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Travelling wave fronts in reaction–diffusion systems with spatio-temporal delays

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Abstract

This paper deals with the existence of travelling wave fronts in reaction–diffusion systems with spatio-temporal delays. Our approach is to use monotone iterations and a nonstandard ordering for the set of profiles of the corresponding wave system. New iterative techniques are established for a class of integral operators when the reaction term satisfies different monotonicity conditions. Following this, the existence of travelling wave fronts for reaction–diffusion systems with spatio-temporal delays is established. Finally, we apply the main results to a single-species diffusive model with spatio-temporal delay and obtain some existence criteria of travelling wave fronts by choosing different kernels.

MSC: 35K57; 34K10; 35B20

Keywords: Travelling wave fronts; Reaction–diffusion systems; Integral operator; Iterative techniques; Quasimonotonicity; Spatio-temporal delays

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1. Introduction

The theory of travelling wave solutions of parabolic differential equations is one of the fastest developing areas of modern mathematics and has attracted much attention due to its significant nature in biology, chemistry, epidemiology and physics, (see [4,8,23,27,31,32,34]). Travelling wave solutions are solutions of special type and can be usually characterized as solutions invariant with respect to transition in space. From the physical point of view, travelling waves describe transition processes. These transition processes (from one equilibrium to another) usually "forget" their initial conditions and the properties of the medium itself.

Among the basic questions in the theory of travelling waves, the existence of travelling wave solutions is an important objective. The case of a *scalar* reaction–diffusion equation has been rather well studied, basically due to applicability of comparison theorems of a special kind for parabolic equations and of phase space analysis for the ordinary differential equations, see [4,8,23,31,34]. For *systems* of reaction–diffusion equations modeling various biological phenomena, many results have been established in [23,27,31]. Since comparison theorems are, in general, not applicable for reaction–diffusion systems and the phase space analysis becomes more complicated, some new approaches, such as the Conley index and degree theory methods, have been developed in [23,27,31].

Recently, many researchers have paid attention to travelling wave solutions for reaction–diffusion equations with time delays, for example, see [1,11,17–19,22,26,29,32, 33,36]. In a pioneering work, Schaaf [26] systematically studied two scalar reaction–diffusion equations with a single discrete delay for the so-called Huxley nonlinearity as well as Fisher nonlinearity by using the phase space analysis, the maximum principle for parabolic functional differential equations and the general theory for ordinary functional differential equations. For reaction–diffusion systems with quasimonotonicity and a single discrete delay, Zou and Wu [36] established the existence of travelling wave fronts by first truncating the unbounded domain and then passing to a limit. Wu and Zou [33] further considered more general reaction–diffusion systems with a single delay and obtained some results on the existence of travelling wave fronts, where the well-known monotone iteration techniques for elliptic systems with advanced arguments in [20,24] are used. The results are applicable to delayed Fisher-KPP equation, Belousov–Zhabotinskii model with delay, and some other models, see [11,1,17,29], etc. Following Wu and Zou [33], Ma [22] employed the Schauder's fixed point theorem to an operator used [33] in a properly chosen subset of the Banach space $C(\mathbb{R}, \mathbb{R}^n)$ equipped with the so-called exponential decay norm. The subset is constructed in terms of a pair of upper-lower solutions, which is less restrictive than the upper-lower solutions required in [33]. This makes the searching for the pair of upper-lower solutions slightly easier. Since Ma [22] only considered delayed systems with quasimonotone reaction terms, Huang and Zou [18] extended the results of Ma [22] to a class of delayed systems with nonquasimonotone reaction terms.

In ecology, since populations take time to move in space and usually were not at the same position in space at previous times, sometimes it is not sufficient only to include a discrete delay or a finite delay in a population model. Motivated by this, Britton

[5,6] considered comprehensively the two factors and introduced the so-called *spatio-temporal delay* or *nonlocal delay*, that is, the delay term involves a weighted spatio-temporal average over the whole of the infinite spatial domain and the whole of the previous times. Since then, great progress has been made on the existence of travelling wave fronts in reaction–diffusion equations with spatio-temporal delays, see [1–3,9,10, 12–16,21,25,28,30,35]. There are three methods which have been used to prove the existence of travelling wave solutions in these works. The first one is the perturbation theory of ordinary differential equations coupled with Fredholm alterative, see [1] for an age-structured reaction–diffusion model with nonlocal delay and Gourley [9] for a nonlocal Fisher equation. The second one is the geometric singular perturbation theory of Fenichel [7], see [2,10,16,25,30]. More precisely, if the corresponding undelayed system under consideration has a travelling wave solution, then, by choosing special kernels and applying the geometric singular perturbation theory, the reaction–diffusion system with spatio-temporal delay also has a travelling wave solution when the delay is sufficiently small. The third one is the monotone iteration approach of Wu and Zou [33], we refer to [21,28,35] for several special reaction–diffusion models with distributed delay or spatio-temporal delay, where the quasimonotonicity condition is required. However, as pointed out by Al-Omari and Gourley [1], the approach of Wu and Zou [33] cannot be applied directly to reaction–diffusion systems with distributed delay or spatio-temporal delay since it requires that the delayed term remains local in space.

It is natural to ask if the above results of Lan and Wu [19] and Wu and Zou [33] can be extended to general reaction–diffusion equations with distributed delays and spatio-temporal delays. A prototype of such equations takes the form

$$\frac{\partial u(t,x)}{\partial t} = D\frac{\partial^2 u(t,x)}{\partial x^2} + f\left(u(t,x),(g*u)(t,x)\right), \tag{1.1}$$

where $t \geqslant 0$, $x \in \mathbb{R}$, $D = \operatorname{diag}(d_1,\ldots,d_n)$, $d_i > 0$, $i = 1,\ldots n$, $n \in \mathbb{N}$; $u(t,x) = (u_1(t,x),\ldots,u_n(t,x))^T$, $f \in C\left(\mathbb{R}^{2n},\mathbb{R}^n\right)$, and

$$(g*u)(t,x) = \int_{-\infty}^{t}\int_{-\infty}^{+\infty} g(t-s,x-y)\,u(s,y)\,dy\,ds \tag{1.2}$$

or

$$(g*u)(t,x) = \int_{-\infty}^{t} g(t-s)\,u(s,x)\,ds. \tag{1.3}$$

The purpose of this paper is to establish the existence of travelling wave fronts of (1.1). *A travelling wave front* is a solution $u(t,x) = \varphi(x+ct)$, where $c > 0$ is a given constant and $\varphi \in BC^2(\mathbb{R},\mathbb{R}^n)$ (see Section 2) is an increasing function satisfying the following functional differential system:

$$-D\varphi''(t) + c\varphi'(t) = f(\varphi(t), (g * \varphi)(t)), \quad t \in \mathbb{R}, \tag{1.4}$$

and the conditions

$$\varphi(-\infty) = \mathbf{0} \quad \text{and} \quad \varphi(+\infty) = \mathbf{K} \text{ with } \mathbf{0} <_s \mathbf{K}, \tag{1.5}$$

where the notation $<_s$ is defined in Section 2 and

$$(g * \varphi)(t) = \int_0^{+\infty} \int_{-\infty}^{+\infty} g(s, y)\, \varphi(t - y - cs)\, dy\, ds \tag{1.6}$$

or

$$(g * \varphi)(t) = \int_0^{+\infty} g(s)\, \varphi(t - cs)\, ds. \tag{1.7}$$

Our main idea is to change the existence problem for the functional differential system (1.4) into a fixed point problem for an integral operator of the form

$$A\varphi(t) \equiv \int_{-\infty}^{+\infty} k(t, s)\, ((F\varphi)(s) + \gamma\varphi(s))\, ds = \varphi(t), \quad t \in \mathbb{R}, \tag{1.8}$$

where $F : BC[\mathbf{0}, \mathbf{K}] \subset BC(\mathbb{R}, \mathbb{R}^n) \to L^\infty(\mathbb{R}, \mathbb{R}^n)$ is a suitable map. (The symbols and the precise definitions of conceptions mentioned in this section will be given later in this paper.) New iterative techniques are established for the map A, which can be used to treat the existence of travelling waves for (1.1). The main difficulty in establishing the theory is that the map A may not be continuous and it is not clear if $\overline{A(E)}$ is compact in $BC(\mathbb{R}, \mathbb{R}^n)$ for all bounded sets E in $BC[\mathbf{0}, \mathbf{K}]$. This idea was first used by Lan and Wu [19] to study the existence of travelling wave solutions of the scalar reaction–diffusion equation with and without delay of the form

$$\frac{\partial u(t, x)}{\partial t} = D\frac{\partial^2 u(t, x)}{\partial x^2} + f(u(t, x), u(t - r_1, x), \ldots, u(t - r_n, x)). \tag{1.9}$$

Since the approach of Lan and Wu [19] is only applicable to scalar reaction–diffusion equations without and with discrete delays, we must search for new techniques for our reaction–diffusion systems with spatio-temporal delays. To overcome the difficulty, we introduce and employ the so-called M-continuity for F and show that the closure of $P_a^b A(E)$ is compact in $C([a, b], \mathbb{R}^n)$ for each bounded subset E, where P_a^b maps each element in $BC(\mathbb{R}, \mathbb{R}^n)$ to its restriction to $[a, b]$, which are extensions of Lan and

Wu [19]. In order to describe the monotonicity of reaction nonlinear terms, in addition to γ-increasing introduced in [19], we also introduce two new concepts: γ^*-increasing and γ^{**}-increasing, which include the cases that the reaction terms do not satisfy the quasimonotonicity condition, in particular, the later is new since it was not considered [19,33]. These, together with Lebesgue's dominated convergence theorem and uniform convergence of the integral $\int_{-\infty}^{\infty} g(t,x)\,dx$ in $t \in [0,a]$, where $a > 0$, enable us to prove that the iterative sequences involved are convergent in some sense. Thus, we can apply the monotone iteration technique coupled with the upper-lower solutions and a nonstandard ordering in the profile set to deal with the existence of travelling wave fronts of reaction–diffusion systems with spatio-temporal delays. Here we need to point out that uniform convergence of the integral $\int_{-\infty}^{\infty} g(t,x)\,dx$ in $t \in [0,a]$, $a > 0$, is not a more restrictive condition. In fact, many known kernel functions $g(t,x)$, such as

$$g(t,x) = \frac{1}{\tau} e^{-\frac{t}{\tau}} \delta(x), \quad \tau > 0, \tag{1.10}$$

$$g(t,x) = \delta(t) \frac{1}{\sqrt{4\pi\rho}} e^{-\frac{x^2}{4\rho}}, \quad \rho > 0, \tag{1.11}$$

and

$$g(t,x) = \frac{1}{\tau} e^{-\frac{t}{\tau}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}, \quad \tau > 0 \tag{1.12}$$

satisfy this condition, which will be verified in Section 5.

The rest of this paper is organized as follows. Section 2 is devoted to some preliminary discussions. We introduce a class of maps, which are bounded, M-continuous, γ-increasing, γ^*-increasing and γ^{**}-increasing, respectively, and provide some basic properties. In Section 3, we develop a monotone iteration scheme and apply it to establish the existence of solutions for the second-order system of functional differential equations if the nonlinear term satisfies one of the γ-increasing, γ^*-increasing or γ^{**}-increasing conditions. Following this, we establish the existence of travelling wave fronts in Section 4. In the last section, we apply our results to a diffusive single-species model with spatio-temporal delay. By constructing a pair of the upper and lower solutions, the existence of travelling wave front are obtained by choosing different kernel functions, such as (1.10)–(1.12).

2. Preliminaries

In this section, we introduce some definitions and lemmas, which will be needed in the sequel.

We denote by $C(\mathbb{R}, \mathbb{R}^n)$, $C((-\infty, b], \mathbb{R}^n)$ and $C([a,b], \mathbb{R}^n)$ the space of all continuous vector functions defined on $\mathbb{R}$, $(-\infty, b]$ and $[a,b]$ with sup-norm, respectively.

