \leqslant 1 \quad \text{if } |\xi| \geqslant M_1.$$

Note that $\bar{U}(\cdot) \leqslant U(\cdot + \xi^*)$ and $\bar{U}(\cdot) \not\equiv U(\cdot + \xi^*)$, by Theorem 2.3, it follows that $\bar{U}(\cdot) < U(\cdot + \xi^*)$ on $\mathbb{R}$. Consequently, by the continuity of $\bar{U}$ and U, there exists a small constant $\rho \in (0, \bar{\delta}]$ with $\rho \leqslant \frac{1}{2\sigma_0} e^{-\beta_0\tau}$, such that

$$\bar{U}(\xi) < U\big(\xi + \xi^* - 2\sigma_0\rho e^{\beta_0\tau}\big) \quad \text{if } \xi \in [-M_1 - 1 - \xi^*, M_1 + 1 - \xi^*]. \tag{3.8}$$

When $|\xi + \xi^*| \geqslant M_1 + 1$, we have

$$\begin{aligned} &U\big(\xi + \xi^* - 2\sigma_0\rho e^{\beta_0\tau}\big) - \bar{U}(\xi) > U\big(\xi + \xi^* - 2\sigma_0\rho e^{\beta_0\tau}\big) - U(\xi + \xi^*) \\ &\quad = -2\sigma_0\rho e^{\beta_0\tau} U'\big(\xi + \xi^* - 2\theta\sigma_0\rho e^{\beta_0\tau}\big) \geqslant -\rho, \end{aligned}$$

which, together with (3.8), implies that for any $s \in [-\tau, 0]$ and $x \in \mathbb{R}$,

$$\begin{aligned} \bar{U}(x - cs) < \min\big\{ &U\big(x - cs + \xi^* - 2\sigma_0\rho e^{\beta_0\tau} + \sigma_0\rho\big(e^{\beta_0\tau} - 1\big) \\ &+ \sigma_0\rho\big(1 - e^{-\beta_0 s}\big)\big) + \rho e^{-\beta_0 s}, 1\big\}. \end{aligned}$$

Therefore, the comparison theorem and Lemma 3.2 imply that for any $t \geqslant 0$ and $x \in \mathbb{R}$,

$$\bar{U}(x - ct) < \min\big\{U\big(x - ct + \xi^* - 2\sigma_0\rho e^{\beta_0\tau} + \sigma_0\rho\big(e^{\beta_0\tau} - e^{-\beta_0 t}\big)\big) + \rho e^{-\beta_0 t}, 1\big\}. \tag{3.9}$$

In (3.9), keeping $\xi = x - ct$ fixed and letting $t \to \infty$, we obtain $\bar{U}(\xi) < U(\xi + \xi^* - \sigma_0\rho e^{\beta_0\tau})$ for all $\xi \in \mathbb{R}$. This contradicts the definition of ξ^*. Hence, $\xi_* = \xi^*$ and this completes the proof. $\square$

4. Asymptotic stability of traveling wave fronts

In this section, we establish the asymptotic stability of traveling wave fronts by using the squeezing technique, which has been used in Chen [13], Chen and Guo [14] and Smith and Zhao [42].

Let $\delta_0 = \min\{\frac{a^-}{2}, \frac{1-a^+}{2}\}$ and let $\zeta(\cdot) \in C^\infty(\mathbb{R}, \mathbb{R})$ be a fixed function with the following properties:

$$\begin{aligned} &\zeta(s) = 0 \quad \text{if } s \leqslant 0; \qquad \zeta(s) = 1 \quad \text{if } s \geqslant 4; \\ &0 < \zeta'(s) < 1; \qquad \big|\zeta''(s)\big| \leqslant 1 \quad \text{if } s \in (0, 4). \end{aligned} \tag{4.1}$$

Lemma 4.1. *Assume that* (H1) *and* (H2) *hold. Then, for any $\delta \in (0, \delta_0]$, there exist two positive numbers $\epsilon = \epsilon(\delta)$ and $C = C(\delta)$ such that, for every $\xi \in \mathbb{R}$, the functions v^+ and v^- defined by*

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$$v^+(x,t) := \min\bigl\{(1+\delta) - \bigl[1 - \bigl(a^- - 2\delta\bigr)e^{-\epsilon t}\bigr]\zeta\bigl(-\epsilon(x-\xi+Ct)\bigr), 1\bigr\},$$
$$v^-(x,t) := \max\bigl\{-\delta + \bigl[1 - \bigl(1 - a^+ - 2\delta\bigr)e^{-\epsilon t}\bigr]\zeta\bigl(\epsilon(x-\xi-Ct)\bigr), 0\bigr\}$$

are a supersolution and a subsolution of (1.1) *on* $[0,\infty)$*, respectively.*

Proof. We only prove that $v^+(x,t)$ is a supersolution of (1.1) since the proof for $v^-(x,t)$ is analogous. Given $\delta \in (0, \delta_0]$, we define

$$m_1 = m_1(\delta) = \max\bigl\{\partial_2 g(u,v):\ (u,v) \in [\delta,1] \times \bigl[S(\delta), S(1+\delta)\bigr]\bigr\} > 0,$$
$$m_2 = m_2(\delta) = \min\bigl\{\zeta'(s):\ \delta/2 \leqslant \zeta(s) \leqslant 1 - \delta/2\bigr\} > 0,$$
$$\kappa_1 = \max\bigl\{S'(u):\ u \in (\delta, 1+\delta)\bigr\}.$$

Let $h_0 = \int_0^\tau \int_{-\infty}^\infty h(y,s)|y|\,dy\,ds$. Then there exists $\epsilon = \epsilon(\delta) > 0$ such that

$$\bigl(a^- - 2\delta\bigr)e^{\epsilon\tau} < 1, \tag{4.2}$$
$$\max\bigl\{g\bigl(u, S(u)\bigr):\ u \in \bigl[\delta, a^- - \delta/2\bigr]\bigr\} + \bigl(\epsilon + d\epsilon^2 + m_1\kappa_1\epsilon(\tau + h_0)\bigr) < 0. \tag{4.3}$$

We further choose $C = C(\delta) > |B|$ such that

$$\bigl(C - |B|\bigr)\epsilon\bigl(1 - a^-\bigr)m_2 - \max\bigl\{\bigl|g\bigl(u, S(u)\bigr)\bigr|:\ u \in [\delta, 1]\bigr\} - \bigl(\epsilon + d\epsilon^2 + m_1\kappa_1\epsilon(\tau + h_0)\bigr) > 0. \tag{4.4}$$

Let

$$\hat{v}^+(x,t) = (1+\delta) - \bigl[1 - \bigl(a^- - 2\delta\bigr)e^{-\epsilon t}\bigr]\zeta\bigl(-\epsilon(x-\xi+Ct)\bigr).$$

By a direct computation and (4.2), it follows that for all $t \geqslant -\tau$,

$$\frac{\partial \hat{v}^+(x,t)}{\partial t} \geqslant -\epsilon\bigl(a^- - 2\delta\bigr)e^{\epsilon\tau}\zeta\bigl(-\epsilon(x-\xi+Ct)\bigr) \geqslant -\epsilon$$

and

$$\frac{\partial \hat{v}^+(x,t)}{\partial x} = \epsilon\bigl[1 - \bigl(a^- - 2\delta\bigr)e^{-\epsilon t}\bigr]\zeta'\bigl(-\epsilon(x-\xi+Ct)\bigr) \leqslant \epsilon.$$

Define

$$B^+ = \bigl\{(x,t):\ \hat{v}^+(x,t) > 1\bigr\} \quad \text{and} \quad B^- = \bigl\{(x,t):\ \hat{v}^+(x,t) < 1\bigr\}.$$

Then for any $(x,t) \in B^+$, it is easy to show that

$$\frac{\partial v^+(x,t)}{\partial t} - d\Delta v^+(x,t) - B\frac{\partial v^+(x,t)}{\partial x} - g\bigl(v^+(x,t), \bigl(h * S\bigl(v^+\bigr)\bigr)(x,t)\bigr) \geqslant 0. \tag{4.5}$$

In the following, we show that (4.5) holds for any $(x,t)\in B^-$. Let $(x,t)\in B^-$. Then $v^+(x,t)=\hat{v}^+(x,t)\in[\delta,1]$. Therefore, we have

$$\begin{aligned}
&g\big(v^+(x,t),\big(h*S\big(v^+\big)\big)(x,t)\big)\\
&\quad\leqslant g\big(v^+(x,t),S\big(v^+(x,t)\big)\big)\\
&\qquad+\big[g\big(v^+(x,t),\big(h*S\big(\hat{v}^+\big)\big)(x,t)\big)-g\big(v^+(x,t),S\big(v^+(x,t)\big)\big)\big]\\
&\quad=g\big(v^+(x,t),S\big(v^+(x,t)\big)\big)+\partial_2 g\big(v^+(x,t),S^*(x,t)\big)\\
&\qquad\times\int_0^\tau\int_{-\infty}^{\infty}h(y,s)S'\big(v^*(x-y,t-s)\big)\big(\hat{v}^+(x-y,t-s)-\hat{v}^+(x,t)\big)\,dy\,ds\\
&\quad=g\big(v^+(x,t),S\big(v^+(x,t)\big)\big)+\partial_2 g\big(v^+(x,t),S^*(x,t)\big)\int_0^\tau\int_{-\infty}^{\infty}\bigg\{h(y,s)\\
&\qquad\times S'\big(v^*(x-y,t-s)\big)\bigg[-y\frac{\partial\hat{v}^+(x-\theta_1 y,t-s)}{\partial x}-s\frac{\partial\hat{v}^+(x,t-\theta_2 s)}{\partial t}\bigg]\bigg\}\,dy\,ds\\
&\quad\leqslant g\big(v^+(x,t),S\big(v^+(x,t)\big)\big)+m_1\kappa_1\epsilon(\tau+h_0),
\end{aligned}$$

where $S^*(x,t)$ is between $S(v^+(x,t))$ and $(h*S(\hat{v}^+))(x,t)$, $v^*(x-y,t-s)$ is between $\hat{v}^+(x-y,t-s)$ and $\hat{v}^+(x,t)$. It then follows that

$$\begin{aligned}
&\frac{\partial v^+(x,t)}{\partial t}-d\Delta v^+(x,t)-B\frac{\partial v^+(x,t)}{\partial x}-g\big(v^+(x,t),\big(h*S\big(v^+\big)\big)(x,t)\big)\\
&\quad=\epsilon C\big[1-\big(a^--2\delta\big)e^{-\epsilon t}\big]\zeta'\big(-\epsilon(x-\xi+Ct)\big)-\epsilon\big(a^--2\delta\big)e^{-\epsilon t}\zeta\big(-\epsilon(x-\xi+Ct)\big)\\
&\qquad+d\epsilon^2\big[1-\big(a^--2\delta\big)e^{-\epsilon t}\big]\zeta''\big(-\epsilon(x-\xi+Ct)\big)\\
&\qquad-\epsilon B\big[1-\big(a^--2\delta\big)e^{-\epsilon t}\big]\zeta'\big(-\epsilon(x-\xi+Ct)\big)-g\big(v^+(x,t),\big(h*S\big(v^+\big)\big)(x,t)\big)\\
&\quad\geqslant\epsilon\big(C-|B|\big)\big(1-a^-\big)\zeta'\big(-\epsilon(x-\xi+Ct)\big)-\epsilon-d\epsilon^2\\
&\qquad-g\big(v^+(x,t),S\big(v^+(x,t)\big)\big)-m_1\kappa_1\epsilon(\tau+h_0)\\
&\quad=\epsilon\big(C-|B|\big)\big(1-a^-\big)\zeta'\big(-\epsilon(x-\xi+Ct)\big)\\
&\qquad-\big[g\big(v^+(x,t),S\big(v^+(x,t)\big)\big)+\big(\epsilon+d\epsilon^2+m_1\kappa_1\epsilon(\tau+h_0)\big)\big].
\end{aligned}\tag{4.6}$$

We distinguish between two cases:
Case (i): $\zeta(-\epsilon(x-\xi+Ct))>1-\delta/2$. It then follows that

$$\delta<v^+(x,t)=\hat{v}^+(x,t)<(1+\delta)-\big[1-\big(a^--2\delta\big)\big](1-\delta/2)<a^--\delta/2.$$

Therefore, by (4.3) and (4.6), we have

$$
\begin{aligned}
&\frac{\partial v^+(x,t)}{\partial t} - d\Delta v^+(x,t) - B\frac{\partial v^+(x,t)}{\partial x} - g\big(v^+(x,t), \big(h * S\big(v^+\big)\big)(x,t)\big) \\
&\quad \geqslant \epsilon\big(C - |B|\big)\big(1 - a^-\big)\zeta'\big(-\epsilon(x - \xi + Ct)\big) \\
&\qquad - \big[g\big(v^+(x,t), S\big(v^+(x,t)\big)\big) + \big(\epsilon + d\epsilon^2 + m_1\kappa_1\epsilon(\tau + h_0)\big)\big] \\
&\quad \geqslant -\big[\max\big\{g\big(u, S(u)\big)\colon\ u \in \big[\delta, a^- - \delta/2\big]\big\} + \big(\epsilon + d\epsilon^2 + m_1\kappa_1\epsilon(\tau + h_0)\big)\big] \geqslant 0.
\end{aligned}
$$