Let

$$BC(\mathbb{R}, \mathbb{R}^n) = \left\{ \begin{array}{c} x \in C(\mathbb{R}, \mathbb{R}^n) : x = (x_1(t), \ldots, x_n(t))^T, x_i(t) \in C(\mathbb{R}), \\ \max_{1 \leqslant i \leqslant n} \sup\{|x_i(t)| : t \in \mathbb{R}\} < \infty \end{array} \right\}$$

and

$$BC^2(\mathbb{R}, \mathbb{R}^n) = \left\{ \begin{array}{l} x \in BC(\mathbb{R}, \mathbb{R}^n) : x', x'' \in BC(\mathbb{R}, \mathbb{R}^n), \\ x' = (x_1'(t), \ldots, x_n'(t))^T, x'' = (x_1''(t), \ldots, x_n''(t))^T \end{array} \right\}.$$

Obviously, $BC(\mathbb{R}, \mathbb{R}^n)$ and $BC^2(\mathbb{R}, \mathbb{R}^n)$ are Banach spaces with the norms

$$\|x\|_{BC(\mathbb{R},\mathbb{R}^n)} = \max_{1 \leqslant i \leqslant n} \sup\{|x_i(t)| : t \in \mathbb{R}\}$$

and

$$\|x\|_{BC^2(\mathbb{R},\mathbb{R}^n)} = \max\left\{\|x\|_{BC(\mathbb{R},\mathbb{R}^n)}, \|x'\|_{BC(\mathbb{R},\mathbb{R}^n)}, \|x''\|_{BC(\mathbb{R},\mathbb{R}^n)}\right\},$$

respectively. For simplicity, we write $\|\cdot\| = \|\cdot\|_{BC(\mathbb{R},\mathbb{R}^n)}$. We also denote by

$$L^\infty\left(\mathbb{R}, \mathbb{R}^n\right) = L^\infty(\mathbb{R}) \times \cdots \times L^\infty(\mathbb{R})$$

with the norm

$$\|x\|_{L^\infty(\mathbb{R},\mathbb{R}^n)} = \max_{1 \leqslant i \leqslant n} \|x_i\|_{L^\infty(\mathbb{R})}.$$

The following lemma provides relations between $BC(\mathbb{R}, \mathbb{R}^n)$ and $C([a, b], \mathbb{R}^n)$, which is an extension of Lemma 2.1 in [19]. Its proof is straightforward and omitted.

Lemma 2.1. *Let $\{x^m\}$ be a sequence in the Banach space $BC(\mathbb{R}, \mathbb{R}^n)$, where $m \in \mathbb{N}$.*

(i) $\|x\| = \sup\left\{\|x\|_{C([a,b],\mathbb{R}^n)} : -\infty < a < b < \infty\right\}$ *for* $x \in (BC(\mathbb{R}, \mathbb{R}^n)$.
(ii) *If* $\{x^m\} \cup \{x\} \subset BC(\mathbb{R}, \mathbb{R}^n)$ *and* $\|x^m - x\| \to 0$ $(m \to \infty)$, *then* $\|x^m - x\|_{C([a,b],\mathbb{R}^n)} \to 0$ $(m \to \infty)$ *for* $a, b \in \mathbb{R}$ *with* $a < b$.
(iii) *If* $\{x^m\} \cup \{x\} \subset BC(\mathbb{R}, \mathbb{R}^n)$ *and* $\|x^m - x\|_{C([a,b],\mathbb{R}^n)} \to 0$ $(m \to \infty)$ *for* $a, b \in \mathbb{R}$ *with* $a < b$, *then* $x^m(t) \to x(t)$ $(m \to \infty)$ *for each* $t \in \mathbb{R}$.

In the rest of this paper, we use the usual notations for the standard ordering in $\mathbb{R}^n$. That is, for $\alpha = (\alpha_1, \ldots, \alpha_n)^T \in \mathbb{R}^n$ and $\beta = \left(\beta_1, \ldots, \beta_n\right)^T \in \mathbb{R}^n$, we denote $\alpha \leqslant \beta$ if $\alpha_i \leqslant \beta_i, i = 1, \ldots, n$ and $\alpha < \beta$ if $\alpha \leqslant \beta$ but $\alpha \neq \beta$. In particular, we denote $\alpha <_s \beta$ if $\alpha_i < \beta_i, i = 1, \ldots, n$. For $u, v \in L^\infty\left(\mathbb{R}, \mathbb{R}^n\right)$, we denote $u \leqslant v$ if $u_i(t) \leqslant v_i(t)$, $i = 1, 2, \ldots, n$, a.e. on $\mathbb{R}$ and $u < v$ if $u \neq v$. For given $\alpha, \beta \in \mathbb{R}^n$ with $\alpha <_s \beta$, let

$BC\left[\alpha,\beta\right]=\left\{x\in BC(\mathbb{R},\mathbb{R}^n):\alpha\leqslant x(t)\leqslant\beta,t\in\mathbb{R}\right\}$. If $Tx\leqslant Ty$ for $x,y\in BC\left[\alpha,\beta\right]$ with $x\leqslant y$, we say the map $T:BC\left[\alpha,\beta\right]\subset BC(\mathbb{R},\mathbb{R}^n)\to L^\infty\left(\mathbb{R},\mathbb{R}^n\right)$ is increasing. For any $\mu\in\mathbb{R}^n$, let $\widehat{\mu}$ denote a constant vector function on $t\in\mathbb{R}$ taking the vector μ.

Now, we introduce several concepts of γ-increasing, γ^*-increasing and γ^{**}-increasing maps, where the concept of a γ-increasing map was introduced by Lan and Wu [19].

Definition 2.2. A map $T:BC\left[\alpha,\beta\right]\subset BC(\mathbb{R},\mathbb{R}^n)\to L^\infty\left(\mathbb{R},\mathbb{R}^n\right)$ is said to be *γ-increasing* if there exists a matrix $\gamma=\operatorname{diag}\left(\gamma_1,\ldots,\gamma_n\right)$ with $\gamma_i>0,i=1,\ldots n$, such that $Ty+\gamma y\geqslant Tx+\gamma x$ for $x,y\in BC\left[\alpha,\beta\right]$ with $x\leqslant y$.

Definition 2.3. A map $T:BC\left[\alpha,\beta\right]\subset BC(\mathbb{R},\mathbb{R}^n)\to L^\infty\left(\mathbb{R},\mathbb{R}^n\right)$ is said to be *γ^*-increasing* if there exists a matrix $\gamma=\operatorname{diag}\left(\gamma_1,\ldots,\gamma_n\right)$ with $\gamma_i>0,i=1,\ldots n$, such that $Ty+\gamma y\geqslant Tx+\gamma x$, where $x,y\in BC\left[\alpha,\beta\right]$ with $x\leqslant y$ satisfy that $e^{\gamma t}\left[y\left(t\right)-x\left(t\right)\right]$ is increasing in $t\in\mathbb{R}$.

Definition 2.4. A map $T:BC\left[\alpha,\beta\right]\subset BC(\mathbb{R},\mathbb{R}^n)\to L^\infty\left(\mathbb{R},\mathbb{R}^n\right)$ is said to be *γ^{**}-increasing* if there exists a matrix $\gamma=\operatorname{diag}\left(\gamma_1,\ldots,\gamma_n\right)$ with $\gamma_i>0,i=1,\ldots n$, such that $Ty+\gamma y\geqslant Tx+\gamma x$, where $x,y\in BC\left[\alpha,\beta\right]$ with $x\leqslant y$ satisfy that $e^{\gamma t}\left[y\left(t\right)-x\left(t\right)\right]$ is increasing in $t\in\mathbb{R}$ and $e^{-\gamma t}\left[y\left(t\right)-x\left(t\right)\right]$ is decreasing in $t\in\mathbb{R}$.

In order to establish our iterative techniques, we need the so-called M-continuity for a map.

Definition 2.5. A map $T:BC\left[\alpha,\beta\right]\subset BC(\mathbb{R},\mathbb{R}^n)\to L^\infty\left(\mathbb{R},\mathbb{R}^n\right)$ is said to be *M-continuous* on $BC\left[\alpha,\beta\right]$ if $\{x^m\}\cup\{x\}\subset BC\left[\alpha,\beta\right]$ and $\|x^m-x\|_{C([a,b],\mathbb{R}^n)}\to 0$ $(m\to\infty)$ for $a,b\in\mathbb{R}$ with $a<b$ imply $\left(Tx^m\right)(t)\to(Tx)(t)$ $(m\to\infty)$ a.e. on $\mathbb{R}$.

3. Systems of the second-order functional differential equations

In this section, we consider the existence of solutions for systems of the second-order functional differential equations of the form

$$-D\varphi''\left(t\right)+c\varphi'\left(t\right)=\Phi\left(\varphi_t\right)\quad\text{a.e. on }\mathbb{R},\tag{3.1}$$

where $D=\operatorname{diag}\left(d_1,d_2,\ldots,d_n\right)$, $d_i>0$, $i=1,2,\ldots,n$; $c\in\mathbb{R}$, $\Phi:BC[\alpha,\beta]\subset BC(\mathbb{R},\mathbb{R}^n)\to\mathbb{R}^n$ is bounded and $\varphi_t\left(\cdot\right)\in BC[\alpha,\beta]$ is defined by $\varphi_t\left(s\right)=\varphi\left(t+s\right)$, $\varphi\in BC[\alpha,\beta]$, $s\in\mathbb{R}$; $\alpha,\beta\in\mathbb{R}^n$ with $\alpha<_s\beta$.

Let

$$Y=\left\{x\in BC(\mathbb{R},\mathbb{R}^n):x',x''\in L^\infty\left(\mathbb{R},\mathbb{R}^n\right)\right\}.$$

Then Y is a Banach space with the norm

$$\|x\|_Y=\max\left\{\|x\|,\|x'\|_{L^\infty(\mathbb{R},\mathbb{R}^n)},\|x''\|_{L^\infty(\mathbb{R},\mathbb{R}^n)}\right\}.$$

In particular, if we set $Y_0 = \left\{x \in BC(\mathbb{R}) : x', x'' \in L^\infty(\mathbb{R})\right\}$, then $Y = Y_0 \times \cdots \times Y_0$.

Now, by a solution to (3.1) we mean a function $\varphi \in Y$ and satisfies (3.1).

Let $\gamma_i > 0, i = 1, 2, \ldots, n$. We write

$$\lambda_{i1} = \frac{c - \sqrt{c^2 + 4\gamma_i d_i}}{2d_i}, \quad \lambda_{i2} = \frac{c + \sqrt{c^2 + 4\gamma_i d_i}}{2d_i}.$$

Then $\lambda_{i1} < 0 < \lambda_{i2}$ and $-d_i\lambda_{ij}^2 + c\lambda_{ij} + \gamma_i = 0$, where $i = 1, 2, \ldots, n$; $j = 1, 2$. Let $\rho_i = d_i(\lambda_{i2} - \lambda_{i1})$. Define a matrix map $k(t, s)$ by

$$k(t, s) = \operatorname{diag}(k_1(t, s), \ldots, k_n(t, s)),$$

where

$$k_i(t, s) = \rho_i^{-1} \begin{cases} e^{\lambda_{i1}(t-s)} & \text{for } s \leqslant t, \\ e^{\lambda_{i2}(t-s)} & \text{for } s \geqslant t, \end{cases}$$

and $i = 1, 2, \ldots, n$.

We now consider the linear integral operator $L : L^\infty\left(\mathbb{R}, \mathbb{R}^n\right) \to Y$ defined by

$$(L\varphi)(t) = \int_{-\infty}^{+\infty} k(t, s)\,\varphi(s)\, ds, \tag{3.2}$$

where

$$(L\varphi)(t) = \left(\left(L_1\varphi_1\right)(t), \left(L_2\varphi_2\right)(t), \ldots, \left(L_n\varphi_n\right)(t)\right)^T \tag{3.3}$$

and

$$\left(L_i\varphi_i\right)(t) = \int_{-\infty}^{+\infty} k_i(t, s)\,\varphi_i(s)\,ds.$$

By Lan and Wu [19, Theorem 3.1, p. 179], we know L_i maps $L^\infty(\mathbb{R})$ onto Y_0, so we obtain the following theorem:

Theorem 3.1. *The map L defined in* (3.2) *maps $L^\infty\left(\mathbb{R}, \mathbb{R}^n\right)$ onto Y and is linear, bounded, and one to one. Moreover, L maps $BC(\mathbb{R}, \mathbb{R}^n)$ onto $BC^2(\mathbb{R}, \mathbb{R}^n)$ and is linear, bounded, and one to one.*

Similar to that of [19], we now introduce the concept of G-compactness and show that the map L is G-compact. The concept of G-compactness is sufficient for us to establish our iterative scheme.

We define a map $P_a^b : BC(\mathbb{R}, \mathbb{R}^n) \to C([a, b], \mathbb{R}^n)$ by $P_a^b x(t) = x(t)|_{[a,b]}$, where $x(t)|_{[a,b]}$ denotes the restriction of $x(t)$ to $[a, b]$.

Definition 3.2. A map $T : L^\infty(\mathbb{R}, \mathbb{R}^n) \to BC(\mathbb{R}, \mathbb{R}^n)$ is said to be *G-compact* if $\overline{P_a^b T(E)}$ is compact in $C([a, b], \mathbb{R}^n)$ for all $a, b \in \mathbb{R}$ with $a < b$ and every bounded subset $E \subset L^\infty(\mathbb{R}, \mathbb{R}^n)$.

Lemma 3.3. *Assume $T_i : L^\infty(\mathbb{R}) \to BC(\mathbb{R})$ is G-compact, $i = 1, 2, \ldots, n$, then the map $T : L^\infty(\mathbb{R}, \mathbb{R}^n) \to BC(\mathbb{R}, \mathbb{R}^n)$ defined by*

$$(Tx)(t) = ((T_1x_1)(t), (T_2x_2)(t), \ldots, (T_nx_n)(t))^T$$

is G-compact.

Proof. Let $E \subset L^\infty(\mathbb{R}, \mathbb{R}^n)$ be a bounded subset and $P_i : x(t) \to x_i(t)$ be a project operator. Let $E_i = P_iE$. Then $E \subset E_1 \times \cdots \times E_n$ and E_i is a bounded subset in $L^\infty(\mathbb{R})$. For $x(t) \in E$, we have

$$\begin{aligned}(Tx)(t) &= ((T_1x_1)(t), (T_2x_2)(t), \ldots, (T_nx_n)(t))^T \\ &\in T_1(E_1) \times T_2(E_2) \times \cdots \times T_n(E_n)\end{aligned}$$

and

$$\begin{aligned}P_a^b(Tx)(t) &= \left(P_a^b(T_1x_1)(t), P_a^b(T_2x_2)(t), \ldots, P_a^b(T_nx_n)(t)\right)^T \\ &\in P_a^bT_1(E_1) \times P_a^bT_2(E_2) \times \cdots \times P_a^bT_n(E_n).\end{aligned}$$

Thus,

$$P_a^bT(E) \subset P_a^bT_1(E_1) \times P_a^bT_2(E_2) \times \cdots \times P_a^bT_n(E_n)$$

and

$$\overline{P_a^bT(E)} \subset \overline{P_a^bT_1(E_1)} \times \overline{P_a^bT_2(E_2)} \times \cdots \times \overline{P_a^bT_n(E_n)}.$$

Since $T_i : L^\infty(\mathbb{R}) \to BC(\mathbb{R})$ is G-compact, $\overline{P_a^bT_i(E_i)}$ is compact in $C([a, b], \mathbb{R})$, then

$$\overline{P_a^bT_1(E_1)} \times \overline{P_a^bT_2(E_2)} \times \cdots \times \overline{P_a^bT_n(E_n)}$$

is compact in $C([a, b], \mathbb{R}^n)$ and $\overline{P_a^bT(E)}$ is compact in $C([a, b], \mathbb{R}^n)$ too. By Definition 3.2, we know that $T : L^\infty(\mathbb{R}, \mathbb{R}^n) \to BC(\mathbb{R}, \mathbb{R}^n)$ is G-compact. The proof is complete. $\square$

Theorem 3.4. *The map L defined in* (3.2) *maps $L^{\infty}\left(\mathbb{R}, \mathbb{R}^n\right)$ into $BC(\mathbb{R}, \mathbb{R}^n)$ and is G-compact.*

Proof. By Lan and Wu [19, Theorem 3.2, p. 179], we know that $L_i : L^{\infty}(\mathbb{R}) \to BC(\mathbb{R})$ is G-compact. Following (3.3) and Lemma 3.3, L is G-compact. The proof is complete. □

Now we define an integral operator A by

$$(A\varphi)(t) \equiv \int_{-\infty}^{+\infty} k(t,s)\left((F\varphi)(s) + \gamma\varphi(s)\right) ds, \quad t \in \mathbb{R},$$

where $F : BC[\alpha, \beta] \to L^{\infty}\left(\mathbb{R}, \mathbb{R}^n\right)$ is defined by $(F\varphi)(t) = \Phi(\varphi_t)$. Then we have the following result:

Lemma 3.5. (i) *Let $\phi \in Y$ and $\varphi \in BC[\alpha, \beta]$. Then $\phi = A\varphi$ if and only if $-D\phi'' + c\phi' + \gamma\phi = F\varphi + \gamma\varphi$.*

(ii) *φ is a solution of* (3.1) *if and only if $\varphi \in Y$ and $\varphi = A\varphi$.*

Proof. We only prove (i), the proof of part (ii) is similar and omitted. Suppose that

$$-D\phi'' + c\phi' + \gamma\phi = F\varphi + \gamma\varphi$$

for $\phi \in Y$ and $\varphi \in BC[\alpha, \beta]$. In view of

$$-D(A\varphi)''(t) + c(A\varphi)'(t) + \gamma(A\varphi)(t) = (F\varphi)(t) + \gamma\varphi(t),$$

we have

$$-D(\phi - A\varphi)''(t) + c(\phi - A\varphi)'(t) + \gamma(\phi - A\varphi)(t) = \mathbf{0}.$$

Let

$$(\phi - A\varphi)(t) = w(t) = (w_1(t), \ldots, w_n(t))^T.$$

Then

$$-d_i w_i''(t) + c w_i'(t) + \gamma_i w_i(t) = 0.$$

Thus,

$$w_i(t) = a_{i1} e^{\lambda_{i1} t} + a_{i2} e^{\lambda_{i2} t}.$$

Since $w_i(t) \in BC(\mathbb{R})$, it follows that $a_{i1} = a_{i2} = 0$, $i = 1, \ldots, n$. Hence, $w(t) = \mathbf{0}$ for $t \in \mathbb{R}$, that is, $\phi = A\varphi$. The converse is obvious. The proof is complete. □

Lemma 3.6. (i) *Assume that* $F : BC[\alpha, \beta] \to L^\infty\left(\mathbb{R}, \mathbb{R}^n\right)$ *is* γ*-increasing and bounded. If* $\varphi(t) \in BC[\alpha, \beta]$ *is increasing, then* $(A\varphi)(t)$ *is also increasing.*

(ii) *Assume that* $F : BC[\alpha, \beta] \to L^\infty\left(\mathbb{R}, \mathbb{R}^n\right)$ *is* γ^**-increasing and bounded. If* $\varphi(t) \in BC[\alpha, \beta]$ *is increasing such that* $e^{\gamma t}\left[\varphi(t+s) - \varphi(t)\right]$ *is increasing in* $t \in \mathbb{R}$ *for every* $s > 0$, *then* $(A\varphi)(t)$ *is also increasing in* $t \in \mathbb{R}$ *and for* $c > 1 - \min\left\{\gamma_i d_i; i = 1, \ldots, n\right\}$, $e^{\gamma t}\left[(A\varphi)(t+s) - (A\varphi)(t)\right]$ *is increasing in* $t \in \mathbb{R}$ *for every* $s > 0$.

(iii) *Assume that* $F : BC[\alpha, \beta] \to L^\infty\left(\mathbb{R}, \mathbb{R}^n\right)$ *is* γ^{**}*-increasing and bounded, where* γ *satisfies* $\min\left\{\gamma_i d_i; i = 1, \ldots, n\right\} - 1 > 0$. *If* $\varphi(t) \in BC[\alpha, \beta]$ *is increasing such that* $e^{\gamma t}\left[\varphi(t+s) - \varphi(t)\right]$ *is increasing in* $t \in \mathbb{R}$ *and* $e^{-\gamma t}\left[\varphi(t+s) - \varphi(t)\right]$ *is decreasing in* $t \in \mathbb{R}$ *for every* $s > 0$, *then* $(A\varphi)(t)$ *is increasing in* $t \in \mathbb{R}$ *and for* c *with*

$$1 - \min\left\{\gamma_i d_i; i = 1, \ldots, n\right\} < c < \min\left\{\gamma_i d_i; i = 1, \ldots, n\right\} - 1, \tag{3.4}$$

$e^{\gamma t}\left[(A\varphi)(t+s) - (A\varphi)(t)\right]$ *is increasing and* $e^{-\gamma t}\left[(A\varphi)(t+s) - (A\varphi)(t)\right]$ *is decreasing in* $t \in \mathbb{R}$ *for every* $s > 0$.

Proof. We only show (iii), the proofs of (i) and (ii) are similar. Let $\Delta t > 0$. Noting that $(F\varphi)(s + \Delta t) = (F\varphi_{\Delta t})(s)$ and employing a change of variable, for $t \in \mathbb{R}$, we have

$$\begin{aligned}
&(A\varphi)(t+\Delta t) - (A\varphi)(t) \\
&\quad = \int_{-\infty}^{t} k(t,s)\left((F\varphi)(s+\Delta t) + \gamma\varphi(s+\Delta t)\right) ds \\
&\qquad + \int_{t}^{\infty} k(t,s)\left((F\varphi)(s+\Delta t) + \gamma\varphi(s+\Delta t)\right) ds \\
&\qquad - \int_{-\infty}^{t} k(t,s)\left((F\varphi)(s) + \gamma\varphi(s)\right) ds - \int_{t}^{\infty} k(t,s)\left((F\varphi)(s) + \gamma\varphi(s)\right) ds \\
&\quad = \int_{-\infty}^{\infty} k(t,s)\left[\left((F\varphi)(s+\Delta t) + \gamma\varphi(s+\Delta t)\right) - \left((F\varphi)(s) + \gamma\varphi(s)\right)\right] ds \\
&\quad = \int_{-\infty}^{\infty} k(t,s)\left[\left((F\varphi_{\Delta t})(s) + \gamma\varphi_{\Delta t}(s)\right) - \left((F\varphi)(s) + \gamma\varphi(s)\right)\right] ds.
\end{aligned}$$

Since $\varphi(t) \in BC[\alpha, \beta]$ is increasing and satisfies that $e^{\gamma t}\left[\varphi(t+\Delta t) - \varphi(t)\right]$ is increasing and $e^{-\gamma t}\left[\varphi(t+\Delta t) - \varphi(t)\right]$ in $t \in \mathbb{R}$, then $e^{\gamma t}\left[\varphi_{\Delta t}(t) - \varphi(t)\right]$ is increasing, $e^{-\gamma t}\left[\varphi_{\Delta t}(t) - \varphi(t)\right]$ is decreasing and $\varphi_{\Delta t}(t) \geqslant \varphi(t)$ in $t \in \mathbb{R}$. Note that F is γ^{**}-increasing, it follows that

$$\left((F\varphi_{\Delta t})(s) + \gamma\varphi_{\Delta t}(s)\right) - \left((F\varphi)(s) + \gamma\varphi(s)\right) \geqslant \mathbf{0}.$$

Hence, $(A\varphi)(t+\Delta t) - (A\varphi)(t) \geqslant \mathbf{0}$, which implies that $(A\varphi)(t)$ is also increasing.