Case (ii): $\zeta(-\epsilon(x - \xi + Ct)) \in [\delta/2, 1 - \delta/2]$. By (4.4) and (4.6), we have

$$
\begin{aligned}
&\frac{\partial v^+(x,t)}{\partial t} - d\Delta v^+(x,t) - B\frac{\partial v^+(x,t)}{\partial x} - g\big(v^+(x,t), \big(h * S\big(v^+\big)(x,t)\big)\big) \\
&\quad \geqslant \epsilon\big(C - |B|\big)\big(1 - a^-\big)\zeta'\big(-\epsilon(x - \xi + Ct)\big) \\
&\qquad - \big[g\big(v^+(x,t), S\big(v^+(x,t)\big)\big) + \big(\epsilon + d\epsilon^2 + m_1\kappa_1\epsilon(\tau + h_0)\big)\big] \\
&\quad \geqslant \epsilon\big(C - |B|\big)\big(1 - a^-\big)m_2 \\
&\qquad - \big[\max\big\{\big|g\big(u, S(u)\big)\big|\colon\ u \in [\delta, 1]\big\} + \big(\epsilon + d\epsilon^2 + m_1\kappa_1\epsilon(\tau + h_0)\big)\big] \\
&\quad \geqslant 0.
\end{aligned}
$$

Now we conclude that (4.5) holds for all $(x,t) \in B^+ \cup B^-$. The remainder is to show that (2.4) holds for v^+. Define $\tilde{v}^+(x,t) = v^+(x - Bt, t)$. To complete the proof, it is sufficient to prove that $\tilde{v}^+(x,t)$ is a supersolution of (2.10). Take $\tilde{B}^+ = \{(x,t)\colon\ (x - Bt, t) \in B^+\}$ and $\tilde{B}^- = \{(x,t)\colon\ (x - Bt, t) \in B^-\}$. Notice that for every $t_0 \geqslant 0$, if $\hat{v}^+(x - Bt_0, t_0) = 1$, then there must be $0 < \zeta(-\epsilon(x - Bt_0 - \xi + Ct_0)) < 1$. Thus, there exists a unique $x^+(t_0)$ such that $0 < \zeta(-\epsilon(x^+(t_0) - Bt_0 - \xi + Ct_0)) < 1$, $\hat{v}^+(x^+(t_0) - Bt_0, t_0) = 1$, $(x, t_0) \in \tilde{B}^+$ for any $x > x^+(t_0)$, $(x, t_0) \in \tilde{B}^-$ for any $x < x^+(t_0)$ and

$$
\frac{\partial \tilde{v}^+(x^+(t_0) - 0, t_0)}{\partial x} = \lim_{x \to x^+(t_0) - 0} \epsilon\big[1 - \big(a^- - 2\delta\big)e^{-\epsilon t_0}\big]\zeta'\big(-\epsilon(x - Bt_0 - \xi + Ct_0)\big) > 0.
$$

Consequently, we can show that $\tilde{v}^+(x,t)$ is a supersolution of (2.10) by a similar argument to that of Lemma 3.2. The details are omitted. The proof is complete. □

Let $U(x - ct)$ be a monotone traveling wave solution of (1.1). In view of Lemma 3.2, we define the following two functions

$$
\begin{aligned}
w^+(x,t,\xi_0,\delta) &= \min\big\{U\big(x - ct + \xi_0 + \sigma_0\delta\big(1 - e^{-\beta_0 t}\big)\big) + \delta e^{-\beta_0 t}, 1\big\}, \\
w^-(x,t,\xi_0,\delta) &= \max\big\{U\big(x - ct + \xi_0 - \sigma_0\delta\big(1 - e^{-\beta_0 t}\big)\big) - \delta e^{-\beta_0 t}, 0\big\}, \\
&x \in \mathbb{R},\ t \in [-\tau, \infty),\ \xi_0 \in \mathbb{R},\ \text{and } \delta \in [0, \infty),
\end{aligned}
$$

where σ_0 and β_0 are as in Lemma 3.2. By the proof of Lemma 3.2 we can choose $\beta_0 > 0$ as small as we wish. Thus we assume that β_0 has been chosen such that $3e^{\beta_0\tau} < 4$ throughout this section.

Remark 4.2. If $u(x,s)$ satisfies $0 \leqslant u(x,s) \leqslant 1$ for all $x \in \mathbb{R}$ and $s \in [-\tau, 0]$, then for all $x \in \mathbb{R}$ and $s \in [-\tau, 0]$, $w^-(x,s,\xi_0,\delta) \leqslant u(x,s)$ is equivalent to

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$$U\big(x-cs+\xi_0-\sigma_0\delta\big(1-e^{-\beta_0 s}\big)\big)-\delta e^{-\beta_0 s}\leqslant u(x,s)$$

and $w^+(x,s,\xi_0,\delta)\geqslant u(x,s)$ is equivalent to

$$U\big(x-cs+\xi_0+\sigma_0\delta\big(1-e^{-\beta_0 s}\big)\big)+\delta e^{-\beta_0 s}\geqslant u(x,s).$$

Lemma 4.3. *Let $U(x-ct)$ be a monotone traveling wave solution of (1.1). Then there exists a positive number ε such that, if $u(x,t)$ is a solution of (1.1) on $[0,\infty)$ with initial data $0\leqslant u(x,s)\leqslant 1$ for all $x\in\mathbb{R}$ and $s\in[-\tau,0]$, and for some $\xi\in\mathbb{R}$, $\eta>0$, $0<\delta<\min(\bar{\delta}/2,\frac{1}{\sigma_0})$ and $T\geqslant 0$, there holds*

$$w_0^-(x,-cT+\xi,\delta)(s)\leqslant u_T(x)(s)\leqslant w_0^+(x,-cT+\xi+\eta,\delta)(s),\quad s\in[-\tau,0],\ x\in\mathbb{R},$$

then for every $t\geqslant T+\tau+1$, there exist $\hat{\xi}(t)$, $\hat{\delta}(t)$ and $\hat{\eta}(t)$ such that

$$w_0^-\big(x,-ct+\hat{\xi}(t),\hat{\delta}(t)\big)(s)\leqslant u_t(x)(s)\leqslant w_0^+\big(x,-ct+\hat{\xi}(t)+\hat{\eta}(t),\hat{\delta}(t)\big)(s)$$

for $s\in[-\tau,0]$, $x\in\mathbb{R}$, with $\hat{\xi}(t)$, $\hat{\delta}(t)$ and $\hat{\eta}(t)$ satisfying

$$\begin{aligned}&\hat{\xi}(t)\in\big[\xi-\sigma_0\delta-2\sigma_0\big(\delta+\varepsilon\min(\eta,1)\big)e^{\beta_0\tau},\xi+\eta+\sigma_0\delta\big],\\&\hat{\delta}(t)=\big(\delta e^{-\beta_0}+\varepsilon\min(\eta,1)\big)e^{-\beta_0((t-(T+\tau+1)))}\end{aligned}$$

and

$$\hat{\eta}(t)\in\big[0,\eta+\big(3e^{\beta_0\tau}-4\big)\sigma_0\varepsilon\min(\eta,1)+3e^{\beta_0\tau}\sigma_0\delta\big].$$

Proof. By Lemma 3.2, $w^+(x,t,-cT+\xi+\eta,\delta)$ and $w^-(x,t,-cT+\xi,\delta)$ are super- and sub-solutions of (1.1), respectively. Clearly, $v(x,t)=u(x,T+t)$, $t\geqslant 0$, is also a solution of (1.1) with $v_0(x)(s)=u_T(x)(s)$, $s\in[-\tau,0]$, $x\in\mathbb{R}$. Then, by Theorem 2.3, there holds

$$\begin{aligned}&w^-(x,t,-cT+\xi,\delta)\leqslant u(x,T+t)\leqslant w^+(x,t,-cT+\xi+\eta,\delta),\\&\quad x\in\mathbb{R},\ t\in[0,\infty).\end{aligned}$$

That is

$$\begin{aligned}&\max\big\{U\big(x-c(T+t)+\xi-\sigma_0\delta\big(1-e^{-\beta_0 t}\big)\big)-\delta e^{-\beta_0 t},0\big\}\leqslant u(x,T+t)\\&\quad\leqslant\min\big\{U\big(x-c(T+t)+\xi+\eta+\sigma_0\delta\big(1-e^{-\beta_0 t}\big)\big)+\delta e^{-\beta_0 t},1\big\}\end{aligned}\tag{4.7}$$

for all $x\in\mathbb{R}$, $t\in[0,\infty)$. Let $z=cT-\xi$. Again by Theorem 2.3, we have that for any $J\geqslant 0$, all $x\in\mathbb{R}$ with $|x-z|\leqslant J$ and all $t>0$,

$$u(x,T+t)-w^-(x,t,-cT+\xi,\delta)\geqslant\Theta(J,t)\int_z^{z+1}\big(u(y,T)-w^-(y,0,-cT+\xi,\delta)\big)\,dy.\tag{4.8}$$

By Lemma 3.1, $\lim_{|r|\to\infty} U'(r) = 0$. Then we can fix a positive number M such that $U'(r) \leqslant \frac{1}{2\sigma_0}$ for all $|r| \geqslant M$. Let $J = M + |c|(1+\tau) + 1$, $\bar\eta = \min(\eta, 1)$, and

$$\varepsilon_1 = \frac{1}{2}\min\bigl\{U'(x):\ |x| \leqslant 2\bigr\} > 0.$$

Since

$$w^+(y, 0, -cT + \xi + \bar\eta, \delta) > U(y - cT + \xi + \bar\eta),$$
$$w^-(y, 0, -cT + \xi, \delta) < U(y - cT + \xi),$$

we have

$$\begin{aligned}
&\int_z^{z+1} \bigl[w^+(y, 0, -cT + \xi + \bar\eta, \delta) - w^-(y, 0, -cT + \xi, \delta)\bigr]\,dy \\
&\quad > \int_z^{z+1} \bigl[U(y - cT + \xi + \bar\eta) - U(y - cT + \xi)\bigr]\,dy \\
&\quad = \int_0^1 \bigl[U(y + \bar\eta) - U(y)\bigr]\,dy \geqslant 2\varepsilon_1\bar\eta.
\end{aligned}$$

Hence, at least one of the following is true:

(i) $\int_z^{z+1} [u(y, T) - w^-(y, 0, -cT + \xi, \delta)]\,dy \geqslant \varepsilon_1\bar\eta$;
(ii) $\int_z^{z+1} [w^+(y, 0, -cT + \xi + \bar\eta, \delta) - u(y, T)]\,dy \geqslant \varepsilon_1\bar\eta$.

In what follows, we consider only the case (i). The case (ii) is similar and thus omitted.

For any $s \in [-\tau, 0]$, $|x - z| \leqslant J$, letting $t = 1 + \tau + s \geqslant 1$ in (4.8), we have

$$\begin{aligned}
&u(x, T + 1 + \tau + s) \\
&\quad \geqslant w^-(x, 1 + \tau + s, -cT + \xi, \delta) + \Theta_0(J)\varepsilon_1\bar\eta \\
&\quad \geqslant U\bigl(x - z - c(1 + \tau + s) - \sigma_0\delta\bigl(1 - e^{-\beta_0(1+\tau+s)}\bigr)\bigr) - \delta e^{-\beta_0(1+\tau+s)} + \Theta_0(J)\varepsilon_1\bar\eta,
\end{aligned}$$

where $\Theta_0(J) = \min_{s\in[-\tau,0]} \Theta(J, 1 + \tau + s)$. Let

$$J_1 = J + |c|(1+\tau) + 3, \qquad \varepsilon = \min\left\{\min_{|x|\leqslant J_1} \frac{\Theta_0(J)\varepsilon_1}{2\sigma_0 U'(x)}, \frac{1}{3\sigma_0}, \bar\delta/2\right\}.$$

By the mean value theorem, it then follows that for all $|x - z| \leqslant J$, $s \in [-\tau, 0]$,

$$\begin{aligned}&U\big(x-z-c(1+\tau+s)+2\sigma_0\varepsilon\bar{\eta}-\sigma_0\delta\big(1-e^{-\beta_0(1+\tau+s)}\big)\big)\\&\quad-U\big(x-z-c(1+\tau+s)-\sigma_0\delta\big(1-e^{-\beta_0(1+\tau+s)}\big)\big)\\&=U'(\mu_1)2\sigma_0\varepsilon\bar{\eta}\leqslant\Theta_0(J)\varepsilon_1\bar{\eta}.\end{aligned}$$

Hence,

$$\begin{aligned}&u(x,T+1+\tau+s)\\&\quad\geqslant U\big(x-c(T+1+\tau+s)+\xi+2\sigma_0\varepsilon\bar{\eta}-\sigma_0\delta\big(1-e^{-\beta_0(1+\tau+s)}\big)\big)\\&\qquad-\delta e^{-\beta_0(1+\tau+s)}.\end{aligned}\tag{4.9}$$

The remainder of proof is similar to that of [42, Lemma 3.1] and is omitted. We only need to notice that $\delta e^{-\beta_0}+\varepsilon\bar{\eta}<\bar{\delta}$ and Remark 4.2. The proof is complete. □

By Remark 4.2 and Lemmas 3.2, 4.1 and 4.3, we can obtain the following Lemma 4.4 and Theorem 4.5. Their proofs are only duplications of proofs of [42, Lemma 3.2, Theorem 3.3], so we omit them.