Let $s > 0$ and

$$H_i(\varphi)(t) = (F_i\varphi)(t) + \gamma_i\varphi_i(t), \quad i \in \{1, \ldots, n\}.$$

Then for $i \in \{1, \ldots, n\}$,

$$\begin{aligned}
&\frac{d}{dt}\left\{e^{-\gamma_i t}\int_{-\infty}^{+\infty} k_i(t,\xi)\left[\left((F_i\varphi)(\xi+s) + \gamma_i\varphi_i(\xi+s)\right) - \left((F_i\varphi)(\xi) + \gamma_i\varphi_i(\xi)\right)\right]d\xi\right\} \\
&\quad = \frac{d}{dt}\left\{\rho_i^{-1}e^{(-\gamma_i+\lambda_{i1})t}\int_{-\infty}^{t} e^{-\lambda_{i1}\xi}\left[H_i(\varphi)(\xi+s) - H_i(\varphi)(\xi)\right]d\xi\right\} \\
&\qquad + \frac{d}{dt}\left\{\rho_i^{-1}e^{(-\gamma_i+\lambda_{i2})t}\int_{t}^{+\infty} e^{-\lambda_{i2}\xi}\left[H_i(\varphi)(\xi+s) - H_i(\varphi)(\xi)\right]d\xi\right\} \\
&\quad = \left(-\gamma_i + \lambda_{i1}\right)\rho_i^{-1}e^{(-\gamma_i+\lambda_{i1})t}\int_{-\infty}^{t} e^{-\lambda_{i1}\xi}\left[H_i(\varphi)(\xi+s) - H_i(\varphi)(\xi)\right]d\xi \\
&\qquad + \rho_i^{-1}e^{(-\gamma_i+\lambda_{i1})t}e^{-\lambda_{i1}t}\left[H_i(\varphi)(t+s) - H_i(\varphi)(t)\right] \\
&\qquad + \left(-\gamma_i + \lambda_{i2}\right)\rho_i^{-1}e^{(-\gamma_i+\lambda_{i2})t}\int_{t}^{+\infty} e^{-\lambda_{i2}\xi}\left[H_i(\varphi)(\xi+s) - H_i(\varphi)(\xi)\right]d\xi \\
&\qquad - \rho_i^{-1}e^{(-\gamma_i+\lambda_{i2})t}e^{-\lambda_{i2}t}\left[H_i(\varphi)(t+s) - H_i(\varphi)(t)\right] \\
&\quad = \left(-\gamma_i + \lambda_{i1}\right)\rho_i^{-1}e^{(-\gamma_i+\lambda_{i1})t}\int_{-\infty}^{t} e^{-\lambda_{i1}\xi}\left[H_i(\varphi)(\xi+s) - H_i(\varphi)(\xi)\right]d\xi \\
&\qquad + \left(-\gamma_i + \lambda_{i2}\right)\rho_i^{-1}e^{(-\gamma_i+\lambda_{i2})t}\int_{t}^{+\infty} e^{-\lambda_{i2}\xi}\left[H_i(\varphi)(\xi+s) - H_i(\varphi)(\xi)\right]d\xi.
\end{aligned}$$

Noting that for $c < \min\left\{\gamma_i d_i;\ i = 1, \ldots, n\right\} - 1$ and each $i \in \{1, \ldots, n\}$,

$$-\gamma_i + \lambda_{i1} = -\gamma_i + \frac{c - \sqrt{c^2 + 4\gamma_i d_i}}{2d_i} < 0,$$

$$\begin{aligned}
-\gamma_i + \lambda_{i2} &= -\gamma_i + \frac{c + \sqrt{c^2 + 4\gamma_i d_i}}{2d_i} \\
&= -\frac{\gamma_i\left(\gamma_i d_i - c - 1\right)}{2d_i\gamma_i - c + \sqrt{c^2 + 4\gamma_i d_i}} < 0,
\end{aligned}$$

and

$$H_i(\varphi)(\xi+s) - H_i(\varphi)(\xi) = \left((F_i\varphi_s)(\xi) + \gamma_i\varphi_{is}(\xi)\right) - \left((F_i\varphi)(\xi) + \gamma_i\varphi_i(\xi)\right) \geqslant 0,$$

we have that $e^{-\gamma t}\left[(A\varphi)(t+s)-(A\varphi)(t)\right]$ is decreasing in $t\in\mathbb{R}$ for every $s>0$. Similarly, we can show that for $c>1-\min\left\{\gamma_i d_i;\ i=1,\ldots,n\right\}$, $e^{\gamma t}[(A\varphi)(t+s)-(A\varphi)(t)]$ is increasing in $t\in\mathbb{R}$ for every $s>0$. The proof is complete. □

Lemma 3.7. *Assume that* $F: BC[\alpha,\beta]\to L^{\infty}\left(\mathbb{R},\mathbb{R}^n\right)$ *is bounded and that* $\phi,\psi\in BC[\alpha,\beta]\cap Y$ *satisfy*:

(C_0) $-D\phi''+c\phi'\leqslant F\phi$ *and* $-D\psi''+c\psi'\geqslant F\psi$.

Then,

(i) $\phi\leqslant A\phi$ *and* $A\psi\leqslant\psi$;
(ii) *for* $c<\min\left\{\gamma_i d_i;\ i=1,\ldots,n\right\}-1$, $e^{-\gamma t}\left[\psi(t)-(A\psi)(t)\right]$ *and* $e^{-\gamma t}[(A\phi)(t)-\phi(t)]$ *are decreasing in* $t\in\mathbb{R}$;
(iii) *for* $c>1-\min\left\{\gamma_i d_i;\ i=1,\ldots,n\right\}$, $e^{\gamma t}\left[\psi(t)-(A\psi)(t)\right]$ *and* $e^{\gamma t}\left[(A\phi)(t)-\phi(t)\right]$ *are increasing in* $t\in\mathbb{R}$.

Proof. (i) Let $w=A\phi-\phi$ and $-Dw''+cw'+\gamma w=r(t)$. In view of

$$-D\phi''+c\phi'+\gamma\phi\leqslant F\phi+\gamma\phi$$

and

$$-D(A\phi)''+c(A\phi)'+\gamma(A\phi)=F\phi+\gamma\phi,$$

we know $r(t)\geqslant\mathbf{0}$ a.e. on $\mathbb{R}$. From

$$-Dw''+cw'+\gamma w=r(t),$$

we get

$$w_i(t)=a_{i1}e^{\lambda_{i1}t}+a_{i2}e^{\lambda_{i2}t}+\int_{-\infty}^{+\infty}k_i(t,s)\,r_i(s)\,ds.$$

Since $w_i(t)$ is bounded, we have $a_{i1}=a_{i2}=0$. Consequently,

$$w_i(t)=\int_{-\infty}^{+\infty}k_i(t,s)\,r_i(s)\,ds\geqslant 0,\quad t\in\mathbb{R},\ \ i=1,\ldots,n.$$

Thus, we proved that $\phi\leqslant A\phi$. Similarly, we can prove that $\psi\geqslant A\psi$.

(ii) By the proof of (i), $(A\phi)(t)-\phi(t)=w(t)\geqslant\mathbf{0}$, then

$$\begin{aligned}&\frac{d}{dt}\left\{e^{-\gamma_i t}\left[\int_{-\infty}^{+\infty}k_i(t,\xi)\,r_i(\xi)\,d\xi\right]\right\}\\&=\frac{d}{dt}\left\{\rho_i^{-1}e^{(-\gamma_i+\lambda_{i1})t}\int_{-\infty}^{t}e^{-\lambda_{i1}\xi}r_i(\xi)\,d\xi\right\}\end{aligned}$$

$$
\begin{aligned}
&+\frac{d}{dt}\left\{\rho_i^{-1}e^{(-\gamma_i+\lambda_{i2})t}\int_t^{+\infty} e^{-\lambda_{i2}\xi} r_i(\xi)\,d\xi\right\}\\
&=\left(-\gamma_i+\lambda_{i1}\right)\rho_i^{-1}e^{(-\gamma_i+\lambda_{i1})t}\int_{-\infty}^{t} e^{-\lambda_{i1}\xi} r_i(\xi)\,d\xi\\
&\quad+\left(-\gamma_i+\lambda_{i2}\right)\rho_i^{-1}e^{(-\gamma_i+\lambda_{i2})t}\int_t^{+\infty} e^{-\lambda_{i2}\xi} r_i(\xi)\,d\xi\\
&\leqslant 0, \quad i=1,\dots,n.
\end{aligned}
$$

This implies that $e^{-\gamma t}\left[(A\phi)(t)-\phi(t)\right]$ is decreasing in $t\in\mathbb{R}$.

The other cases are similar and omitted. The proof is complete. □

Lemma 3.8. *Assume that $F : BC[\alpha,\beta]\to L^\infty\left(\mathbb{R},\mathbb{R}^n\right)$ is bounded and $\phi,\psi\in BC[\alpha,\beta]$ with $\phi\leqslant\psi$.*

(i) *If F is γ-increasing, then $(A\psi)(t)\geqslant(A\phi)(t)$ in $t\in\mathbb{R}$;*
(ii) *If F is γ^*-increasing and $e^{\gamma t}\left[\psi(t)-\phi(t)\right]$ is increasing in $t\in\mathbb{R}$, then for $c>1-\min\left\{\gamma_i d_i; i=1,\dots,n\right\}$, $(A\psi)(t)\geqslant(A\phi)(t)$ and $e^{\gamma t}\left[(A\psi)(t)-(A\phi)(t)\right]$ is increasing in $t\in\mathbb{R}$;*
(iii) *If F is γ^{**}-increasing, $e^{\gamma t}\left[\psi(t)-\phi(t)\right]$ is increasing and $e^{-\gamma t}\left[\psi(t)-\phi(t)\right]$ is decreasing in $t\in\mathbb{R}$, where γ satisfies $\min\left\{\gamma_i d_i; i=1,\dots,n\right\}-1>0$, then $(A\psi)(t)\geqslant(A\phi)(t)$, $e^{\gamma t}\left[(A\psi)(t)-(A\phi)(t)\right]$ is increasing and $e^{-\gamma t}[(A\psi)(t)-(A\phi)(t)]$ is decreasing in $t\in\mathbb{R}$, where c satisfies* (3.4).

Proof. We only show that (iii) holds. Let

$$w(t)=(A\psi)(t)-(A\phi)(t), \quad t\in\mathbb{R}.$$

Then we have

$$-Dw''(t)+cw'(t)+\gamma w(t)=(F\psi)(t)+\gamma\psi(t)-(F\phi)(t)-\gamma\phi(t)\geqslant\mathbf{0} \quad \text{on } \mathbb{R}.$$

Denote

$$g(t)=-Dw''(t)+cw'(t)+\gamma w(t).$$

Then $g(t)\geqslant\mathbf{0}$ a.e. on $\mathbb{R}$ and $g(t)\in L^\infty\left(\mathbb{R},\mathbb{R}^n\right)$. By an argument similar to that of Lemma 3.7, we get $(A\psi)(t)\geqslant(A\phi)(t)$ and

$$\frac{d}{dt}\left\{e^{-\gamma t}\left[(A\psi)(t)-(A\phi)(t)\right]\right\}\leqslant\mathbf{0} \text{ on } \mathbb{R} \text{ for } c<\min\left\{\gamma_i d_i; i=1,\dots,n\right\}-1.$$

This implies that $e^{-\gamma t}\left[(A\psi)(t)-(A\phi)(t)\right]$ is decreasing in $t \in \mathbb{R}$. Similarly, we have that $e^{\gamma t}\left[(A\psi)(t)-(A\phi)(t)\right]$ is increasing in $t \in \mathbb{R}$. The proof is complete. □

By Lemma 3.5, we see that a solution of (3.1) is a fixed point of A. Hence, we have the following main results in this section.

Theorem 3.9. *Assume that $F : BC[\alpha,\beta] \to L^{\infty}\left(\mathbb{R},\mathbb{R}^n\right)$ is bounded and M-continuous. Assume further that $\phi,\psi \in Y \cap BC[\alpha,\beta]$ with $\phi \leqslant \psi$ satisfy* (C$_0$).