Lemma 4.4. *Let $U(x-ct)$ be a monotone traveling wave solution of* (1.1) *and let $\varphi\in[0,1]_C$ be such that*

$$\liminf_{x\to\infty}\min_{s\in[-\tau,0]}\varphi(x,s)>a^+,\qquad\limsup_{x\to-\infty}\max_{s\in[-\tau,0]}\varphi(x,s)<a^-.$$

Then, for any $\delta>0$, there exist $T=T(\varphi,\delta)>0$, $\xi=\xi(\varphi,\delta)\in\mathbb{R}$ and $\eta=\eta(\varphi,\delta)>0$ such that

$$w_0^-(x,-cT+\xi,\delta)(s)\leqslant u_T(x,\varphi)(s)\leqslant w_0^+(x,-cT+\xi+\eta,\delta)(s),\quad s\in[-\tau,0],\ x\in\mathbb{R}.$$

Theorem 4.5. *Assume that* (H1) *and* (H2) *hold. Assume further that* (1.1) *has a monotone traveling wave solution $U(x-ct)$. Then $U(x-ct)$ is globally asymptotically stable with phase shift in the sense that there exists $k>0$ such that for any $\varphi\in[0,1]_C$ with*

$$\liminf_{x\to\infty}\min_{s\in[-\tau,0]}\varphi(x,s)>a^+,\qquad\limsup_{x\to-\infty}\max_{s\in[-\tau,0]}\varphi(x,s)<a^-,$$

the solution $u(x,t,\varphi)$ of (1.1) *with the initial value φ satisfies*

$$\big|u(x,t,\varphi)-U(x-ct+\xi)\big|\leqslant Ke^{-kt},\quad x\in\mathbb{R},\ t\geqslant 0,$$

for some $K=K(\varphi)>0$ and $\xi=\xi(\varphi)\in\mathbb{R}$.

Remark 4.6. Theorems 3.3 and 4.5 are still available for equation without delay.

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5. Existence of traveling wave fronts

In this section, we consider the case $a^+ = a^-$, namely, $f_0(u) = g(u, S(u))$ has only three zeros. For the case that $f_0(u)$ has more than three zeros, we refer to [20,44]. In the following, we denote $a = a^+ = a^-$. Moreover, we list the following conditions:

(H3) $g(u, S(u)) < 0$ for $u \in (0, a)$, $g(u, S(u)) > 0$ for $u \in (a, 1)$ and $\partial_1 g(a, S(a)) + \partial_2 g(a, S(a))S'(a) > 0$.

As discussed in Lemma 2.3, we can show that if $U(x - ct)$ is a traveling wave front of (1.1) with wave speed c, then $U(x - (B + c)t)$ is a traveling wave front of (2.10) with wave speed $B + c$. Inversely, if $V(x - ct)$ is a traveling wave front of (2.10), then $V(x + (B - c)t)$ is a traveling wave front of (1.1) with wave speed $-B + c$. Following this fact, we only need to consider the existence of traveling wave fronts of (2.10). Define $h_c(y) = \int_0^\tau \bar{h}(y + cs, s)\,ds$, where c is a real constant. In the first part of this section, we show the existence of traveling wave fronts of the following equation

$$\frac{\partial u(x,t)}{\partial t} = d\Delta u(x,t) + g\big(u(x,t), \big(h_c * S(u)\big)(x,t)\big), \quad x \in \mathbb{R},\ t \in [0, \infty), \tag{5.1}$$

where

$$\big(h_c * S(u)\big)(x,t) = \int_{-\infty}^{\infty} h_c(y) S\big(u(x - y, t)\big)\,dy.$$

Lemma 5.1. *Assume that* (H1) *and* (H2) *hold. Then for any* $\delta \in (0, \bar{\delta}_0]$, $\bar{\delta}_0 = \min\{\frac{a}{2}, \frac{1-a}{2}\}$, *there exist two positive constants* $\epsilon_0 = \epsilon_0(\delta) > 0$ *and* $C_0 = C_0(\delta) > 0$ *such that*

(i) *the functions* $v_0^+(x,t)$ *and* $v_0^-(x,t)$ *defined by*

$$v_0^+(x,t) = \min\big\{(1+\delta) - \big[1 - (a - 2\delta)e^{-\epsilon_0 t}\big]\zeta\big(-\epsilon_0(x - \xi + C_0 t)\big), 1\big\},$$
$$v_0^-(x,t) = \max\big\{-\delta + \big[1 - (1 - a - 2\delta)e^{-\epsilon_0 t}\big]\zeta\big(\epsilon_0(x - \xi - C_0 t)\big), 0\big\}$$

are a supersolution of (5.1) *for* $c \leqslant 0$ *and a subsolution of* (5.1) *for* $c \geqslant 0$, *respectively*;

(ii) *the functions* $v_c^+(x,t)$ *and* $v_c^-(x,t)$ *defined by*

$$v_c^+(x,t) = \min\big\{(1+\delta) - \big[1 - (a - 2\delta)e^{-\epsilon_c t}\big]\zeta\big(-\epsilon_c(x - \xi + C_c t)\big), 1\big\},$$
$$v_c^-(x,t) = \max\big\{-\delta + \big[1 - (1 - a - 2\delta)e^{-\epsilon_c t}\big]\zeta\big(\epsilon_c(x - \xi - C_c t)\big), 0\big\}$$

are a supersolution of (5.1) *and a subsolution of* (5.1) *for any* $c \in \mathbb{R}$, *respectively, where* $\epsilon_c = \frac{\epsilon_0}{1+|c|}$ *and* $C_c = (1 + |c|)C_0$.

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Proof. We only prove for $v_0^+(x,t)$ and $v_c^+(x,t)$. Define

$$\hat{v}_0^+(x,t) = (1+\delta) - \big[1-(a-2\delta)e^{-\epsilon_0 t}\big]\zeta\big(-\epsilon_0(x-\xi+C_0 t)\big),$$
$$\hat{v}_c^+(x,t) = (1+\delta) - \big[1-(a-2\delta)e^{-\epsilon_c t}\big]\zeta\big(-\epsilon_c(x-\xi+C_c t)\big),$$
$$\hat{v}_c^-(x,t) = -\delta + \big[1-(1-a-2\delta)e^{-\epsilon_c t}\big]\zeta\big(\epsilon_c(x-\xi-C_c t)\big).$$

By a similar argument to that of Lemma 4.1 for $v^+(x,t)$, it is sufficient to show that for all $(x,t)\in B_i^+\cup B_i^-$,

$$\frac{\partial v_i^+(x,t)}{\partial t} - d\Delta v_i^+(x,t) - g\big(v_i^+(x,t), \big(h_c * S\big(v_i^+\big)\big)(x,t)\big) \geqslant 0, \tag{5.2}$$

respectively, where $i=0,c$ and $B_i^\pm$ are defined by

$$B_i^+ = \big\{(x,t)\colon\ \hat{v}_i^+(x,t) > 1\big\}, \qquad B_i^- = \big\{(x,t)\colon\ \hat{v}_i^+(x,t) < 1\big\}.$$

Obviously, (5.2) holds for $(x,t)\in B_i^+$. Thus we only need to show that (5.2) holds for $(x,t)\in B_i^-$.

For fixed $\delta\in(0,\bar{\delta}_0]$, let

$$\varrho_1 = \varrho_1(\delta) = \max\big\{\partial_2 g(u,v)\colon\ (u,v)\in[0,1]\times\big[S(-\delta), S(1+\delta)\big]\big\},$$
$$\varrho_2 = \varrho_2(\delta) = \max\big\{S'(u)\colon\ u\in[\delta, 1+\delta]\big\},$$
$$\varrho_0 = \varrho_0(\delta) = \varrho_1\varrho_2,$$
$$m_1 = m_1(\delta) = \min\left\{-f_0(u)\colon\ u\in\left[\delta, a-\frac{1}{2}\delta\right]\right\} > 0.$$

Then we can choose two positive constants $\epsilon^* = \epsilon^*(\delta) > 0$ and $M_0 = M_0(\delta) > 0$, with $\epsilon^* < \delta$ sufficiently small and M_0 sufficiently large, such that

$$m_1 - \varrho_0\epsilon^* - 2\varrho_0\left[\int_0^\tau\int_{-\infty}^{\infty} - \int_0^\tau\int_{-M_0}^{M_0} \bar{h}(y,s)\,dy\,ds\right] > 0.$$

Take $\mu = \mu(\epsilon^*)\in(0,1)$ sufficiently small such that

$$0 \leqslant \zeta(x) < \frac{\epsilon^*}{2} \quad \text{if } x<\mu,$$
$$1-\frac{\epsilon^*}{2} < \zeta(x) \leqslant 1 \quad \text{if } x > 4-\mu,$$

$\varpi = \varpi(\mu) > 0$ sufficiently small such that

$$(1-\varpi)\left(4-\frac{\mu}{2}\right) > 4-\mu,$$

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and $\epsilon_0 = \epsilon_0(\delta) > 0$ sufficiently small such that

$$\epsilon_0 M_0 \leqslant \varpi(4-\mu), \qquad \epsilon_0\tau < \varpi\left(4-\frac{\mu}{2}\right),$$

$$-\epsilon_0 - d\epsilon_0^2 + m_1 - \varrho_0\epsilon^* - 2\varrho_0\left[\int_0^\tau\int_{-\infty}^{\infty} - \int_0^\tau\int_{-M_0}^{M_0} \bar h(y,s)\,dy\,ds\right] > 0.$$

Set

$$m_0 = \min\left\{\zeta'(x)\colon\ \frac{\mu}{2} \leqslant x \leqslant 4-\frac{\mu}{2}\right\} > 0.$$

Take $C_0 = C_0(\delta) > 0$ such that

$$\epsilon_0 C_0(1-a+2\delta)m_0 - \epsilon_0 - d\epsilon_0^2 - \max\left\{\left|g\big(u, S(v)\big)\right|\colon\ (u,v)\in[\delta,1]^2\right\} > 0.$$

Note that ϵ_0 and C_0 are independent of c.

Assume $(x,t)\in B_0^-$. Then we have that for all $t \geqslant 0$,

$$\begin{aligned}
&g\big(v_0^+(x,t), \big(h_c * S\big(v_0^+\big)\big)(x,t)\big)\\
&\quad\leqslant g\big(v_0^+(x,t), S\big(v_0^+(x,t)\big)\big)\\
&\qquad + \big[g\big(v_0^+(x,t), \big(h_c * S\big(\hat v_0^+\big)\big)(x,t)\big) - g\big(v_0^+(x,t), S\big(v_0^+(x,t)\big)\big)\big]\\
&\quad= g\big(v_0^+(x,t), S\big(v_0^+(x,t)\big)\big) + \partial_2 g\big(v_0^+(x,t), S^*(x,t)\big)\\
&\qquad\times \int_{-\infty}^{\infty} h_c(y) S'\big(v_0^*(y)\big)\big(\hat v_0^+(x-y,t) - v_0^+(x,t)\big)\,dy\\
&\quad= g\big(v_0^+(x,t), S\big(v_0^+(x,t)\big)\big) + \partial_2 g\big(v_0^+(x,t), S^*(x,t)\big)\\
&\qquad\times \int_{-\infty}^{\infty}\int_0^\tau \bar h(y+cs,s) S'\big(v_0^*(y)\big)\big(\hat v_0^+(x-y,t) - \hat v_0^+(x,t)\big)\,ds\,dy\\
&\quad= g\big(v_0^+(x,t), S\big(v_0^+(x,t)\big)\big) + \partial_2 g\big(v_0^+(x,t), S^*(x,t)\big)\\
&\qquad\times \int_0^\tau\int_{-\infty}^{\infty} \bar h(y,s) S'\big(v_0^*(y-cs)\big)\big(\hat v_0^+(x-y+cs,t) - \hat v_0^+(x,t)\big)\,dy\,ds\\
&\quad\leqslant g\big(v_0^+(x,t), S\big(v_0^+(x,t)\big)\big)\\
&\qquad + \varrho_0\int_0^\tau\int_{-\infty}^{\infty} \bar h(y,s)\big|\zeta\big(-\epsilon_0(x-y+cs+\xi+C_0t)\big) - \zeta\big(-\epsilon_0(x+\xi+C_0t)\big)\big|\,dy\,ds,
\end{aligned}$$

where $v_0^*(y)$ is between $\hat{v}_0^+(x-y,t)$ and $\hat{v}_0^+(x,t)$, $S^*(x,t)$ is between $S(v_0^+(x,t))$ and $(h_c * S(\hat{v}_0^+))(x,t)$. It then follows that

$$\begin{aligned}
&\frac{\partial v_0^+(x,t)}{\partial t} - d\Delta v_0^+(x,t) - g\big(v_0^+(x,t), \big(h_c * S\big(v_0^+\big)\big)(x,t)\big) \\
&\quad = \epsilon_0 C_0\big[1-(a-2\delta)e^{-\epsilon_0 t}\big]\zeta'\big(-\epsilon_0(x-\xi+C_0t)\big) - \epsilon_0(a-2\delta)e^{-\epsilon_0 t}\zeta\big(-\epsilon_0(x-\xi+C_0t)\big) \\
&\qquad + d\epsilon_0^2\big[1-(a-2\delta)e^{\epsilon_0 t}\big]\zeta''\big(-\epsilon_0(x-\xi+C_0t)\big) - g\big(v_0^+(x,t), \big(h_c * S\big(\hat{v}_0^+\big)\big)(x,t)\big) \\
&\quad \geqslant -\epsilon_0 - d\epsilon_0^2 - g\big(v_0^+(x,t), S\big(v_0^+(x,t)\big)\big) - \partial_2 g\big(v_0^+(x,t), S^*(x,t)\big) \\
&\qquad \times \int_0^\tau \int_{-\infty}^{\infty} \bar{h}(y,s) S'\big(v_0^*(y-cs)\big)\big(\hat{v}_0^+(x-y+cs,t) - \hat{v}_0^+(x,t)\big)\,ds\,dy \\
&\quad \geqslant -\epsilon_0 - d\epsilon_0^2 - g\big(v_0^+(x,t), S\big(v_0^+(x,t)\big)\big) \\
&\qquad - \varrho_0 \int_0^\tau \int_{-\infty}^{\infty} \bar{h}(y,s)\big|\zeta\big(-\epsilon_0(x-y+cs-\xi+C_0t)\big) - \zeta\big(-\epsilon_0(x-\xi+C_0t)\big)\big|\,dy\,ds.
\end{aligned}$$

Let $\eta = \epsilon_0(x-\xi+C_0t)$, we consider two cases.