(i) *If F is γ-increasing, then* (3.1) *has two solutions $\phi_*,\psi^* \in BC[\alpha,\beta]$ with $\phi \leqslant \phi_* \leqslant \psi^* \leqslant \psi$. If ϕ and ψ are increasing, then ϕ_* and ψ^* are increasing. Moreover, for $a,b \in \mathbb{R}$ with $a<b$,*

$$\|\phi^m-\phi_*\|_{C([a,b],\mathbb{R}^n)} \to 0 \text{ and } \|\psi^m-\psi^*\|_{C([a,b],\mathbb{R}^n)} \to 0, \tag{3.5}$$

where $\phi^m = A\phi^{m-1}$, $\psi^m = A\psi^{m-1}$ and

$$\phi=\phi^0 \leqslant \phi^1 \leqslant \cdots \leqslant \phi^m \leqslant \cdots \leqslant \psi^m \leqslant \cdots \leqslant \psi^1 \leqslant \psi^0=\psi. \tag{3.6}$$

(ii) *If F is γ^*-increasing and $e^{\gamma t}\left[\psi(t)-\phi(t)\right]$ is increasing in $t \in \mathbb{R}$, then for $c>1-\min\left\{\gamma_i d_i; i=1,\ldots,n\right\}$,* (3.1) *has two solutions $\phi_*,\psi^* \in BC[\alpha,\beta]$ with $\phi \leqslant \phi_* \leqslant \psi^* \leqslant \psi$, which satisfy* (3.5) *and* (3.6)*. Furthermore, if ϕ and ψ are increasing such that $e^{\gamma t}\left[\psi(t+s)-\psi(t)\right]$ and $e^{\gamma t}\left[\phi(t+s)-\phi(t)\right]$ are increasing in $t \in \mathbb{R}$ for every $s>0$, then ϕ_* and ψ^* are increasing.*

(iii) *If F is γ^{**}-increasing, $e^{\gamma t}\left[\psi(t)-\phi(t)\right]$ is increasing and $e^{-\gamma t}\left[\psi(t)-\phi(t)\right]$ is decreasing in $t \in \mathbb{R}$, where γ satisfies $\min\left\{\gamma_i d_i; i=1,\ldots,n\right\}-1>0$, then for c satisfying* (3.4), (3.1) *has two solutions $\phi_*,\psi^* \in BC[\alpha,\beta]$ with $\phi \leqslant \phi_* \leqslant \psi^* \leqslant \psi$, which satisfy* (3.5) *and* (3.6)*. Furthermore, if ϕ and ψ are increasing in $t \in \mathbb{R}$, $e^{\gamma t}\left[\psi(t+s)-\psi(t)\right]$ and $e^{\gamma t}\left[\phi(t+s)-\phi(t)\right]$ are increasing in $t \in \mathbb{R}$ for every $s>0$ and $e^{-\gamma t}\left[\psi(t+s)-\psi(t)\right]$ and $e^{-\gamma t}\left[\phi(t+s)-\phi(t)\right]$ are decreasing in $t \in \mathbb{R}$ for every $s>0$, then ϕ_* and ψ^* are increasing.*

Proof. (i) By Theorem 3.4 and Lemma 3.8(i), A maps $BC[\alpha,\beta]$ into $BC\left(\mathbb{R},\mathbb{R}^n\right)$ and is increasing, so Lemma 3.7 and condition (C$_0$) imply that (3.6) holds. Then there exist $\phi_*,\psi^* \in L^{\infty}\left(\mathbb{R},\mathbb{R}^n\right)$ such that

$$\phi^m(t) \to \phi_*(t) \quad \text{and} \quad \psi^m(t) \to \psi^*(t) \quad \text{for each } \; t \in \mathbb{R}.$$

Obviously, $\phi \leqslant \phi_* \leqslant \psi^* \leqslant \psi$ follow from (3.6). By Theorem 3.4, $\overline{P_a^b LH\left\{\phi^m\right\}} = \overline{P_a^b A\left\{\phi^m\right\}}$ is compact in $C([a,b],\mathbb{R}^n)$ for $a,b \in \mathbb{R}$ with $a<b$. It follows that there exists $y \in C([a,b],\mathbb{R}^n)$ such that $\|A\phi^m-y\|_{C([a,b],\mathbb{R}^n)} \to 0$. Hence, we have $\phi_*(t)=y(t)$ for $t \in [a,b]$ and thus, $\phi_* \in BC(\mathbb{R},\mathbb{R}^n)$. Since F is bounded and

M-continuous, it follows that for each $t \in R$,

$$k_i(t,s)\left(\left(F_i\phi^m\right)(s)+\gamma_i\phi_i^m(s)\right)\to k_i(t,s)\left(\left(F_i\phi_*\right)(s)+\gamma_i\phi_{*i}(s)\right)\ \text{ for } s\in\mathbb{R},$$

$$\left|k_i(t,s)\left(\left(F_i\phi^m\right)(s)+\gamma_i\phi_i^m(s)\right)\right|\leqslant\eta k_i(t,s)\ \text{ for } s\in\mathbb{R} \text{ and some } \eta>0.$$

It follows from Lebesgue's dominated convergence theorem that

$$L_i\left(\left(F_i\phi^m\right)(t)+\gamma_i\phi_i^m(t)\right)\to L_i\left(\left(F_i\phi_*\right)(t)+\gamma_i\phi_{*i}(t)\right)$$

for each $t \in \mathbb{R}$ and

$$\phi_{*i}=L_i\left(\left(F_i\phi_*\right)(t)+\gamma_i\phi_{*i}(t)\right),\quad i=1,2,\ldots,n.$$

Hence,

$$\phi_*=L\left(\left(F\phi_*\right)(t)+\gamma\phi_*(t)\right)=A\phi_*.$$

A similar argument shows that $\psi^*=A\psi^*$. If ϕ and ψ are increasing, by Lemma 3.6(i), ϕ^m and ψ^m are increasing, $m\in\mathbb{N}$. Consequently, ϕ_* and ψ^* are increasing. The proof of (i) is complete.

(ii) Fix $c>1-\min\left\{\gamma_i d_i; i=1,\ldots,n\right\}$. Let $\psi^m=A\psi^{m-1}$ and $\phi^m=A\phi^{m-1}$, $m\in\mathbb{N}$. By Lemmas 3.7 and 3.8, $\psi^1=A\psi^0=A\psi$ and $\phi^1=A\phi^0=A\phi$ satisfy

(I) $\phi(t)\leqslant\left(\phi^1\right)(t)\leqslant\left(\psi^1\right)(t)\leqslant\psi(t)$ for $t\in\mathbb{R}$;
(II) $e^{\gamma t}\left[\left(\phi^1\right)(t)-\phi(t)\right]$ is increasing in $t\in\mathbb{R}$;
(III) $e^{\gamma t}\left[\left(\psi^1\right)(t)-\left(\phi^1\right)(t)\right]$ is increasing in $t\in\mathbb{R}$;
(IV) $e^{\gamma t}\left[\psi(t)-\left(\psi^1\right)(t)\right]$ is increasing in $t\in\mathbb{R}$.

By induction and the above lemmas, we obtain two sequences of vector functions $\left\{\psi^m\right\}_{m=1}^\infty$ and $\left\{\phi^m\right\}_{m=1}^\infty$ with the following properties:

(a) $\phi(t)\leqslant\left(\phi^m\right)(t)\leqslant\left(\phi^{m+1}\right)(t)\leqslant\left(\psi^{m+1}\right)(t)\leqslant\left(\psi^m\right)(t)\leqslant\psi(t)$ for $t\in\mathbb{R}$ and $m\in\mathbb{N}$;
(b) for $m\in\mathbb{N}$, $e^{\gamma t}\left[\left(\phi^{m+1}\right)(t)-\left(\phi^m\right)(t)\right]$ is increasing in $t\in\mathbb{R}$;
(c) for $m\in\mathbb{N}$, $e^{\gamma t}\left[\left(\psi^m\right)(t)-\left(\phi^m\right)(t)\right]$ is increasing in $t\in\mathbb{R}$;
(d) for $m\in\mathbb{N}$, $e^{\gamma t}\left[\left(\psi^m\right)(t)-\left(\psi^{m+1}\right)(t)\right]$ is increasing in $t\in\mathbb{R}$.

By Lemma 3.5, we have

$$-D\left(\psi^m\right)''+c\left(\psi^m\right)'+\gamma\psi^m = F\left(\psi^{m-1}\right)+\gamma\psi^{m-1} \quad (m\in\mathbb{N}),$$
$$-D\left(\phi^m\right)''+c\left(\phi^m\right)'+\gamma\phi^m = F\left(\phi^{m-1}\right)+\gamma\phi^{m-1} \quad (m\in\mathbb{N})$$

and

$$\phi=\phi^0\leqslant\phi^1\leqslant\cdots\leqslant\phi^m\leqslant\cdots\leqslant\psi^m\leqslant\cdots\leqslant\psi^1\leqslant\psi^0=\psi.$$

Then the existence of ϕ_* and ψ^* follow from a similar argument to that in (i).

If ϕ and ψ are increasing and satisfy that

$$e^{\gamma t}\left[\psi(t+s)-\psi(t)\right] \quad\text{and}\quad e^{\gamma t}\left[\phi(t+s)-\phi(t)\right]$$

are increasing in $t\in\mathbb{R}$ for every $s>0$, then from Lemma 3.6(ii) and by induction, we see that for $m\in\mathbb{N}$, ψ^m and ϕ^m are increasing in $t\in\mathbb{R}$ and

$$e^{\gamma t}\left[\left(\psi^m\right)(t+s)-\left(\psi^m\right)(t)\right] \quad\text{and}\quad e^{\gamma t}\left[\left(\phi^m\right)(t+s)-\left(\phi^m\right)(t)\right]$$

are increasing in $t\in\mathbb{R}$ for every $s>0$. Therefore, ϕ_* and ψ^* are increasing. The proof of (ii) is complete.

(iii) Fix c with (3.4). Let $\psi^m=A\psi^{m-1}$ and $\phi^m=A\phi^{m-1}$, $m\in\mathbb{N}$. By Lemmas 3.7 and 3.8, $\psi^1=A\psi^0=A\psi$ and $\phi^1=A\phi^0=A\phi$ satisfy

(I) $\phi(t)\leqslant\left(\phi^1\right)(t)\leqslant\left(\psi^1\right)(t)\leqslant\psi(t)$ for $t\in\mathbb{R}$;

(II) $e^{\gamma t}\left[\left(\phi^1\right)(t)-\phi(t)\right]$ is increasing and $e^{-\gamma t}\left[\left(\phi^1\right)(t)-\phi(t)\right]$ is decreasing in $t\in\mathbb{R}$;

(III) $e^{\gamma t}\left[\left(\psi^1\right)(t)-\left(\phi^1\right)(t)\right]$ is increasing and $e^{-\gamma t}\left[\left(\psi^1\right)(t)-\left(\phi^1\right)(t)\right]$ is decreasing in $t\in\mathbb{R}$;

(IV) $e^{\gamma t}\left[\psi(t)-\left(\psi^1\right)(t)\right]$ is increasing and $e^{-\gamma t}\left[\psi(t)-\left(\psi^1\right)(t)\right]$ is decreasing in $t\in\mathbb{R}$.

By induction and the above lemmas, we obtain two sequences of vector functions $\left\{\psi^m\right\}_{m=1}^{\infty}$ and $\left\{\phi^m\right\}_{m=1}^{\infty}$ which satisfy that for $m\in\mathbb{N}$,

(a) $\phi(t)\leqslant\left(\phi^m\right)(t)\leqslant\left(\phi^{m+1}\right)(t)\leqslant\left(\psi^{m+1}\right)(t)\leqslant\left(\psi^m\right)(t)\leqslant\psi(t)$ for $t\in\mathbb{R}$;

(b) $e^{\gamma t}\left[\left(\phi^{m+1}\right)(t)-\phi^m(t)\right]$ is increasing and $e^{-\gamma t}\left[\left(\phi^{m+1}\right)(t)-\phi^m(t)\right]$ is decreasing in $t\in\mathbb{R}$;

(c) $e^{\gamma t}\left[\left(\psi^{m+1}\right)(t)-\left(\phi^{m+1}\right)(t)\right]$ is increasing and $e^{-\gamma t}\left[\left(\psi^{m+1}\right)(t)-\left(\phi^{m+1}\right)(t)\right]$ is decreasing in $t\in\mathbb{R}$;

(d) $e^{\gamma t}\left[\left(\psi^{m}\right)(t)-\left(\psi^{m+1}\right)(t)\right]$ is increasing and $e^{-\gamma t}\left[\left(\psi^{m}\right)(t)-\left(\psi^{m+1}\right)(t)\right]$ is decreasing in $t\in\mathbb{R}$.

The remainder of the proof is similar to that of (ii) and is omitted. The proof of (iii) is complete. □

4. Existence of travelling wave fronts

In this section, we shall consider the existence of travelling wave fronts in reaction–diffusion systems with spatio-temporal delays of the form

$$\frac{\partial u(t,x)}{\partial t}=D\frac{\partial^2 u(t,x)}{\partial x^2}+f\left(u(t,x),(g_1*u)(t,x),\ldots,(g_m*u)(t,x)\right), \tag{4.1}$$

where $t\geqslant 0$, $x\in\mathbb{R}$, $D=\operatorname{diag}(d_1,\ldots,d_n)$, $d_i>0$, $i=1,\ldots n$, $n\in\mathbb{N}$; $u(t,x)=(u_1(t,x),\ldots,u_n(t,x))^T$, $f\in C\left(\mathbb{R}^{(m+1)n},\mathbb{R}^n\right)$, and

$$\left(g_j*u\right)(t,x)=\int_{-\infty}^{t}\int_{-\infty}^{+\infty}g_j(t-s,x-y)\,u(s,y)\,dy\,ds,$$

the kernel $g_j(t,x)$ is any integrable nonnegative function satisfying

$$g_j(t,-x)=g_j(t,x)\ \text{ and }\ \int_0^{+\infty}\int_{-\infty}^{+\infty}g_j(s,y)\,dy\,ds=1,\ j=1,\ldots m,\ m\in\mathbb{N}. \tag{4.2}$$

In the following, we shall apply our theory developed in Section 3 to establish the existence of travelling wave fronts for system (4.1) and give an iterative scheme to compute the travelling wave fronts.