Case (i): $\eta = \epsilon_0(x-\xi+C_0t) < -4 + \frac{\mu}{2}$. Then

$$\zeta\big(-\epsilon_0(x-\xi+C_0t)\big) > \zeta\left(4-\frac{\mu}{2}\right) > 1 - \frac{\epsilon^*}{2},$$

so

$$\begin{aligned}
\delta \leqslant v_0^+(x,t) = \hat{v}_0^+(x,t) &\leqslant (1+\delta) - \big[1-(a-2\delta)\big]\left(1-\frac{\epsilon^*}{2}\right) \\
&\leqslant \left(1-\frac{\epsilon^*}{2}\right)(a-\delta) + \frac{\epsilon^*}{2} < a - \frac{1}{2}\delta.
\end{aligned}$$

By the choice of ϵ_0 and ϖ, we see that

$$\frac{\eta\varpi}{\epsilon_0} = \varpi(x-\xi+C_0t) \leqslant \frac{\varpi(-4+\frac{\mu}{2})}{\epsilon_0} < \frac{\varpi(-4+\mu)}{\epsilon_0} \leqslant -M_0.$$

Let $y \in [\varpi(x-\xi+C_0t), -\varpi(x-\xi+C_0t)]$. Then for $c \leqslant 0$,

$$\begin{aligned}
\epsilon_0(x-y+cs-\xi+C_0t) &< \epsilon_0(1-\varpi)(x-\xi+C_0t) + \epsilon_0 cs \\
&\leqslant (1-\varpi)\left(-4+\frac{\mu}{2}\right) < -4+\mu.
\end{aligned}$$

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Noting that $\eta = \epsilon_0(x - \xi + C_0 t) < -4 + \frac{\mu}{2} < -4 + \mu$, we have

$$\int_0^\tau \int_{\frac{\varpi\eta}{\epsilon_0}}^{-\frac{\varpi\eta}{\epsilon_0}} \bar{h}(y,s)\big|\zeta\big(-\epsilon_0(x - y + cs - \xi + C_0 t)\big) - \zeta\big(-\epsilon_0(x - \xi + C_0 t)\big)\big|\, dy\, ds \leqslant \epsilon^*.$$

Therefore,

$$\begin{aligned}
&\frac{\partial v_0^+(x,t)}{\partial t} - d\Delta v_0^+(x,t) - g\left(v_0^+(x,t), \int_{-\infty}^{\infty} h_c(y) S\big(v_0^+(x-y,t)\big)\, dy\right) \\
&\quad \geqslant -\epsilon_0 - d\epsilon_0^2 - g\big(v_0^+(x,t), S\big(v_0^+(x,t)\big)\big) \\
&\qquad - \varrho_0 \int_0^\tau \int_{-\infty}^{\infty} \bar{h}(y,s)\big|\zeta\big(-\epsilon_0(x - y + cs - \xi + C_0 t)\big) - \zeta\big(-\epsilon_0(x - \xi + C_0 t)\big)\big|\, dy\, ds \\
&\quad \geqslant -\epsilon_0 - d\epsilon_0^2 + m_1 - 2\varrho_0 \left[\int_0^\tau \int_{-\infty}^{\infty} - \int_0^\tau \int_{\frac{\varpi\eta}{\epsilon_0}}^{-\frac{\varpi\eta}{\epsilon_0}} \bar{h}(y,s)\, dy\, ds\right] \\
&\qquad - \varrho_0 \int_0^\tau \int_{\frac{\varpi\eta}{\epsilon_0}}^{-\frac{\varpi\eta}{\epsilon_0}} \bar{h}(y,s)\big|\zeta\big(-\epsilon_0(x - y + cs - \xi + C_0 t)\big) - \zeta\big(-\epsilon_0(x - \xi + C_0 t)\big)\big|\, dy\, ds \\
&\quad \geqslant -\epsilon_0 - d\epsilon_0^2 + m_1 - 2\varrho_0 \left[\int_0^\tau \int_{-\infty}^{\infty} - \int_0^\tau \int_{-M_0}^{M_0} \bar{h}(y,s)\, dy\, ds\right] - \varrho_0 \epsilon^* \geqslant 0.
\end{aligned}$$

Case (ii): $-\frac{\mu}{2} \geqslant \epsilon_0(x - \xi + C_0 t) \geqslant -4 + \frac{\mu}{2}$. Then

$$\begin{aligned}
&\frac{\partial v_0^+(x,t)}{\partial t} - d\Delta v_0^+(x,t) - g\left(v_0^+(x,t), \int_{-\infty}^{\infty} h_c(y) S\big(v_0^+(x-y,t)\big)\, dy\right) \\
&\quad \geqslant \epsilon_0 C_0 (1 - a + 2\delta)\zeta'\big(-\epsilon_0(x - \xi + C_0 t)\big) - \epsilon_0 - d\epsilon_0^2 \\
&\qquad - \max\big\{\big|g\big(u, S(v)\big)\big|:\ (u,v) \in [\delta,1]^2\big\} \geqslant 0.
\end{aligned}$$

Now, we conclude that for $(x,t) \in B_0^-$, if $c \leqslant 0$, then (5.2) holds for $v_0^+(x,t)$.

Noting that $\epsilon_c \leqslant \epsilon_0$, $\epsilon_c C_c = \epsilon_0 C_0$, we can prove that for any $c \in \mathbb{R}$ and any $(x,t) \in B_c^-$, (5.2) holds for $v_c^+(x,t)$.

The proof is complete. □

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Remark 5.2. One observes that the functions $\hat{v}_c^+$ and $\hat{v}_c^-$ have the following properties:

$$\begin{cases} \hat{v}_c^+(x,0) = 1+\delta \quad \text{if } x \geqslant \xi, \quad \hat{v}_c^+(x,0) \geqslant a-\delta \quad \text{for all } x \in \mathbb{R}, \\ \hat{v}_c^+(x,t) \leqslant \delta + (a-2\delta)e^{-\epsilon_c t} \quad \text{for all } t>0,\ x \leqslant \xi - C_c t - 4\epsilon_c^{-1}, \\ \hat{v}_c^-(x,0) = -\delta \quad \text{if } x \leqslant \xi, \quad \hat{v}_c^-(x,0) \leqslant a+\delta \quad \text{for all } x \in \mathbb{R}, \\ \hat{v}_c^-(x,t) \geqslant 1-\delta-(1-a-2\delta)e^{-\epsilon_c t} \quad \text{for all } t>0,\ x \geqslant \xi + C_c t + 4\epsilon_c^{-1}. \end{cases} \tag{5.3}$$

Remark 5.3. By the local regularity result of parabolic equations and the semigroup proposition of solutions, if $u_0(x) \in C^3(\mathbb{R})$ satisfies $0 \leqslant u_0 \leqslant 1$ and $\|u_0(\cdot)\|_{C^3(\mathbb{R})} < \infty$, the solution $u(x,t)$ of (5.1) with initial value $u(\cdot,0) = u_0(\cdot)$ satisfies $\sup_{t\in[0,\infty)} \|u(\cdot,t)\|_{C^2(\mathbb{R})} < \infty$. See also the condition (C4) and Theorem 5.1 of Chen [13].

Lemma 5.4. *Assume that* (H1)–(H3) *hold. Let $H(x)$ be a continuous function equal to* 1 *when $x > \frac{1}{k}$, kx when $0 \leqslant x \leqslant \frac{1}{k}$ and* 0 *when $x < 0$, $k > 1$ is a sufficiently large constant.*

(i) *Let v_I^1 and v_I^2 be the solutions to the following linear evolution problems*:

$$\begin{cases} \dfrac{\partial v_I^1(x,t)}{\partial t} = d\Delta v_I^1(x,t) + \partial_1 g\big(a, S(a)\big)v_I^1(x,t) + \partial_2 g\big(a,S(a)\big)S'(a)\big(h_c * v_I^1\big)(x,t), \\ v_I^1(x,0) = H(x), \end{cases} \tag{5.4}$$

and

$$\begin{cases} \dfrac{\partial v_I^2(x,t)}{\partial t} = d\Delta v_I^2(x,t) + \partial_1 g\big(a, S(a)\big)v_I^2(x,t) + \partial_2 g\big(a,S(a)\big)S'(a)\big(h_c * v_I^2\big)(x,t), \\ v_I^2(x,0) = H(x) - H\left(-x + \dfrac{1}{k}\right) = -1 + 2H(x). \end{cases}$$

Then there exist constants $r_1 > 0$, which is independent of k and c, and $x_c \in \mathbb{R}$ such that

$$v_I^1(x_c, r_1) = 3, \qquad v_I^2(x_c, r_1) = -3.$$

(ii) *There exists a small positive constant $\delta_1 > 0$, which is independent of k and c, such that for any $\delta \in (0,\delta_1)$, the solutions v_{II}^1 and v_{II}^2 to*

$$\begin{cases} \dfrac{\partial v_{II}^1(x,t)}{\partial t} = d\Delta v_{II}^1(x,t) + g\big(v_{II}^1(x,t), \big(h_c * S\big(v_{II}^1\big)\big)(x,t)\big), \\ v_{II}^1(x,0) = a + \delta H(x) \end{cases}$$

and

$$\begin{cases} \dfrac{\partial v_{II}^2(x,t)}{\partial t} = d\Delta v_{II}^2(x,t) + g\big(v_{II}^2(x,t), \big(h_c * S\big(v_{II}^2\big)\big)(x,t)\big), \\ v_{II}^2(x,0) = a + \delta\left[H(x) - H\left(-x + \dfrac{1}{k}\right)\right] \end{cases}$$

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satisfy

$$v_{II}^1(x_c, r_1) \geqslant a + 2\delta, \qquad v_{II}^2(x_c, r_1) \leqslant a - 2\delta.$$

(iii) *For any $\delta \in (0, \delta_1)$, there exists a large positive constant r_0, which is independent of k and c, such that for $r_c = (1 + |c|)r_0$, the solutions v_{III}^1 and v_{III}^2 to*

$$\begin{cases} \dfrac{\partial v_{III}^1(x,t)}{\partial t} = d\Delta v_{III}^1(x,t) + g\big(v_{III}^1(x,t), \big(h_c * S\big(v_{III}^1\big)\big)(x,t)\big), \\ v_{III}^1(x,0) = a + \delta H(x) - aH(-r_c - x) \end{cases}$$

and

$$\begin{cases} \dfrac{\partial v_{III}^2(x,t)}{\partial t} = d\Delta v_{III}^2(x,t) + g\big(v_{III}^2(x,t), \big(h_c * S\big(v_{III}^2\big)\big)(x,t)\big), \\ v_{III}^2(x,0) = a + \delta\left[H(x) - H\left(-x + \dfrac{1}{k}\right)\right] + (1 - a - \delta)H(x - r_c) \end{cases}$$

satisfy

$$v_{III}^1(x_c, r_1) \geqslant a + \delta, \qquad v_{III}^2(x_c, r_1) \leqslant a - \delta.$$

(iv) *Let $u(x,t)$ be the solution of* (5.1) *with initial data $u_0(x) \in C^3(\mathbb{R})$ satisfying $0 \leqslant u_0 \leqslant 1$ and $\|u_0(\cdot)\|_{C^3(\mathbb{R})} < \infty$. For any $\delta \in (0, \delta_1)$, assume that for some finite $\xi_-(0)$ and $\xi_+(0)$,*

$$u_0\big(\xi_-(0)\big) \leqslant a - \delta, \qquad u_0\big(\xi_+(0)\big) \geqslant a + \delta,$$

then for every $t > 0$, there exist $\xi_-(t)$ and $\xi_+(t)$ such that

$$u\big(\xi_-(t), t\big) = a - \delta, \qquad u\big(\xi_+(t), t\big) = a + \delta,$$
$$\xi_+(t) - \xi_-(t) \leqslant \max\big\{\xi_+(0) - \xi_-(0) + 2\big(1 + |c|\big)\big(4\epsilon_0^{-1}(\delta) + C_0(\delta)r_1\big), 2r_c + 2\big\},$$

where $\epsilon_0(\delta)$ and $C_0(\delta)$ are as in Lemma 5.1.

Proof. (i) Since $\partial_2 g(a, S(a))S'(a) \geqslant 0$, then the existence and uniqueness of (5.4) with initial value $\varphi(\cdot, 0) \in X$ follows from a similar argument to that in Theorem 2.3, and the solution is a classical solution for all $t > 0$. Moreover, Eq. (5.4) satisfies the comparison principle.

Denote $f_0'(a) = \partial_1 g(a, S(a)) + \partial_2 g(a, S(a))S'(a)$ by γ. Then $e^{\gamma t}$ is an exact solution to (5.4) with initial value 1. Further, since $-H(-x + \frac{1}{k}) = -1 + H(x)$, by the existence and uniqueness of the initial value problem, one has that $v_I^2(x,t) = -e^{\gamma t} + 2v_I^1(x,t)$ in $\mathbb{R} \times [0, \infty)$. Thus, we need only study $v_I^1(x,t)$.

Since $0 \leqslant v_I^1(\cdot, 0) \leqslant 1$, comparing v_I^1 with 0 and $e^{\gamma t}$ then yields $0 \leqslant v_I^1(\cdot, t) \leqslant e^{\gamma t}$ for all $t \geqslant 0$. In addition, since for each $\eta > 0$, $v_I^1(\cdot + \eta, 0) \geqslant v_I^1(\cdot, 0)$, we have $v_I^1(\cdot + \eta, t) \geqslant v_I^1(\cdot, t)$ for all $t \geqslant 0$. Namely, v_I^1 is nondecreasing in x. Obviously, v_I^1 is continuous in x, too.