Assume $u(t,x)=\varphi(x+ct)$ and replace $x+ct$ with t, then we can write (4.1) in the form

$$-D\varphi''(t)+c\varphi'(t)=f\left(\varphi(t),(g_1*\varphi)(t),\ldots,(g_m*\varphi)(t)\right),\quad t\in\mathbb{R}, \tag{4.3}$$

where

$$\left(g_j*\varphi\right)(t)=\int_0^{+\infty}\int_{-\infty}^{+\infty}g_j(s,y)\,\varphi(t-y-cs)\,dy\,ds,\qquad j=1,\ldots m.$$

By a travelling wave front with a wave speed $c>0$ to (4.1), we mean an increasing function $\varphi\in BC^2(\mathbb{R},\mathbb{R}^n)$ and a number $c>0$ which satisfy (4.3) and the following boundary condition:

$$\varphi(-\infty) = \mathbf{0} \quad \text{and} \quad \varphi(+\infty) = \mathbf{K} = (K_1, \ldots, K_n)^T \text{ with } \mathbf{0} <_s \mathbf{K}. \tag{4.4}$$

Now we make an assumption on the kernels $g_j(t, x)$, $j = 1, \ldots m$, and then list several propositions of convolutions $g_j * \varphi$, $j = 1, \ldots m$. The assumption is as follows:

(H$_0$) $\int_{-\infty}^{+\infty} g_j(t, x)\, dx$ is uniformly convergent for $t \in [0, a]$, $a > 0$, $j = 1, \ldots m$. In other word, if given $\varepsilon > 0$, then there exists $M > 0$ such that $\int_M^{+\infty} g_j(t, x)\, dx < \varepsilon$ for any $t \in [0, a]$.

Proposition 4.1. *Assume that $\varphi(t) \in BC(\mathbb{R}, \mathbb{R}^n)$ and satisfies*

$$\lim_{t\to-\infty} \varphi(t) = \alpha_0 \quad \textit{and} \quad \lim_{t\to+\infty} \varphi(t) = \beta_0.$$

If $g_j(t, x)$ satisfies (H$_0$), *then*

$$\lim_{t\to-\infty} \left(g_j * \varphi\right)(t) = \alpha_0 \quad \textit{and} \quad \lim_{t\to+\infty} \left(g_j * \varphi\right)(t) = \beta_0, \quad j = 1, \ldots, m.$$

Proof. Fix $j \in \{1, \ldots, m\}$. Let

$$h(t) = \int_{-\infty}^{+\infty} g_j(t, x)\, dx.$$

Then by (4.2), we have $\int_0^{\infty} h(t)\, dt = 1$. Given $\varepsilon > 0$, there exists $A > 0$ such that

$$\int_A^{\infty} h(t)\, dt < \varepsilon.$$

Note that (H$_0$) holds, then there exists $B > 0$ such that for any $t \in [0, A]$,

$$\int_B^{+\infty} g_j(t, x)\, dx < \frac{\varepsilon}{A}.$$

By $\lim_{t\to+\infty} \varphi(t) = \beta_0$, there exists $T > 0$ such that $\left|\varphi(t) - \beta_0\right| < \varepsilon$ for any $t > T$. Consequently, we have for any $t > T + B + cA$ that

$$\begin{aligned}
&\left|\int_0^A \int_{-B}^{+B} g_j(s, y)\left[\varphi(t - y - cs) - \beta_0\right] dy\, ds\right| \\
&\quad\leqslant \int_0^A \int_{-B}^{+B} g_j(s, y)\left|\varphi(t - y - cs) - \beta_0\right| dy\, ds \\
&\quad\leqslant \varepsilon \int_0^A \int_{-B}^{+B} g_j(s, y)\, dy\, ds \leqslant \varepsilon.
\end{aligned}$$

In terms of $g_j(t,-x) = g_j(t,x)$, it follows that

$$\begin{aligned}\left|\left(g_j * \varphi\right)(t) - \beta_0\right| &= \left|\int_0^{+\infty}\int_{-\infty}^{+\infty} g_j(s,y)\left[\varphi(t-y-cs) - \beta_0\right] dy\, ds\right| \\ &\leqslant \int_A^{+\infty}\int_{-\infty}^{+\infty} g_j(s,y)\left|\varphi(t-y-cs) - \beta_0\right| dy\, ds \\ &\quad + \int_0^A\int_{-B}^{B} g_j(s,y)\left|\varphi(t-y-cs) - \beta_0\right| dy\, ds \\ &\quad + \int_0^A\int_{B}^{+\infty} g_j(s,y)\left|\varphi(t-y-cs) - \beta_0\right| dy\, ds \\ &\quad + \int_0^A\int_{-\infty}^{-B} g_j(s,y)\left|\varphi(t-y-cs) - \beta_0\right| dy\, ds \\ &\leqslant (6\|\varphi\| + 1)\,\varepsilon,\end{aligned}$$

which implies that $\lim_{t\to+\infty}\left(g_j * \varphi\right)(t) = \beta_0$. Similarly, we can show the other. The proof is complete. □

Proposition 4.2. *Assume that* $\left(\varphi^k\right)(t)$ *and* $\varphi(t) \in BC(\mathbb{R}, \mathbb{R}^n)$ *are uniformly bounded for* $k \in \mathbb{N}$ *and satisfy* $\lim_{k\to+\infty}\left\|\varphi^k - \varphi\right\|_{C([a,b],\mathbb{R}^n)} = 0$ *for* $a, b \in \mathbb{R}$ *with* $a < b$. *If* $g_j(t,x)$ *satisfies* (H$_0$), *then*

$$\lim_{k\to+\infty}\left(g_j * \varphi^k\right)(t) = \left(g_j * \varphi\right)(t)$$

for each $t \in \mathbb{R}$, $j = 1, \ldots, m$.

Proof. Fix $j \in \{1, \ldots, m\}$ and $t \in \mathbb{R}$, then there exists $N > 0$ such that $t \in [-N, N]$. Let $M > 0$ satisfy $\left\|\varphi^k\right\| < M$ and $\|\varphi\| < M$. By a similar argument to that of Proposition 4.1, for given $\varepsilon > 0$, there exists $A > 0$ such that

$$\int_A^{\infty} h(t)\, dt < \varepsilon,$$

and there exists $B > 0$ such that for any $t \in [0, A]$,

$$\int_B^{+\infty} g_j(t,x)\, dx < \frac{\varepsilon}{A}.$$

From

$$\lim_{k\to+\infty}\left\|\varphi^k - \varphi\right\|_{C([a,b],\mathbb{R}^n)} = 0 \quad \text{for any } a, b \in \mathbb{R} \text{ with } a < b,$$

we know that there exists $K \in \mathbb{N}$ such that

$$\left\| \varphi^k - \varphi \right\|_{C([a,b],\mathbb{R}^n)} < \varepsilon \quad \text{for any } k > K,$$

where $-a = b = N + B + cA$. Consequently, for any $k > K$, we have

$$\begin{aligned}
&\left| \int_0^A \int_{-B}^{+B} g_j(s, y) \left[\left(\varphi^k \right)(t - y - cs) - \varphi(t - y - cs) \right] dy\, ds \right| \\
&\quad \leqslant \int_0^A \int_{-B}^{+B} g_j(s, y) \left| \left(\varphi^k \right)(t - y - cs) - \varphi(t - y - cs) \right| dy\, ds \leqslant \varepsilon.
\end{aligned}$$

In view of $g_j(s, -x) = g_j(s, x)$, we know that for any $k > K$,

$$\begin{aligned}
&\left| \left(g_j * \varphi^k \right)(t) - \left(g_j * \varphi \right)(t) \right| \\
&\quad = \left| \int_0^{+\infty} \int_{-\infty}^{+\infty} g_j(s, y) \left[\left(\varphi^k \right)(t - y - cs) - \varphi(t - y - cs) \right] dy\, ds \right| \\
&\quad \leqslant \int_A^{+\infty} \int_{-\infty}^{+\infty} g_j(s, y) \left| \left(\varphi^k \right)(t - y - cs) - \varphi(t - y - cs) \right| dy\, ds \\
&\qquad + \int_0^A \int_{-B}^{B} g_j(s, y) \left| \left(\varphi^k \right)(t - y - cs) - \varphi(t - y - cs) \right| dy\, ds \\
&\qquad + \int_0^A \int_{B}^{+\infty} g_j(s, y) \left| \left(\varphi^k \right)(t - y - cs) - \varphi(t - y - cs) \right| dy\, ds \\
&\qquad + \int_0^A \int_{-\infty}^{-B} g_j(s, y) \left| \left(\varphi^k \right)(t - y - cs) - \varphi(t - y - cs) \right| dy\, ds \\
&\quad \leqslant (6M + 1)\,\varepsilon,
\end{aligned}$$

which implies that $\lim_{k \to +\infty} \left(g_j * \varphi^k \right)(t) = \left(g_j * \varphi \right)(t)$. Noting that $t \in \mathbb{R}$ and $j \in \{1, \ldots, m\}$ are arbitrary, so the conclusion follows. The proof is complete. □

Proposition 4.3. *If* $\varphi(t) \in BC(\mathbb{R}, \mathbb{R}^n)$ *and* (H_0) *holds, then* $\left(g_j * \varphi \right)(t) \in BC(\mathbb{R}, \mathbb{R}^n)$, $j \in 1, \ldots, m$.

Proof. Fix $j \in \{1, \ldots, m\}$ and $t \in \mathbb{R}$. Then there exist $N > 0$ such that $t \in [-N, N]$. Let $M > 0$ satisfy $\|\varphi\| < M$. As above, for given $\varepsilon > 0$, there exists $A > 0$ such that

$$\int_A^{\infty} h(t)\, dt < \varepsilon,$$

and there exists $B > 0$ such that for any $s \in [0, A]$,

$$\int_B^{+\infty} g_j(s, x)\, dx < \frac{\varepsilon}{A}.$$

Since $\varphi(s)$ is uniformly continuous in $s \in [-(N + B + cA + 1), N + B + cA + 1]$, there exists δ with $0 < \delta < 1$ such that $|\varphi(s + \theta) - \varphi(s)| < \varepsilon$ for any $\theta \in (-\delta, \delta)$ and any $s \in [-(N + B + cA), N + B + cA]$. Following this, we have for any $\theta \in (-\delta, \delta)$ that

$$\begin{aligned}
&\left|\int_0^A \int_{-B}^{+B} g_j(s, y)\left[\varphi(t + \theta - y - cs) - \varphi(t - y - cs)\right] dy\, ds\right| \\
&\quad \leqslant \int_0^A \int_{-B}^{+B} g_j(s, y)\, |\varphi(t + \theta - y - cs) - \varphi(t - y - cs)|\, dy\, ds \leqslant \varepsilon.
\end{aligned}$$

Now, for any $\theta \in (-\delta, \delta)$,

$$\begin{aligned}
&\left|\left(g_j * \varphi\right)(t + \theta) - \left(g_j * \varphi\right)(t)\right| \\
&\quad = \left|\int_0^{+\infty} \int_{-\infty}^{+\infty} g_j(s, y)\left[\varphi(t + \theta - y - cs) - \varphi(t - y - cs)\right] dy\, ds\right| \\
&\quad \leqslant (6M + 1)\,\varepsilon,
\end{aligned}$$

which implies that

$$\lim_{\theta \to 0} \left(g_j * \varphi\right)(t + \theta) = \left(g_j * \varphi\right)(t).$$

Noting that $t \in \mathbb{R}$ and $j \in \{1, \ldots, m\}$ are arbitrary, we complete the proof. □

For the sake of convenience, we list some kernel functions which have been frequently used in the references.