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Now, we show that $\lim_{x\to-\infty} v_I^1(x,t) = 0$ and $\lim_{x\to\infty} v_I^1(x,t) = e^{\gamma t}$. Let $L_2 = \max\{|\partial_2 g(u,v)|:\ (u,v)\in[0,1]\times[S(-9),S(10)]\}$, $L_3 = \max\{S'(u):\ u\in[-9,10]\}$ and $\lambda = L_1 + L_2L_3$, where L_1 is defined in Theorem 2.3. Let

$$\rho(\varepsilon_0) = \frac{1}{\gamma}\big[d\varepsilon_0^2 + L_2L_3\varepsilon_0(\bar h_0 + \tau)\big] < \frac{1}{81}, \tag{5.5}$$

$\varepsilon_c = \frac{\varepsilon_0}{1+|c|}$, $w(x,t) = \rho(\varepsilon_0)e^{2\gamma t} + \zeta(\varepsilon_c x)e^{\gamma t}$ and $\bar h_0 = \int_0^\tau\int_{-\infty}^{\infty}\bar h(y,s)|y|\,dy\,ds$. Then

$$\begin{aligned}
&\frac{\partial w(x,t)}{\partial t} - d\Delta w(x,t) - \partial_1 g\big(a,S(a)\big)w(x,t) - \partial_2 g\big(a,S(a)\big)S'(a)(h_c * w)(x,t)\\
&\quad= 2\gamma\rho(\varepsilon_0)e^{2\gamma t} + \gamma\zeta(\varepsilon_c x)e^{\gamma t} - d\varepsilon_c^2\zeta''(\varepsilon_c x)e^{\gamma t} - \partial_1 g\big(a,S(a)\big)\big[\rho(\varepsilon_0)e^{2\gamma t} + \zeta(\varepsilon_c x)e^{\gamma t}\big]\\
&\qquad - \partial_2 g\big(a,S(a)\big)S'(a)\rho(\varepsilon_0)e^{2\gamma t} - \partial_2 g\big(a,S(a)\big)S'(a)e^{\gamma t}\int_{-\infty}^{\infty} h_c(y)\zeta\big(\varepsilon_c(x-y)\big)\,dy\\
&\quad\geqslant \gamma\rho(\varepsilon_0)e^{2\gamma t} - d\varepsilon_0^2 e^{\gamma t} - \partial_2 g\big(a,S(a)\big)S'(a)e^{\gamma t}\int_{-\infty}^{\infty} h_c(y)\big[\zeta\big(\varepsilon_c(x-y)\big) - \zeta(\varepsilon_c x)\big]\,dy\\
&\quad\geqslant \gamma\rho(\varepsilon_0)e^{2\gamma t} - d\varepsilon_0^2 e^{\gamma t} - \partial_2 g\big(a,S(a)\big)S'(a)e^{\gamma t}\int_{-\infty}^{\infty} \varepsilon_c|y|h_c(y)\,dy\\
&\quad= \gamma\rho(\varepsilon_0)e^{2\gamma t} - d\varepsilon_0^2 e^{\gamma t} - \partial_2 g\big(a,S(a)\big)S'(a)\varepsilon_c e^{\gamma t}\int_{-\infty}^{\infty}\int_0^{\tau} |y|\bar h(y+cs,s)\,ds\,dy\\
&\quad\geqslant \gamma\rho(\varepsilon_0)e^{2\gamma t} - d\varepsilon_0^2 e^{\gamma t} - \partial_2 g\big(a,S(a)\big)S'(a)\varepsilon_c e^{\gamma t}\int_0^{\tau}\int_{-\infty}^{\infty} |y|\bar h(y,s)\,dy\,ds\\
&\qquad - \partial_2 g\big(a,S(a)\big)S'(a)\varepsilon_c|c|e^{\gamma t}\int_0^{\tau}\int_{-\infty}^{\infty} s\bar h(y,s)\,dy\,ds\\
&\quad\geqslant \gamma\rho(\varepsilon_0)e^{2\gamma t} - d\varepsilon_0^2 e^{\gamma t} - \partial_2 g\big(a,S(a)\big)S'(a)\varepsilon_0 e^{\gamma t}(\bar h_0 + \tau) \geqslant 0.
\end{aligned}$$

This implies that w is a supersolution of Eq. (5.4). Since $v_I^1(x,0) = H(x) \leqslant w(x + 4/\varepsilon_c, 0)$, the comparison yields $v_I^1(x,t) \leqslant w(x + 4/\varepsilon_c, t)$ in $\mathbb{R}\times[0,\infty)$, namely, $v_I^1(x,t) \leqslant \rho(\varepsilon_0)e^{2\gamma t} + \zeta(\varepsilon_c(x + 4/\varepsilon_c))e^{\gamma t}$. Consequently,

$$0 \leqslant v_I^1(x,t) \leqslant \rho(\varepsilon_0)e^{2\gamma t} \quad \text{for } x \leqslant -4/\varepsilon_c. \tag{5.6}$$

Similarly, we can show that $e^{\gamma t} - w(-x,t)$ is a subsolution. Since $v_I^1(x,0) = H(x) \geqslant 1 - w(-(x-1-4/\varepsilon_c),0)$, we have $v_I^1(x,t) \geqslant e^{\gamma t} - w(-(x-1-4/\varepsilon_c),t)$, that is, $v_I^1(x,t) \geqslant e^{\gamma t} - \rho(\varepsilon_0)e^{2\gamma t} - \zeta(-\varepsilon_c(x-1-4/\varepsilon_c))e^{\gamma t}$. We can conclude that

$$v_I^1(x,t) \geqslant e^{\gamma t} - \rho(\varepsilon_0)e^{2\gamma t} \quad \text{for } x \geqslant 1 + 4/\varepsilon_c. \tag{5.7}$$

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Note that ε_0 may be as small as we wish. Combining (5.6), (5.7) and $0 \leqslant v_I^1(\cdot,t) \leqslant e^{\gamma t}$, we have

$$\lim_{x\to-\infty} v_I^1(x,t)=0 \quad \text{and} \quad \lim_{x\to\infty} v_I^1(x,t)=e^{\gamma t}.$$

Set r_1 such that $e^{\gamma r_1}=9$. Then by the choice of ε_0, we have

$$0 \leqslant v_I^1(x,r_1) \leqslant 1 \quad \text{for } x \leqslant 4/\varepsilon_c,$$
$$9 \leqslant v_I^1(x,r_1) \leqslant 10 \quad \text{for } x \geqslant 1+4/\varepsilon_c.$$

Thus, by the monotonicity and continuity of $v_I^1(\cdot,r_1)$, there exists $x_c \in (-4/\varepsilon_c, 1+4/\varepsilon_c)$ such that

$$v_I^1(x_c,r_1)=3.$$

Consequently, $v_I^2(x_c,r_1)=-3$.

(ii) Note that there exists $K_2>0$, K_2 is independent of c, such that for any $u \in C^0(\mathbb{R})$ with $0 \leqslant a+u \leqslant 1$,

$$\bigl|g\bigl(a+u, h_c * S(a+u)\bigr) - \partial_1 g\bigl(a,S(a)\bigr)u - \partial_2 g\bigl(a,S(a)\bigr)S'(a)h_c * u\bigr| \leqslant K_2\|u\|^2_{C^0(\mathbb{R})}. \tag{5.8}$$

Then let $K=4K_2e^{2\gamma r_1}$ and $\delta_1=\min\{\frac{a}{10}, \frac{1-a}{10}, \frac{1}{2Ke^{(\gamma+1)r_1}}, \bar{\delta}_0\}$. For $\delta \in (0,\delta_1)$, consider the function $w(x,t)=a+\delta v_I^2(x,t)+K\delta^2e^{(\gamma+1)t}$. Clearly, for any $x\in\mathbb{R}$ and $t\in[0,r_1]$, we have $0\leqslant w(x,t)\leqslant 1$. We can calculate, for $t\in[0,r_1]$, that

$$\begin{aligned}
&\frac{\partial w(x,t)}{\partial t} - d\Delta w(x,t) - g\bigl(w(x,t), \bigl(h_c * S(w)\bigr)(x,t)\bigr)\\
&= \delta\frac{\partial v_I^2(x,t)}{\partial t} + (\gamma+1)K\delta^2e^{(\gamma+1)t} - \delta d\Delta v_I^2(x,t) - g\bigl(w(x,t), \bigl(h_c * S(w)\bigr)(x,t)\bigr)\\
&= K\delta^2e^{(\gamma+1)t} - g\bigl(w(x,t), \bigl(h_c * S(w)\bigr)(x,t)\bigr) + \partial_1 g\bigl(a,S(a)\bigr)\bigl[\delta v_I^2(x,t)+K\delta^2e^{(\gamma+1)t}\bigr]\\
&\quad + \partial_2 g\bigl(a,S(a)\bigr)S'(a)\int_{-\infty}^{\infty} h_c(y)\bigl[\delta v_I^2(x-y,t)+K\delta^2e^{(\gamma+1)t}\bigr]\,dy\\
&\geqslant K\delta^2e^{(\gamma+1)t} - K_2\bigl\|\delta v_I^2(x,t)+K\delta^2e^{(\gamma+1)t}\bigr\|^2_{C^0(\mathbb{R})}\\
&\geqslant K\delta^2 - K_2\bigl(\delta e^{\gamma t}+K\delta^2e^{(\gamma+1)t}\bigr)^2 \geqslant 0.
\end{aligned}$$

This implies that $w(x,t)$ is a supersolution of (5.1) on $[0,r_1]$. Thus, by comparison,

$$v_{II}^2(x_c,r_1) \leqslant w(x_c,r_1) \leqslant a+\delta v_I^2(x_c,r_1)+\delta = a-2\delta.$$

In a similar manner, one can show that $w(x,t)=a+\delta v_I^1(x,t)-K\delta^2e^{(\gamma+1)t}$ is a subsolution of (5.1) in $\mathbb{R}\times[0,r_1]$ so that $v_{II}^1(x_c,r_1)\geqslant a+2\delta$.

(iii) For $\delta\in(0,\delta_1)$, let ε_0 be a sufficiently small positive constant such that $\rho(\varepsilon_0)e^{2\lambda r_1}<\delta$ and $\varepsilon_0<\frac{1}{2}$, where $\rho(\varepsilon_0)$ is defined by (5.5). Consider $w(x,t)=\max\{\hat{w}(x,t),0\}$, where $\hat{w}(x,t)=$

$v^1_{II}(x,t)+\psi(x,t)$, $\psi(x,t)=-\rho(\varepsilon_0)e^{2\lambda t}-ae^{\lambda t}\zeta(-\varepsilon_c(x-x_c))$. By Remarks 2.4 and 2.6, $\frac{\partial}{\partial x}v^1_{II}(x,t)>0$ for $t>0$. In view of $\frac{\partial}{\partial x}\psi(x,t)\geqslant 0$, then for $t>0$ and $x\in\mathbb{R}$, $\frac{\partial}{\partial x}\hat{w}(x,t)>0$. Thus, for every $t_0\in(0,r_1]$, there exists a unique $x^-(t_0)$ such that $\hat{w}(x^-(t_0),t-0)>0$, $(x,t_0)\in B_+$ for $x<x^-(t_0)$, $(x,t_0)\in B_-$ for $x>x^-(t_0)$, and

$$\frac{\partial}{\partial x}\hat{w}\big(x^-(t_0)+0,t_0\big)=\lim_{x\to x^-(t_0)+0}\frac{\partial}{\partial x}\hat{w}(x,t_0)>0,$$

where $B_+=\{(x,t)\in\mathbb{R}\times(0,r_1]:\ \hat{w}(x,t)<0\}$, $B_-=\{(x,t)\in\mathbb{R}\times(0,r_1]:\ \hat{w}(x,t)>0\}$. Then by the same argument as in Lemma 3.2, the inequality

$$\frac{\partial w(x,t)}{\partial t}-d\Delta w(x,t)-g\big(w(x,t),\big(h_c*S(w)\big)(x,t)\big)\leqslant 0\quad\text{for } (x,t)\in B_+\cup B_-$$

implies that $w(x,t)$ is a subsolution of (5.1) in $\mathbb{R}\times[0,r_1]$. Now we only prove that the above inequality holds for $(x,t)\in B_-$. If $(x,t)\in B_-$, namely, $w(x,t)=\hat{w}(x,t)$, then

$$\begin{aligned}
&\frac{\partial w(x,t)}{\partial t}-d\Delta w(x,t)-g\big(w(x,t),\big(h_c*S(w)\big)(x,t)\big)\\
&\quad=\frac{\partial v^1_{II}(x,t)}{\partial t}-d\Delta v^1_{II}(x,t)-2\lambda\rho(\varepsilon_0)e^{2\lambda t}-\lambda ae^{\lambda t}\zeta\big(-\varepsilon_c(x-x_c)\big)\\
&\qquad+da\varepsilon_c^2e^{\lambda t}\zeta\big(-\varepsilon_c(x-x_c)\big)-g\big(w(x,t),\big(h_c*S(w)\big)(x,t)\big)\\
&\quad\leqslant-2\lambda\rho(\varepsilon_0)e^{2\lambda t}+da\varepsilon_0^2e^{\lambda t}-\lambda ae^{\lambda t}\zeta\big(-\varepsilon_c(x-x_c)\big)\\
&\qquad+g\big(v^1_{II}(x,t),\big(h*S(v^1_{II})\big)(x,t)\big)-g\big(w(x,t),\big(h_c*S(\hat{w})\big)(x,t)\big)\\
&\quad\leqslant-2\lambda\rho(\varepsilon_0)e^{2\lambda t}+da\varepsilon_0^2e^{\lambda t}-\lambda ae^{\lambda t}\zeta\big(-\varepsilon_c(x-x_c)\big)\\
&\qquad+L_1\big[\rho(\varepsilon_0)e^{2\lambda t}+ae^{\lambda t}\zeta\big(-\varepsilon_c(x-x_c)\big)\big]\\
&\qquad+L_2L_3\rho(\varepsilon_0)e^{2\lambda t}+aL_2L_3e^{\lambda t}\int_{-\infty}^{\infty}h_c(y)\zeta\big(-\varepsilon_c(x-y-x_c)\big)\,dy\\
&\quad\leqslant-\lambda\rho(\varepsilon_0)e^{2\lambda t}+da\varepsilon_0^2e^{\lambda t}\\
&\qquad+aL_2L_3e^{\lambda t}\int_{-\infty}^{\infty}h_c(y)\big|\zeta\big(-\varepsilon_c(x-y-x_c)\big)-\zeta\big(-\varepsilon_c(x-x_c)\big)\big|\,dy\\
&\quad\leqslant-\lambda\rho(\varepsilon_0)e^{2\lambda t}+da\varepsilon_0^2e^{\lambda t}+aL_2L_3\varepsilon_ce^{\lambda t}\int_{-\infty}^{\infty}|y|h_c(y)\,dy\\
&\quad\leqslant\big[-\lambda\rho(\varepsilon_0)+da\varepsilon_0^2+aL_2L_3\varepsilon_0(h_0+\tau)\big]e^{\lambda t}\leqslant 0.
\end{aligned}$$