(A) If $g_j(t, x) = \delta(t)\,\delta(x)$, then $\left(g_j * u\right)(t, x) = u(t, x)$, which is a local version without temporal delay, $j \in \{1, \ldots, m\}$, where $\delta(\cdot)$ is the Dirac delta function.

(B) If $g_j(t,x) = \delta(t)\, p_j(x)$, then

$$\left(g_j * u\right)(t,x) = \int_{-\infty}^{+\infty} p_j(x-y)\, u(t,y)\, dy,$$

which is a nonlocal version without temporal delay, $j \in \{1, \ldots, m\}$.

(C) If $g_j(t,x) = \delta\left(t-\tau_j\right)\delta(x)$, then

$$\left(g_j * u\right)(t,x) = u\left(t-\tau_j, x\right),$$

which is a local version with a discrete temporal delay, $j \in \{1, \ldots, m\}$.

(D) If $g_j(t,x) = \delta\left(t-\tau_j\right) p_j(x)$, then

$$\left(g_j * u\right)(t,x) = \int_{-\infty}^{+\infty} p_j(x-y)\, u\left(t-\tau_j, y\right) dy,$$

which is a nonlocal version with a discrete temporal delay, $j \in \{1, \ldots, m\}$.

(E) If $g_j(t,x) = q_j(t)\,\delta(x)$, then

$$\left(g_j * u\right),(t,x) = \int_{-\infty}^{t} q_j(t-s)\, u(s,x)\, dx,$$

which is a local version with distributed temporal delay, $j \in \{1, \ldots, m\}$.

Let

$$(F\varphi)(t) = \Phi\left(\varphi_t\right) = f\left(\varphi(t), (g_1 * \varphi)(t), \ldots, (g_m * \varphi)(t)\right), \quad t \in \mathbb{R}. \tag{4.5}$$

Following the argument in Section 3, (4.3) can be changed into the following integral equation

$$\varphi(t) = \int_{-\infty}^{+\infty} k(t,s)\left((F\varphi)(s) + \gamma\varphi(s)\right) ds \equiv (A\varphi)(t), \quad t \in \mathbb{R},$$

where k is the same as that in Section 3. Here we list some conditions, which will be used in our results.

(H_1) There exists a matrix $\gamma = \mathrm{diag}\left(\gamma_1, \ldots, \gamma_n\right)$ with $\gamma_i > 0, i = 1, \ldots n$ such that

$$\begin{aligned} &f\left(\varphi_2(t), \left(g_1 * \varphi_2\right)(t), \ldots, \left(g_m * \varphi_2\right)(t)\right) + \gamma\varphi_2(t) \\ &\quad \geqslant f\left(\varphi_1(t), \left(g_1 * \varphi_1\right)(t), \ldots, \left(g_m * \varphi_1\right)(t)\right) + \gamma\varphi_1(t), \end{aligned}$$

where $\varphi_1, \varphi_2 \in C(\mathbb{R}, \mathbb{R}^n)$ satisfy $\mathbf{0} \leqslant \varphi_1(t) \leqslant \varphi_2(t) \leqslant \mathbf{K}$ in $t \in \mathbb{R}$.

(H_1^*) There exists a matrix $\gamma = \operatorname{diag}(\gamma_1, \ldots, \gamma_n)$ with $\gamma_i > 0, i = 1, \ldots n$ such that

$$f\left(\varphi_2(t), (g_1 * \varphi_2)(t), \ldots, (g_m * \varphi_2)(t)\right) + \gamma\varphi_2(t)$$
$$\geqslant f\left(\varphi_1(t), (g_1 * \varphi_1)(t), \ldots, (g_m * \varphi_1)(t)\right) + \gamma\varphi_1(t),$$

where $\varphi_1, \varphi_2 \in C(\mathbb{R}, \mathbb{R}^n)$ satisfy $\begin{cases} \text{(i) } \mathbf{0} \leqslant \varphi_1(t) \leqslant \varphi_2(t) \leqslant \mathbf{K} & \text{in } t \in \mathbb{R}; \\ \text{(ii) } e^{\gamma t}\left[\varphi_2(t) - \varphi_1(t)\right] & \text{is increasing in } t \in \mathbb{R}. \end{cases}$

(H_1^{**}) There exists a matrix $\gamma = \operatorname{diag}(\gamma_1, \ldots, \gamma_n)$ with $\gamma_i > 0, i = 1, \ldots n$ such that

$$f\left(\varphi_2(t), (g_1 * \varphi_2)(t), \ldots, (g_m * \varphi_2)(t)\right) + \gamma\varphi_2(t)$$
$$\geqslant f\left(\varphi_1(t), (g_1 * \varphi_1)(t), \ldots, (g_m * \varphi_1)(t)\right) + \gamma\varphi_1(t),$$

where $\varphi_1, \varphi_2 \in C(\mathbb{R}, \mathbb{R}^n)$ satisfy $\begin{cases} \text{(i) } \mathbf{0} \leqslant \varphi_1(t) \leqslant \varphi_2(t) \leqslant \mathbf{K} \text{ in } t \in \mathbb{R}; \\ \text{(ii) } e^{\gamma t}\left[\varphi_2(t) - \varphi_1(t)\right] \text{ is increasing and} \\ e^{-\gamma t}\left[\varphi_2(t) - \varphi_1(t)\right] \text{ is decreasing in } t \in \mathbb{R}. \end{cases}$

(H_2) $f(\mu, \ldots, \mu) \neq \mathbf{0}$ for $\mathbf{0} < \mu < \mathbf{K}$;
(H_3) $f(\mu, \ldots, \mu) = \mathbf{0}$ when $\mu = \mathbf{0}$ or $\mathbf{K}$.

Lemma 4.4. *If* (H_0) *holds and* $\varphi(t) \in BC(\mathbb{R}, \mathbb{R}^n)$, *then* $(F\varphi)(t) \in BC(\mathbb{R}, \mathbb{R}^n)$, *where* F *is defined by* (4.5).

Lemma 4.5. *Assume that* (H_0) *holds.*

(i) *If f satisfies* (H_1), *then* $F : BC[\mathbf{0}, \mathbf{K}] \to BC(\mathbb{R}, \mathbb{R}^n)$ *is γ-increasing, M-continuous and bounded.*
(ii) *If f satisfies* (H_1^*), *then* $F : BC[\mathbf{0}, \mathbf{K}] \to BC(\mathbb{R}, \mathbb{R}^n)$ *is γ^*-increasing, M-continuous and bounded.*
(iii) *If f satisfies* (H_1^{**}), *then* $F : BC[\mathbf{0}, \mathbf{K}] \to BC(\mathbb{R}, \mathbb{R}^n)$ *is γ^{**}-increasing, M-continuous and bounded.*

Lemma 4.6. *Assume that* (H_0) *holds and* $\varphi \in BC^2(\mathbb{R}, \mathbb{R}^n)$ *is a solution of* (4.3) *satisfying* $\lim_{t\to-\infty} \varphi(t) = \alpha_0$ *and* $\lim_{t\to+\infty} \varphi(t) = \beta_0$. *Then* $f(\alpha_0, \ldots, \alpha_0) = f\left(\beta_0, \ldots, \beta_0\right) = \mathbf{0}$, *that is,* $(F\widehat{\alpha_0})(t) = \left(F\widehat{\beta_0}\right)(t) = \mathbf{0}$ *for any* $t \in \mathbb{R}$.

We remark that Lemmas 4.4–4.6 are obvious. In fact, Lemmas 4.4 and 4.5 follow from the continuity of f and Propositions 4.2 and 4.3. By Proposition 4.1 and a similar argument to that of [33, Proposition 2.1], it is easy to see that Lemma 4.6 is also true.

Now we give definitions of the upper and lower solutions of (4.3) (see [33, Definition 3.2]).

Definition 4.7. A continuous function $\varphi : \mathbb{R} \to \mathbb{R}^n$ is called *an upper solution* of (4.3) if φ' and φ'' exist almost everywhere and are essentially bounded on $\mathbb{R}$, and φ

satisfies

$$-D\varphi''(t) + c\varphi'(t) \geqslant f\left(\varphi(t), (g_1 * \varphi)(t), \ldots, (g_m * \varphi)(t)\right), \qquad \text{a.e. on } \mathbb{R}. \tag{4.6}$$

A lower solution of (4.3) is defined in a similar way by reversing the inequality in (4.6).

Obviously, if ψ and ϕ are upper and lower solutions of (4.3), respectively, then ψ and ϕ satisfy (C_0).

Now we are in a position to state our main results in this section. Let

$$\Gamma = \left\{ \varphi \in Y : \begin{array}{l} \text{(i)}\varphi \text{ is increasing in } \mathbb{R}; \\ \text{(ii) } \mathbf{0} \leqslant \lim_{t\to-\infty} \varphi(t) < \mathbf{K} \text{ and } \lim_{t\to+\infty} \varphi(t) = \mathbf{K}. \end{array} \right\},$$

$$\Gamma^* = \left\{ \varphi \in Y : \begin{array}{l} \text{(i) } \varphi \text{ is increasing in } \mathbb{R}; \\ \text{(ii) } \mathbf{0} \leqslant \lim_{t\to-\infty} \varphi(t) < \mathbf{K} \text{ and } \lim_{t\to+\infty} \varphi(t) = \mathbf{K}; \\ \text{(iii) } e^{\gamma t}\left[\varphi(t+s) - \varphi(t)\right] \text{ is increasing in } t \in \mathbb{R} \\ \quad \text{for every } s > 0. \end{array} \right\},$$

and

$$\Gamma^{**} = \left\{ \varphi \in Y : \begin{array}{l} \text{(i) } \varphi \text{ is increasing in } \mathbb{R}; \\ \text{(ii) } \mathbf{0} \leqslant \lim_{t\to-\infty} \varphi(t) < \mathbf{K} \text{ and } \lim_{t\to+\infty} \varphi(t) = \mathbf{K}; \\ \text{(iii) } e^{\gamma t}\left[\varphi(t+s) - \varphi(t)\right] \text{ is increasing in } t \in \mathbb{R} \text{ and} \\ \quad e^{-\gamma t}\left[\varphi(t+s) - \varphi(t)\right] \text{ is decreasing in } t \in \mathbb{R} \text{ for} \\ \quad \text{every } s > 0. \end{array} \right\}.$$

Theorem 4.8. *Assume that* (H_2), (H_3) *and* (H_0) *hold. Assume further that* ϕ *and* ψ, *where* $\phi \in BC[\mathbf{0}, \mathbf{K}] \cap Y$ *with* $\phi \neq \widehat{\mathbf{0}}$, $\lim_{t\to-\infty} \phi(t) = \mathbf{0}$ *and* $\phi \leqslant \psi$, *are lower and upper solutions of* (4.3), *respectively*.

(i) *If* (H_1) *holds and* $\psi \in \Gamma$, *then* (4.1) *has a travelling wave front* ψ^* *such that* (4.4) *holds and for* $a, b \in \mathbb{R}$ *with* $a < b$,

$$\|\psi^m - \psi^*\|_{C([a,b],\mathbb{R}^n)} \to 0, \tag{4.7}$$

where

$$-D\left(\psi^m\right)'' + c\left(\psi^m\right)' + \gamma\psi^m = F\psi^{m-1} + \gamma\psi^{m-1}\ (m \in \mathbb{N}) \tag{4.8}$$

and

$$\phi \leqslant \psi^* \leqslant \cdots \leqslant \psi^m \leqslant \cdots \leqslant \psi^1 \leqslant \psi^0 = \psi. \tag{4.9}$$

(ii) *If* (H_1^*) *holds,* $\psi \in \Gamma^*$ *and* $e^{\gamma t}\left[\psi(t)-\phi(t)\right]$ *is increasing in* $t \in \mathbb{R}$*, then for* $c > 1 - \min\left\{\gamma_i d_i; i = 1, \ldots, n\right\}$, (4.1) *has a travelling wave front* ψ^* *such that* (4.4) *holds and for* $a, b \in \mathbb{R}$ *with* $a < b$, (4.7), (4.8) *and* (4.9) *hold.*

(iii) *If* (H_1^{**}) *holds,* $\psi \in \Gamma^{**}$, $e^{\gamma t}[\psi(t)-\phi(t)]$ *is increasing in* $t \in \mathbb{R}$ *and* $e^{-\gamma t}[\psi(t)-\phi(t)]$ *is decreasing in* $t \in \mathbb{R}$*, where* $\min\left\{\gamma_i d_i; i = 1, \ldots, n\right\} - 1 > 0$*, then for* $0 < c < \min\left\{\gamma_i d_i; i = 1, \ldots, n\right\} - 1$, (4.1) *has a travelling wave front* ψ^* *such that* (4.4) *holds and for* $a, b \in \mathbb{R}$ *with* $a < b$, (4.7), (4.8) *and* (4.9) *hold.*

In particular, if $\lim_{t\to-\infty}\psi(t) = \mathbf{0}$*, then* $\|\psi^m - \psi^*\| \to 0$.