Thus, $w(x,t)$ is a subsolution of (5.1) in $\mathbb{R}\times[0,r_1]$. Let $r_0=9/\varepsilon_0$ and $r_c=(1+|c|)r_0$. Then

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$$\begin{aligned}\hat{w}(x,0) &= v_{II}^1(x,0) - \rho(\varepsilon_0) - a\zeta\big(-\varepsilon_c(x-x_c)\big)\\ &\leqslant v_{II}^1(x,0) - a\zeta\big(-\varepsilon_c(x-x_c)\big)\\ &\leqslant v_{II}^1(x,0) - aH(-r_c-x)\\ &= v_{III}^1(x,0)\end{aligned}$$

and $w(x,0) \leqslant v_{III}^1(x,0)$. By comparison, we have $v_{III}^1(x,t) \geqslant w(x,t) \geqslant v_{II}^1(x,t) - \rho(\varepsilon_0)e^{2\lambda t} - ae^{\lambda t}\zeta(-\varepsilon_c(x-x_c))$. In particular,

$$v_{III}^1(x_c,r_1) \geqslant v_{II}^1(x_c,r_1) - \rho(\varepsilon_0)e^{2\lambda r_1} \geqslant a + 2\delta - \delta = a+\delta.$$

Similarly, $w(x,t) = \min\{v_{II}^2(x,t) + \rho(\varepsilon_0)e^{2\lambda t} + (1-a-\delta)e^{\lambda t}\zeta(\varepsilon_c(x-x_c)), 1\}$ is a supersolution of (5.1) on $\mathbb{R}\times[0,r_1]$. Since

$$\begin{aligned}&v_{II}^2(x,0) + \rho(\varepsilon_0) + (1-a-\delta)\zeta\big(\varepsilon_c(x-x_c)\big)\\ &\quad\geqslant v_{II}^2(x,0) + (1-a-\delta)\zeta\big(\varepsilon_c(x-x_c)\big)\\ &\quad\geqslant v_{II}^2(x,0) + (1-a-\delta)H(x-r_c)\\ &\quad= v_{III}^2(x,0),\end{aligned}$$

then $v_{III}^2(x,t) \leqslant w(x,t) \leqslant v_{II}^2(x,t) + \rho(\varepsilon_0)e^{2\lambda t} + (1-a-\delta)e^{\lambda t}\zeta(\varepsilon_c(x-x_c))$ on $\mathbb{R}\times[0,r_1]$. Hence,

$$v_{III}^2(x_c,r_1) \leqslant v_{II}^2(x_c,r_1) + \rho(\varepsilon_0)e^{2\lambda r_1} \leqslant a - 2\delta + \delta = a-\delta.$$

(iv) For $\delta \in (0,\delta_1)$, by Lemma 5.1 and (5.3),

$$\begin{aligned}&(1+\delta) - \big[1-(a-2\delta)\big]\zeta\big(-\epsilon_c\big(x-\xi_-(0)\big)\big)\\ &\quad\geqslant u_0(x,0) \geqslant -\delta + \big[1-(1-a-2\delta)\big]\zeta\big(\epsilon_c\big(x-\xi_+(0)\big)\big).\end{aligned}$$

Following the comparison, we have

$$\begin{aligned}&(1+\delta) - \big[1-(a-2\delta)e^{-\epsilon_c t}\big]\zeta\big(-\epsilon_c\big(x-\xi_-(0)+C_c t\big)\big)\\ &\quad\geqslant \min\big\{(1+\delta) - \big[1-(a-2\delta)e^{-\epsilon_c t}\big]\zeta\big(-\epsilon_c\big(x-\xi_-(0)+C_c t\big)\big), 1\big\}\\ &\quad\geqslant u(x,t) \geqslant \max\big\{-\delta + \big[1-(1-a-2\delta)e^{-\epsilon_c t}\big]\zeta\big(\epsilon_c\big(x-\xi_+(0)-C_c t\big)\big), 0\big\}\\ &\quad\geqslant -\delta + \big[1-(1-a-2\delta)e^{-\epsilon_c t}\big]\zeta\big(\epsilon_c\big(x-\xi_+(0)-C_c t\big)\big).\end{aligned}$$

From (5.3), $\xi_+(t) \leqslant \xi_+(0) + C_c t + 4\epsilon_c^{-1}$, $\xi_-(t) \geqslant \xi_-(0) - C_c t - 4\epsilon_c^{-1}$. Hence,

$$\begin{aligned}\xi_+(t) - \xi_-(t) &\leqslant \xi_+(0) - \xi_-(0) + 2C_c t + 8\epsilon_c^{-1}\\ &= \xi_+(0) - \xi_-(0) + 2\big(1+|c|\big)\big(4\epsilon_0^{-1} + C_0 t\big).\end{aligned}$$

In particular, for all $t \in [0, r_1]$, we have

$$\xi_+(t) - \xi_-(t) \leqslant \xi_+(0) - \xi_-(0) + 2\bigl(1 + |c|\bigr)\bigl(4\epsilon_0^{-1} + C_0 r_1\bigr).$$

To finish the proof, we need to prove the following: for every $t_1 \geqslant 0$,

$$\xi_+(t_1 + r_1) - \xi_-(t_1 + r_1) \leqslant \max\bigl\{\xi_+(t_1) - \xi_-(t_1), 2r_c + 2\bigr\}.$$

By translation, we can assume that $u(0, t_1) = a$ so that $\xi_+(t_1) > 0 > \xi_-(t_1)$. By symmetry, we need only consider the case $\xi_+(t_1) \geqslant |\xi_-(t_1)|$. Noting that r_1, δ_1 and r_0 are independent of k, we take $k > \frac{1}{\delta} \sup_{t \in [0,\infty)} \|\frac{\partial}{\partial x} u(\cdot, t)\|_{C^0(\mathbb{R})}$ for $H(x)$ in the following.

Set $r_+ = \max\{\xi_+(t_1), r_c + 1\}$. Then, $u(\cdot + r_+, t_1) \geqslant v_{III}^1(\cdot, 0)$ in $\mathbb{R}$, so that, by comparison,

$$u(x_c + r_+, t_1 + r_1) \geqslant v_{III}^1(x_c, r_1) \geqslant a + \delta,$$

which implies that $\xi_+(t_1 + r_1) \leqslant x_c + r_+$.

Set $r_- = \max\{\xi_+(t_1) - \xi_-(t_1), r_c + 1\}$. Then $u(\cdot + \xi_+(t_1) - r_-, t_1) \leqslant v_{III}^2(\cdot, 0)$ in $\mathbb{R}$. By comparison, we have

$$u\bigl(x_c + \xi_+(t_1) - r_-, t_1 + r_1\bigr) \leqslant v_{III}^2(x_c, r_1) \leqslant a - \delta,$$

which implies that $\xi_-(t_1 + r_1) \geqslant x_c + \xi_+(t_1) - r_-$.

Combining the two estimates for $\xi_+(t_1 + r_1)$ and $\xi_-(t_1 + r_1)$, we have

$$\xi_+(t_1 + r_1) - \xi_-(t_1 + r_1) \leqslant r_+ - \xi_+(t_1) + r_- \leqslant \max\bigl\{\xi_+(t_1) - \xi_-(t_1), 2r_c + 2\bigr\}.$$

This completes the proof. □

Theorem 5.5. *Assume that* (H1)–(H3) *hold, then for every* $c \in \mathbb{R}$, *Eq.* (5.1) *admits a unique monotonic traveling wave front* $(U_c, C(c))$ *satisfying* (1.3).

Proof. Let $v(x, t)$ be the solution of

$$\begin{cases} \dfrac{\partial v(x,t)}{\partial t} = d\Delta v(x,t) + g\bigl(v(x,t), \bigl(h_c * S(v)\bigr)(x,t)\bigr), & (x,t) \in \mathbb{R} \times [0,\infty), \\ v(x,0) = \zeta(x), & x \in \mathbb{R}. \end{cases} \tag{5.9}$$

Here and in the sequel, $\zeta(\cdot)$ always refers to the function ζ satisfying (4.1). First of all, by comparison, we have $0 \leqslant v(x, t) \leqslant 1$ on $\mathbb{R} \times [0, \infty)$.

For all $t > 0$, $s \geqslant 0$, $x \in \mathbb{R}$, and $z \in \mathbb{R}$, by Remark 2.4,

$$v_x(x, t + s) \geqslant \Theta\bigl(|x - z|, t\bigr) \int_z^{z+1} v_x(y, s)\, dy. \tag{5.10}$$

In particular, taking $s = z = 0$, we have $v_x(x, t) \geqslant \Theta(|x|, t)\zeta(1) > 0$ in $\mathbb{R} \times [0, \infty)$. Observe that Lemma 5.1 implies

$$\lim_{x \to \infty} v(x, t) = 1 \quad \text{and} \quad \lim_{x \to -\infty} v(x, t) = 0$$

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for all $t > 0$. It then follows that there exists a unique function $z(\alpha, t)$ defined on $(0, 1) \times [0, \infty)$ such that $v(z(\alpha, t), t) = \alpha$, $\alpha \in (0, 1)$, $t \in [0, \infty)$. Note that $\partial z(\alpha, t)/\partial\alpha = 1/\frac{\partial v(z(\alpha,t),t)}{\partial x} > 0$.

We claim that for every $T \geqslant 0$, $v(\cdot, T)$ is Lipschitz continuous in $\mathbb{R}$. In fact, for any positive ε such that $\varepsilon e^{\lambda T} \leqslant 9$, consider $w(x, t) = \min\{v(x, t) + \varepsilon e^{\lambda t}, 1\}$, λ is defined as in Lemma 5.4. For any $(x, t) \in \mathbb{R} \times (0, T]$ such that $v(x, t) + \varepsilon e^{\lambda t} < 1$, there is

$$\begin{aligned}
&\frac{\partial w(x,t)}{\partial t} - d\Delta w(x,t) - g\big(w(x,t), \big(h_c * S(w)\big)(x,t)\big) \\
&\quad = \varepsilon\lambda e^{\lambda t} + g\big(v(x,t), \big(h_c * S(v)\big)(x,t)\big) - g\big(w(x,t), \big(h_c * S(w)\big)(x,t)\big) \\
&\quad \geqslant \varepsilon\lambda e^{\lambda t} - \varepsilon L_1 e^{\lambda t} - \varepsilon L_2 L_3 e^{\lambda t} \geqslant 0,
\end{aligned}$$

which implies that $w(x, t)$ is a supersolution of (5.1) in $(x, t) \in \mathbb{R} \times (0, T]$ by the same argument as in Lemma 3.2. Now since $v(\cdot, 0) \leqslant v(\cdot + \varepsilon, 0) \leqslant \min\{v(\cdot, 0) + \varepsilon, 1\}$, it follows by comparison that $v(\cdot, t) \leqslant v(\cdot + \varepsilon, t) \leqslant \min\{v(\cdot, t) + \varepsilon e^{\lambda t}, 1\}$ for all $t \in [0, T]$. Hence, $v(\cdot, T)$ is nondecreasing and is Lipschitz continuous in $\mathbb{R}$.

From (5.3), one sees that for every small positive $\delta > 0$, there exist $\epsilon_0 = \epsilon_0(\delta)$ and $C_0 = C_0(\delta)$ such that

$$\begin{cases}
z(a+\delta, t) \leqslant z(a+\delta, r) + 4\epsilon_c^{-1} + C_c(t-r) & \forall 0 \leqslant r < t < \infty, \\
z(a-\delta, t) \geqslant z(a-\delta, r) - 4\epsilon_c^{-1} - C_c(t-r) & \forall 0 \leqslant r < t < \infty, \\
z(1-2\delta, t) \leqslant z(a+\delta, r) + 4\epsilon_c^{-1} + C_c(t-r) & \forall r \geqslant 0,\ t - r \geqslant \epsilon_c^{-1}|\ln\delta|, \\
z(2\delta, t) \leqslant z(a-\delta, r) - 4\epsilon_c^{-1} - C_c(t-r) & \forall r \geqslant 0,\ t - r \geqslant \epsilon_c^{-1}|\ln\delta|,
\end{cases} \tag{5.11}$$

where $\epsilon_c = \frac{\epsilon_0}{1+|c|}$, $C_c = (1 + |c|)C_0$. Here we prove (5.11). We only prove the first inequality. Since $v(\cdot, r)$ is increasing in x for every fixed r, let $\mu = z(a + \delta, r)$, then

$$\begin{cases}
v(x,r) \geqslant a+\delta & \text{for } x \geqslant \mu, \\
v(x,r) < a+\delta & \text{for } x < \mu.
\end{cases}$$

But there are

$$\begin{cases}
v_c^-(x,0) = -\delta & \text{for } x \leqslant \mu, \\
-\delta < v_c^-(x,0) < a+\delta & \text{for } \mu < x < \mu + 4\epsilon_c^{-1}, \\
v_c^-(x,0) = a+\delta & \text{for } x \geqslant \mu + 4\epsilon_c^{-1}.
\end{cases}$$

Hence, $\hat{v}_c^-(x, t - r) < v(x, t)$ for all $t > r \geqslant 0$ and $x \in \mathbb{R}$. Now, if $x_0 = z(a + \delta, t) > z(a + \delta, r) + 4\epsilon_c^{-1} + C_c(t - r)$, then by (5.3),

$$v_c^-(x_0, t-r) \geqslant 1 - \delta - (1 - a - 2\delta)e^{-\epsilon_c(t-r)} \geqslant 1 - \delta - (1 - a - 2\delta) = a + \delta.$$

This contradicts to $v_c^-(x_0, t - r) < v(x_0, t) = v(z(a + \delta, t), t) = a + \delta$, which implies that the first inequality of (5.11) holds.

For $u(\cdot, 0) = \zeta(\cdot)$, there are $0 \leqslant \xi_-(0) < \xi_+(0) \leqslant 4$. Then by using Lemma 5.4 and (5.11), we can derive the following:

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(a) For every $\delta \in (0, \delta_1/2]$, there exist $M^* = M^*(\delta) > 0$ and $L^* = L^*(\delta) > 0$, independent of c, such that

$$z(1-\delta, t) - z(\delta, t) \leqslant M^* + L^*(\delta)|c| \quad \forall t \geqslant 0. \tag{5.12}$$

(b) For every $M > 0$, there exists a constant $\hat{\Theta}(M, c) > 0$ such that

$$\frac{\partial}{\partial x} v\big(x + z(a,t), t\big) \geqslant \hat{\Theta}(M, c) \quad \forall t \geqslant 1,\ x \in [-M, M]. \tag{5.13}$$

The detailed proofs of (a) and (b) are similar to that of [13, (a), (b), p. 144].

Note the comparison functions $W^+(x,t)$ and $W^-(x,t)$ are defined by (5.15) in Lemma 5.6. Since the family $\{v(\cdot + z(a,t), t)\}$ consists of monotonic bounded functions, there exist a sequence $\{t_j\}_{j=1}^{\infty}$ and a nondecreasing function $U_c(\cdot)$ such that $j \to \infty$, $t_j \to \infty$ and $v(\xi + z(a, t_j), t_j) \to U_c(\xi)$ for all $\xi \in \mathbb{R}$. Clearly, $U_c(0) = a$ and $0 \leqslant U_c \leqslant 1$. In addition, from (5.12) we know that for all small $\delta > 0$,

$$U_c\big(M^* + L^*|c|\big) \geqslant 1 - \delta \quad \text{and} \quad U_c\big(-M^* - L^*|c|\big) \leqslant \delta. \tag{5.14}$$

This implies that $\lim_{\xi\to\infty} U(\xi) = 1$ and $\lim_{\xi\to-\infty} U(\xi) = 0$. Furthermore, by virtue of the condition (C_4) and Remark 5.2(2) of Chen [13], we can show that U_c is the profile of a traveling wave front and there exists $C(c) \in \mathbb{R}$ such that $(U_c, C(c))$ is a traveling wave front to (5.1). The remainder of the proof is analogous to Steps 3 and 4 of Chen [13, Theorem 4.1], so we omit them.

So far, we complete the proof of the existence. In view of Remark 4.2, the uniqueness of traveling wave fronts is obvious. □

Lemma 5.6. *Assume that* (H1)–(H3) *hold, and let $v(x,t)$ be a solution of* (5.9). *Then there exist three positive numbers β_1 (which is independent of v), σ_1 and $\bar{\delta}_1$ such that for any $\delta \in (0, \bar{\delta}_1]$ and every $\xi_0 \in \mathbb{R}$, the functions W^+ and W^- defined by*

$$\begin{aligned} W^+(x,t) &:= \min\big\{v\big(x + \xi_0 + \sigma_1\delta\big(1 - e^{-\beta_1 t}\big), t+1\big) + \delta e^{-\beta_1 t}, 1\big\}, \\ W^-(x,t) &:= \max\big\{v\big(x + \xi_0 - \sigma_1\delta\big(1 - e^{-\beta_1 t}\big), t+1\big) - \delta e^{-\beta_1 t}, 0\big\}, \end{aligned} \tag{5.15}$$

are a supersolution and a subsolution of (5.1) *on* $[0, \infty)$, *respectively.*

Proof. We prove only that $W^+(x,t)$ is a supersolution of (5.1) on $[0,\infty)$. Since $\frac{\partial}{\partial x} v(x,t) > 0$ for all $(x,t) \in \mathbb{R} \times (0,\infty)$, then we only need to show that for all $(x,t) \in \mathbb{R} \times (0,\infty)$ satisfying $\hat{W}^+(x,t) < 1$, the inequality

$$\frac{\partial W^+(x,t)}{\partial t} - d\Delta W^+(x,t) - g\big(W^+(x,t), \big(h_c * S\big(W^+\big)\big)(x,t)\big) \geqslant 0 \tag{5.16}$$

holds, where $\hat{W}^+(x,t) = v(x + \xi_0 + \sigma_1\delta(1 - e^{-\beta_1 t}), t+1) + \delta e^{-\beta_1 t}$. In the following, we always assume $(x,t) \in \mathbb{R} \times (0,\infty)$ satisfying $\hat{W}^+(x,t) < 1$.

In Lemma 3.2, by setting $\tau = 0$, we can fix $\beta_1 > 0$ and $\delta_1^* > 0$ such that

$$\partial_1 g(u,v) + \varpi\,\partial_2 g(r,s) < -\beta_1 \tag{5.17}$$

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for all $(u,v,r,s,\varpi)\in[0,\delta_1^*]\times[S(0)-\delta_1^*,S(0)+\delta_1^*]\times[0,\delta_1^*]\times[S(0)-\delta_1^*,S(0)+\delta_1^*]\times[S'(0)-\delta_1^*,S'(0)+\delta_1^*]$ and $(u,v,r,s,\varpi)\in[1-\delta_1^*,1]\times[S(1)-\delta_1^*,S(1)+\delta_1^*]\times[1-\delta_1^*,1]\times[S(1)-\delta_1^*,S(1)+\delta_1^*]\times[S'(1)-\delta_1^*,S'(1)+\delta_1^*]$. By the continuity of $S(v)$ and $S'(v)$, there exists $\hat\delta_1\in(0,\delta_1^*]$ such that for any $\delta\in[-\hat\delta_1,\hat\delta_1]$,

$$\big|S(1+\delta)-S(1)\big|<\frac{\delta_1^*}{3},\qquad \big|S(\delta)-S(0)\big|<\frac{\delta_1^*}{3},$$
$$\big|S'(1+\delta)-S'(1)\big|<\frac{\delta_1^*}{3},\qquad \big|S'(\delta)-S'(0)\big|<\frac{\delta_1^*}{3}.$$

Further, there exists $M_0=M_0(v,\beta_0,\delta_1^*)>0$ such that for any $\xi\in[-\hat\delta_1,1+\hat\delta_1]$,

$$\left|S(\xi)\left[\int_{M_0}^{\infty}+\int_{-\infty}^{-M_0}h_c(y)\,dy\right]\right|<\frac{\delta_1^*}{3},\qquad \left|S'(\xi)\left[\int_{M_0}^{\infty}+\int_{-\infty}^{-M_0}h_c(y)\,dy\right]\right|<\frac{\delta_1^*}{3}.$$

Let $\bar\delta_1=\min\{\hat\delta_1,\frac{\delta_1}{2}\}$. Take

$$\rho_1=\max\bigl\{\big|\partial_1 g(u,v)\big|+\kappa_2\big|\partial_2 g(r,s)\big|\colon\ u,r\in[0,1],\ v\in\bigl[S(0),S(1)\bigr],\ s\in\bigl[S(0),S\bigl(1+\delta_1^*\bigr)\bigr]\bigr\},$$

where $\kappa_1=\max\{S'(u)\colon\ u\in[0,1+\delta_1^*]\}$, and define

$$\sigma_1=\frac{\beta_1+\rho_1}{\hat\Theta(M_0+M^*+L^*|c|)\beta_1},$$

where $\hat\Theta(M_0+M^*+L^*|c|)$ is defined by the previous (b). For any given $\delta\in(0,\bar\delta_1)$, let $\xi(x,t)=x+\xi_0+\sigma_0\delta[1-e^{-\beta_0 t}]$. It then follows that, for any $t>0$,

$$\begin{aligned}
&\frac{\partial W^+(x,t)}{\partial t}-d\Delta W^+(x,t)-g\bigl(W^+(x,t),\bigl(h_c*S\bigl(W^+\bigr)\bigr)(x,t)\bigr)\\
&\quad=\sigma_1\delta\beta_1 e^{-\beta_1 t}\frac{\partial}{\partial x}v\bigl(\xi(x,t),t+1\bigr)+\frac{\partial}{\partial t}v\bigl(\xi(x,t),t+1\bigr)-\beta_1\delta e^{-\beta_1 t}-d\Delta v\bigl(\xi(x,t),t+1\bigr)\\
&\qquad-g\bigl(W^+(x,t),\bigl(h_c*S\bigl(W^+\bigr)\bigr)(x,t)\bigr)\\
&\quad\geqslant\sigma_1\delta\beta_1 e^{-\beta_1 t}\frac{\partial}{\partial x}v\bigl(\xi(x,t),t+1\bigr)-\beta_1\delta e^{-\beta_1 t}\\
&\qquad-g\left(v\bigl(\xi(x,t),t+1\bigr),\int_{-\infty}^{+\infty}h_c(y)S\bigl(v\bigl(\xi(x-y,t),t+1\bigr)\bigr)\,dy\right)\\
&\qquad-g\bigl(W^+(x,t),\bigl(h_c*S\bigl(\hat W^+\bigr)\bigr)(x,t)\bigr)\\
&\quad\geqslant\delta e^{-\beta_1 t}\biggl\{\sigma_1\beta_1\frac{\partial}{\partial x}v\bigl(\xi(x,t),t+1\bigr)-\beta_1
\end{aligned}$$

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$$- \partial_1 g\left(v\big(\xi(x,t),t+1\big)+\theta_1\delta e^{-\beta_0 t}, \int_{-\infty}^{+\infty} h_c(y) S\big(v\big(\xi(x-y,t),t+1\big)\big)\,dy\right)$$

$$- \partial_2 \hat{g}\left(v\big(\xi(x,t),t+1\big)+\delta e^{-\beta_0 t}, \int_{-\infty}^{+\infty} h_c(y) S\big(v\big(\xi(x-y,t),t+1\big)+\theta_2\delta e^{-\beta_1 t}\big)\,dy\right)$$

$$\times \int_{-\infty}^{+\infty} h_c(y) S'\big(v\big(\xi(x-y,t),t+1\big)+\theta_2\delta e^{-\beta_0 t}\big)\,dy\bigg\}. \tag{5.18}$$

Let $\eta(x,t) = x+\xi_0+\sigma_0\delta[1-e^{-\beta_0 t}] - z(a,t+1)$. We consider three cases.

Case (i): $\eta(x,t) > M_0 + M^* + L^*|c|$. Then by the previous (a),

$$v\big(\xi(x,t),t+1\big) = v\big(\eta(x,t)+z(a,t+1),t+1\big) \geqslant v\big(z(1-\delta,t+1),t+1\big) = 1-\delta.$$

For $y \in [-M_0, M_0]$,

$$\begin{aligned} v\big(\xi(x-y,t),t+1\big) &\geqslant v\big(\eta(x,t)-M_0+z(a,t+1),t+1\big) \\ &\geqslant v\big(z(1-\delta,t+1),t+1\big) = 1-\delta. \end{aligned}$$

Therefore,

$$\begin{aligned} &\int_{-\infty}^{+\infty} h_c(y) S\big(v\big(\xi(x-y,t),t+1\big)\big)\,dy \\ &\quad = \left(\int_{-\infty}^{-M_0} + \int_{M_0}^{+\infty}\right) h_c(y) S\big(v\big(\xi(x-y,t),t+1\big)\big)\,dy \\ &\qquad + \int_{-M_0}^{M_0} h_c(y) S\big(v\big(\xi(x-y,t),t+1\big)\big)\,dy \\ &\quad \geqslant -\frac{\delta_1^*}{3} + \int_{-M_0}^{M_0} h_c(y) S(1-\delta)\,dy \geqslant S(1) - \delta_1^*. \end{aligned}$$

Similarly,

$$S(1)+\delta_1^* \geqslant \int_{-\infty}^{+\infty} h_c(y) S\big(v\big(\xi(x-y,t),t+1\big)+\theta_2\delta e^{-\beta_1 t}\big)\,dy \geqslant S(1)-\delta_1^*,$$

$$S'(1)+\delta_1^* \geqslant \int_{-\infty}^{+\infty} h_c(y) S'\big(v\big(\xi(x-y,t),t+1\big)+\theta_2\delta e^{-\beta_0 t}\big)\,dy \geqslant S'(1)-\delta_1^*.$$

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Thus, by (5.17), we have that (5.16) holds.

Case (ii): $|\eta(x,t)| \leqslant M_0 + M^* + L^*|c|$. Then, by the choice of σ_1 and (5.18),

$$\frac{\partial W^+(x,t)}{\partial t} - d\Delta W^+(x,t) - g\big(W^+(x,t), \big(h_c * S\big(W^+\big)\big)(x,t)\big)$$
$$\geqslant \delta e^{-\beta_1 t}\big[\sigma_1\beta_1\hat{\Theta}\big(M_0 + M^* + L^*|c|\big) - \beta_1 - \rho_1\big] \geqslant 0.$$

Case (iii): $\eta(x,t) \leqslant -(M_0 + M^* + L^*|c|)$. The proof is similar to that for the case (i) and is omitted.

This completes the proof. □

Lemma 5.7. *Assume that* (H1)–(H3) *hold. Then the wave speed $C(c)$ of the traveling wave front $(U_c, C(c))$ in Eq.* (5.1) *is a continuous function of $c \in \mathbb{R}$.*

Proof. Without loss of generality, we assume that $U_c(0) = a$ for each $c \in \mathbb{R}$. Then $U_c(x - C(c)t)$ satisfies

$$-C(c)U_c'(\xi) = dU_c''(\xi) + g\left(U_c(\xi), \int_0^\tau \int_{-\infty}^\infty \bar{h}(y,s)S\big(U_c(\xi - y + cs)\big)\,dy\,ds\right),$$

where $\xi = x - C(c)t$. Hence,

$$U_c(\xi) = \frac{1}{d(\lambda_2(C(c)) - \lambda_1(C(c)))}\left[\int_{-\infty}^{\xi} e^{\lambda_1(C(c))(\xi - s)}H(U_c)(s)\,ds + \int_{\xi}^{\infty} e^{\lambda_2(C(c))(\xi - s)}H(U_c)(s)\,ds\right], \tag{5.19}$$

where

$$\lambda_1\big(C(c)\big) = \frac{-C(c) - \sqrt{C^2(c) + 4dL_1}}{2d}, \qquad \lambda_2\big(C(c)\big) = \frac{-C(c) + \sqrt{C^2(c) + 4dL_1}}{2d}$$

and

$$H(U_c)(\xi) = L_1U_c(\xi) + g\left(U_c(\xi), \int_0^\tau \int_{-\infty}^\infty \bar{h}(y,s)S\big(U_c(\xi - y + cs)\big)\,dy\,ds\right).$$

Since $0 \leqslant U_c(\xi) \leqslant 1$ and $\lambda_2(C(c)) - \lambda_1(C(c)) = \sqrt{C^2(c) + 4dL_1}/d \geqslant 2\sqrt{L_1/d}$, it is easy to show that

$$\big|U_c'(\xi)\big| \leqslant \frac{G}{2\sqrt{dL_1}} \quad \text{for every } c \in \mathbb{R} \text{ and } \xi \in \mathbb{R},$$

where $G = 2L_1 + 2\max\{|g(u,v)|\colon (u,v) \in [0,1] \times [S(0), S(1)]\}$.

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Here we first show that for any bounded $c \in \mathbb{R}$, the speed $C(c)$ is also bounded. In fact, consider functions $U_c(x)$ and $v_c^-(x, 0)$ defined in Lemma 5.1 with $\delta = \bar{\delta}_0$, and there exists $x_0 \in \mathbb{R}$ such that $v_c^-(x - x_0, 0) < U_c(x)$ for all $x \in \mathbb{R}$. Then by comparison, we have that $v_c^-(x - x_0, t) < U_c(x - C(c)t)$ for all $x \in \mathbb{R}$ and $t \in [0, \infty)$, that is,

$$-\delta + \left[1 - (1 - a - 2\delta)e^{-\epsilon_c t}\right]\zeta\left(\epsilon_c(x - x_0 - C_c t)\right) < U_c\left(x - C(c)t\right). \tag{5.20}$$

Now we claim that $C(c) \leqslant C_c$. If not, namely, $C(c) > C_c$, then we fix $x - x_0 - C_c t = \xi^*$ with $\zeta(\epsilon_c \xi^*) = 2\delta$, hence, $U_c(x - C(c)t) = U_c(\xi^* + x_0 + (C_c - C(c))t)$. Letting $t \to \infty$ in (5.20), then we have $\delta \leqslant U_c(-\infty)$, which is a contradiction to $U_c(-\infty) = 0$. Thus, we have $C(c) \leqslant C_c = (1 + |c|)C_0$. Similarly, comparing functions $U_c(x)$ and $v_c^+(x, 0)$, we obtain $C(c) \geqslant -C_c = -(1 + |c|)C_0$.

Suppose $c_n \to c$, but $C(c_n)$ does not converge to $C(c)$, then there exists a subsequence $c_{n_k} \to c$ so that $C(c_{n_k}) \to b \neq C(c)$. Let $H^* = \sup\{|c_n|\}$. Since $U_{c_{n_k}}(\cdot)$ is nondecreasing, $U_{c_{n_k}}(0) = a$, and by (a) and (b) in Theorem 5.5, $U_{c_{n_k}}(\cdot)$ also satisfies, for sufficiently small $\delta > 0$,

$$\begin{aligned} &U_{c_{n_k}}(x) \leqslant \delta, \quad \text{if } x \leqslant -M^* - L^* H^* \leqslant -M^* - L^*|c_{n_k}|, \\ &U_{c_{n_k}}(x) \geqslant 1 - \delta, \quad \text{if } x \geqslant M^* + L^* H^* \geqslant M^* + L^*|c_{n_k}|. \end{aligned}$$

By the Arzela–Ascoli theorem and the above inequalities, we can choose a subsequence of $\{c_{n_k}\}$, such that $U_{c_{n_k}}(\cdot)$ converges uniformly to a continuous function $U(\cdot)$ in $\mathbb{R}$. We still denote this subsequence by c_{n_k}. Obviously, $U(\cdot)$ is nondecreasing, $0 \leqslant U(\cdot) \leqslant 1$, and

$$\lim_{x \to -\infty} U(x) = 0 \quad \text{and} \quad \lim_{x \to \infty} U(x) = 1.$$

In Eq. (5.19) with c being replaced by c_{n_k}, we let $k \to \infty$ and apply the dominant convergence theorem to get

$$U(\xi) = \frac{1}{d(\lambda_2(b) - \lambda_1(b))}\left[\int_{-\infty}^{\xi} e^{\lambda_1(b)(\xi - s)} H(U)(s)\,ds + \int_{\xi}^{\infty} e^{\lambda_2(b)(\xi - s)} H(U)(s)\,ds\right].$$

Hence $U(x - bt)$ is a solution of (5.1). Furthermore, by virtue of Theorem 3.3, we have $C(c) = b$, which is a contradiction. This completes the proof. □

Theorem 5.8. *Assume that* (H1)–(H3) *hold. Then* (1.1) *admits a strictly monotonic traveling wave front $U(x - (-B + c^*)t)$ with $|c^*| \leqslant C_0$, where $C_0 = C_0(\bar{\delta}_0)$ is given in Lemma* 5.1.

Proof. It is easy to see that if there exists $c^* \in \mathbb{R}$ such that $C(c^*) = c^*$, and $U(x - c^*t)$ is a monotonic traveling wave front of (5.1), then $U(x - c^*t)$ is also a monotonic traveling wave front of (2.10) and $U(x - (-B + c^*)t)$ is also a monotonic traveling wave front of (1.1) with wave speed $-B + c^*$. Therefore, it suffices to show that the curves $y = -c$ and $y = -C(c)$ have at least one common point in the (c, y) phase.

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For $c \geqslant 0$, let $v_0^-(x,t)$ be the subsolution of (5.1) given in Lemma 5.1 with $\delta = \bar{\delta}_0$. Then there exists a large constant x_0 such that $U_c(\cdot) \geqslant v_1^-(\cdot - x_0, 0)$. Therefore, by the comparison, it follows that $U_c(x - C(c)t) \geqslant v_1^-(x - x_0, t)$ for all $t \geqslant 0$ and $x \in \mathbb{R}$. Thus, we have

$$-\delta + \left[1 - (1 - a - 2\delta)e^{-\epsilon_0 t}\right]\zeta\left(\epsilon_0(x - x_0 - C_0 t)\right) < U_c\left(x - C(c)t\right).$$

By a similar argument to that for (5.20), we have $C(c) \leqslant C_0$. Similarly, we can show that $C(c) \geqslant -C_0$ for $c \leqslant 0$ by comparing $U_c(x - C(c)t)$ with $v_0^+(x,t)$. Thus, we can find a common point of the curves $y = -c$ and $y = -C(c)$ in $c \in [-C_0, C_0]$. The proof is complete. □

Now we consider the influence of advection on the propagation of fronts in Eq. (1.1).

Remark 5.9. Assume that (1.1) has a traveling wave front connecting equilibria 0 and 1 when the advection term is absent, namely, $B = 0$, then when $B \neq 0$, the advection term may cause a shift of the unique wave speed. Moreover, if the wave speed is positive (negative) when $B = 0$, then when $B \neq 0$, the wave speed can be null or negative (positive or null), which is dependent of B. As showed in the beginning of this section, for Eq. (1.1) with $h(y,s) = h(y)\delta(s)$, if the wave speed is c when $B = 0$, then when $B \neq 0$, the wave speed is $-B + c$. But for general kernel $h(y,s)$, the change of the wave speed due to the advection term becomes very complicated because of the effect of the time delay.

6. Applications

In the previous sections, we have studied the existence, uniqueness and asymptotic stability of traveling wave fronts of nonlocal reaction advection diffusion equation (1.1) with distributed delay. This equation is more general than the versions studied by Schaaf [38], Chen [13], Smith and Zhao [42] and Ma and Wu [28]. Obviously, our main results include those in these papers. In particular, our results can be applied to Eqs. (1.8) and (1.10). Note that though we only consider the existence for the case $f_0'(a) > 0$, we can get a similar result for the case $f_0'(a) = 0$ by a perturbation f_ε of f_0 so that $f_\varepsilon'(a) > 0$, see also Remark 5.2(5) in [13]. Note that twice differentiability of g and S in (H1) is only used for determining the inequality (5.8), otherwise, g and S only need to be continuously differentiable. Furthermore, our results are easily extended to the following equation with multi-delay

$$\frac{\partial u}{\partial t} = d\Delta u + B\frac{\partial u}{\partial x} + g\left(u(x,t), \left(h_1 * S_1(u)\right)(x,t), \ldots, \left(h_m * S_m(u)\right)(x,t)\right),$$

where kernels $h_1, \ldots, h_m$ are required as h in (1.1), $m \in \mathbb{N}$.

Now we consider Eqs. (1.8) and (1.10).

Example 6.1. Assume that Eq. (1.8) satisfies the following conditions:

(C1) There exist $0 \leqslant a_1 < a_2 < a_3$ such that $\varepsilon b(a_i) - d_m a_i = 0$, $i = 1, 2, 3$; $\varepsilon b(u) - d_m u < 0$ for $u \in (a_1, a_2)$; $\varepsilon b(u) - d_m u > 0$ for $u \in (a_2, a_3)$.

(C2) $b(\cdot) \in C^2([a_1, a_3])$, $b'(\cdot) \geqslant 0$, $\varepsilon b'(a_1) < d_m$, $\varepsilon b'(a_2) > d_m$, $\varepsilon b'(a_3) < d_m$.

We refer to $b(u) = pu^2 e^{-\beta u}$ which is widely used in the literature and satisfies (C1) and (C2), where $p > 0$ and $\beta > 0$ are appropriate constants, see also Ma and Zhao [30]. Now set $\tilde{u}(x,t) = \frac{1}{a_3 - a_1}(u(x,t) - a_1)$ and denote $\tilde{u}$ still by u, then Eq. (1.8) reduces to the following equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(D_m \frac{\partial u}{\partial x} + Bu\right) - d_m u(x,t) - \frac{d_m a_1}{a_3 - a_1} + \frac{\varepsilon}{a_3 - a_1} \int_{-\infty}^{\infty} J_\alpha(x + B\tau - y) b\big((a_3 - a_1)u(y, t-\tau) + a_1\big)\, dy. \tag{6.1}$$

Thus, we only need to consider Eq. (6.1). In particular, (H1)–(H3) hold for Eq. (6.1). By Theorems 3.3, 4.5 and 5.8, Eq. (6.1) admits a unique monotonic traveling wave front, and hence, Eq. (1.8) admits a unique monotonic traveling wave front $U(x - ct)$ (up to translation) which is asymptotically stable and satisfies $\lim_{\xi \to -\infty} U(\xi) = a_1$ and $\lim_{\xi \to +\infty} U(\xi) = a_3$.

Example 6.2. Assume that Eq. (1.10) satisfies the following conditions:

(C3) There exist $0 \leqslant a_1 < a_2 < a_3$ such that $\varepsilon b(a_i) - d(a_i) = 0$, $i = 1, 2, 3$; $\varepsilon b(u) - d(u) < 0$ for $u \in (a_1, a_2)$; $\varepsilon b(u) - d(u) > 0$ for $u \in (a_2, a_3)$, where $\varepsilon = \int_0^\tau f(s) e^{-\gamma s}\, ds$.

(C4) $b(\cdot), d(\cdot) \in C^2([a_1, a_3])$, $b'(\cdot) \geqslant 0$, $\varepsilon b'(a_1) < d'(a_1)$, $\varepsilon b'(a_2) > d'(a_2)$, $\varepsilon b'(a_3) < d'(a_3)$.

Then by a same argument as that for Eq. (1.8) and by Theorems 3.3, 4.5 and 5.8, we obtain the existence, uniqueness and asymptotic stability of traveling wave fronts of (1.10) connecting equilibria $u \equiv a_1$ and $u \equiv a_3$.

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