Proof. (i) Let $F : BC[\mathbf{0}, \mathbf{K}] \to BC(\mathbb{R}, \mathbb{R}^n)$ be defined by (4.5). It follows from Lemma 4.5 that $F : BC[\mathbf{0}, \mathbf{K}] \to BC(\mathbb{R}, \mathbb{R}^n)$ is bounded, M-continuous and γ-increasing. By the definitions of upper and lower solutions, we have

$$-D\phi'' + c\phi' \leqslant F\phi \quad \text{and} \quad -D\psi'' + c\psi' \geqslant F\psi.$$

Then Theorem 3.9(i) implies that there exists $\psi^* \in BC[\mathbf{0}, \mathbf{K}]$ such that (4.7), (4.8) and (4.9) hold. Since $F\psi^* \in BC(\mathbb{R}, \mathbb{R}^n)$, it follows from Theorem 3.1 that

$$\psi^* = A\psi^* \in BC^2(\mathbb{R}, \mathbb{R}^n),$$

that is,

$$-D\left(\psi^*\right)'' + c\left(\psi^*\right)' = F\psi^*.$$

Since ψ is increasing in $\mathbb{R}$, ψ^* is increasing. Let $\lim_{t\to-\infty}\psi^*(t) = \alpha_0$ and $\lim_{t\to+\infty}\psi^*(t) = \beta_0$, then $f(\alpha_0, \ldots, \alpha_0) = f\left(\beta_0, \ldots, \beta_0\right) = \mathbf{0}$ follows from Lemma 4.6. So the conditions (H_2) and (H_3) mean that $\alpha_0, \beta_0 \in \{\mathbf{0}, \mathbf{K}\}$. Since $\phi \neq \widehat{\mathbf{0}}$ and $\phi \leqslant \psi^*$, we have $\beta_0 > \mathbf{0}$ and thus, $\beta_0 = \mathbf{K}$. Since $\psi(t) \in \Gamma$ and $\psi^* \leqslant \psi$, $\lim_{t\to-\infty}\psi^*(t) \leqslant \lim_{t\to-\infty}\psi(t) < \mathbf{K}$. Hence, $\alpha_0 = \mathbf{0}$. Thus, we complete the proof of (i).

(ii) It follows from Lemma 4.5(ii) that $F : BC[\mathbf{0}, \mathbf{K}] \to BC(\mathbb{R}, \mathbb{R}^n)$ is bounded, M-continuous and γ^*-increasing. By Theorem 3.9(ii), Lemmas 3.6(ii) and 4.5(ii), and employing an argument similar to that of (i), we can complete the proof of (ii).

(iii) It follows from Lemma 4.5(iii) that $F : BC[\mathbf{0}, \mathbf{K}] \to BC(\mathbb{R}, \mathbb{R}^n)$ is bounded, M-continuous and γ^{**}-increasing. Noting that Theorem 3.9(iii), Lemmas 3.6(iii) and 4.5(iii), the conclusion can be proved following a similar argument to that of (i).

If $\lim_{t\to-\infty}\psi(t) = \mathbf{0}$ and $\lim_{t\to\infty}\psi(t) = \mathbf{K}$, then for given $\varepsilon > 0$, there exists $M > 0$ such that

$$\max_{1\leqslant i\leqslant n}\sup_{t\leqslant -M}\left|\psi_i(t)\right| < \frac{\varepsilon}{2}, \quad \max_{1\leqslant i\leqslant n}\sup_{t\geqslant M}\left|\psi_i(t) - K_i\right| < \frac{\varepsilon}{2},$$

$$\max_{1\leqslant i\leqslant n}\sup_{t\leqslant -M}\left|\psi_i^*(t)\right| < \frac{\varepsilon}{2}, \quad \max_{1\leqslant i\leqslant n}\sup_{t\geqslant M}\left|\psi_i^*(t) - K_i\right| < \frac{\varepsilon}{2},$$

so for any $m \in \mathbb{N}$,

$$\max_{1 \leqslant i \leqslant n} \sup_{t \leqslant -M} \left|\psi_i^m(t)\right| < \frac{\varepsilon}{2}, \quad \max_{1 \leqslant i \leqslant n} \sup_{t \geqslant M} \left|\psi_i^m(t) - K_i\right| < \frac{\varepsilon}{2}.$$

Consequently,

$$\max_{1 \leqslant i \leqslant n} \sup_{|t| \geqslant M} \left|\psi_i^m(t) - \psi_i^*(t)\right| < \varepsilon.$$

By (4.7), there exists $N \in \mathbb{N}$ such that

$$\|\psi^m - \psi^*\|_{C([-M,M],\mathbb{R}^n)} < \varepsilon \quad \text{for all } m \geqslant N.$$

Thus, $\|\psi^m - \psi^*\| < \varepsilon$ for all $m \geqslant N$. Therefore, $\|\psi^m - \psi^*\| \to 0$.

The proof is complete. $\square$

Corollary 4.9. *Assume that* (H3) *and* (H0) *hold, and* $f(\mu, \ldots, \mu) \neq \mathbf{0}$ *for* $\mathbf{0} < \delta \leqslant \mu < \mathbf{K}$, *where* $\delta \in \mathbb{R}^n$. *Also assume that* ϕ *and* ψ, *where* $\phi \in BC[\mathbf{0}, \mathbf{K}] \cap Y$ *with* $\sup_{t \in \mathbb{R}} \phi(t) \geqslant \delta$, $\phi \leqslant \psi$ *and* $\lim_{t \to -\infty} \psi(t) = \mathbf{0}$, *are lower and upper solutions of* (4.3), *respectively. Then* (i)–(iii) *of Theorem* 4.8 *hold.*

5. Applications

In [5], Britton proposed a model for a single biological population of the form

$$\frac{\partial u(t,x)}{\partial t} = \triangle u(t,x) + u(t,x)\left[1 + au(t,x) - (1+a)(g \oplus u)(t,x)\right],$$

where $a > 0$, g is a given function and $g \oplus u$ represents a convolution in the spatial variable. In this equation, the term au with $a > 0$ represents an advantage in local aggregation, the term $(1+a)\, g \oplus u$ represents a disadvantage in global population levels being too high because of the resultant depletion of resources. Under recognizing that animals take time to move, he proposed a spatio-temporal average model weighted toward the current time and position of the form

$$\frac{\partial u(t,x)}{\partial t} = \triangle u(t,x) + u(t,x)\left[1 + au(t,x) - (1+a)(g * u)(t,x)\right], \tag{5.1}$$

where

$$(g * u)(t,x) = \int_{-\infty}^{t} \int_{\Omega} g(t-\xi, x-y)\, u(\xi, y)\, dy\, d\xi,$$
$$g * 1 = 1 \quad \text{and} \quad g(t,x) = g(t,-x), \quad x \in \Omega \subseteq \mathbb{R}.$$

In [6], Britton considered three kinds of bifurcations from the uniform steady-state solution $u \equiv 1$, that is, (i) steady spatially periodic structures, (ii) periodic standing wave solutions, and (iii) periodic travelling wave solutions.

In the following, we consider the existence of travelling wave fronts of (5.1), where $\Omega = \mathbb{R}$. This system has two equilibria:

$$u \equiv 0 \quad \text{and} \quad u \equiv 1.$$

Obviously, the travelling wave equation of (5.1) corresponding to (4.3) is

$$-\varphi''(t) + c\varphi'(t) = \varphi(t)\left[1 + a\varphi(t) - (1+a)(g * \varphi)(t)\right], \qquad t \in \mathbb{R}, \tag{5.2}$$

where

$$(g * \varphi)(t) = \int_0^{+\infty} \int_{-\infty}^{+\infty} g(\xi, y)\, \varphi(t - c\xi - y)\, dy\ d\xi.$$

Let $K = 1$. Then the travelling wave front φ of (5.1) satisfies the asymptotic boundary condition

$$\lim_{t \to -\infty} \varphi(t) = 0 \quad \text{and} \quad \lim_{t \to +\infty} \varphi(t) = 1. \tag{5.3}$$

Let

$$\lambda_1 = \frac{c - \sqrt{c^2 - 4(1+a)}}{2} \quad \text{and} \quad \lambda_2 = \frac{c + \sqrt{c^2 - 4(1+a)}}{2}$$

be two real positive roots of the equation

$$\lambda^2 - c\lambda + 1 + a = 0,$$

where $c \geqslant 2\sqrt{1+a}$. Let

$$\lambda_3 = \frac{c + \sqrt{c^2 + 4a}}{2}.$$

Then λ_3 satisfies $\lambda^2 - c\lambda - a = 0$. Now we let

$$\psi(t) = \frac{1}{1 + \alpha e^{-\lambda_1 t}} \quad \text{and} \quad \phi(t) = \min\left\{\varepsilon e^{\lambda_3 t}, \varepsilon\right\}, \ \ 0 < \varepsilon < \frac{1}{1+\alpha}, \tag{5.4}$$

where α is a positive constant.

Lemma 5.1. (i) $\psi(t) = 1/\left(1+\alpha e^{-\lambda_1 t}\right)$ *is increasing in* $t \in \mathbb{R}$ *and satisfies the asymptotic boundary condition* (5.3);

(ii) $\psi(t) \geqslant \phi(t)$ *for* $t \in \mathbb{R}$.

Proof. We only show (ii). For $t \geqslant 0$, $\psi(0) = 1/(1+\alpha)$, $\phi(t) = \varepsilon$. Since $\psi(t) = 1/\left(1+\alpha e^{-\lambda_1 t}\right)$ is increasing, then $\psi(t) \geqslant 1/(1+\alpha) > \varepsilon = \phi(t)$.

For $t < 0$, $\psi(t) = 1/\left(1+\alpha e^{-\lambda_1 t}\right)$, $\phi(t) = \varepsilon e^{\lambda_3 t}$. Since $\lambda_3 > \lambda_1 > 0$, then $e^{\lambda_3 t} < 1$ and $e^{(\lambda_3-\lambda_1)t} < 1$. Consequently,

$$\begin{aligned}\psi(t) - \phi(t) &= \frac{1}{1+\alpha e^{-\lambda_1 t}} - \varepsilon e^{\lambda_3 t} = \frac{1 - \varepsilon e^{\lambda_3 t} - \varepsilon\alpha e^{(\lambda_3-\lambda_1)t}}{1+\alpha e^{-\lambda_1 t}} \\ &\geqslant \frac{1-\varepsilon-\varepsilon\alpha}{1+\alpha e^{-\lambda_1 t}} > 0.\end{aligned}$$

The proof is complete. □

Lemma 5.2. *Assume* $\gamma > \lambda_1$. *For sufficiently small* α *and* ε, *the following statements hold*:

(i) $e^{\gamma t}\left[\psi(t) - \phi(t)\right]$ *is increasing and* $e^{-\gamma t}\left[\psi(t) - \phi(t)\right]$ *is decreasing in* $t \in \mathbb{R}$;

(ii) $e^{\gamma t}\left[\psi(t+s) - \psi(t)\right]$ *is increasing and* $e^{-\gamma t}\left[\psi(t+s) - \psi(t)\right]$ *is decreasing in* $t \in \mathbb{R}$ *for every* $s > 0$.

Proof. (i) First, we show that $e^{\gamma t}\left[\psi(t) - \phi(t)\right]$ is increasing in $t \in \mathbb{R}$.

For $t > 0$, $\phi(t) = \varepsilon$ and $e^{-\lambda_1 t} \leqslant 1$, by a direct calculation, we have

$$\begin{aligned}&\frac{d}{dt}\left\{e^{\gamma t}\left[\frac{1}{1+\alpha e^{-\lambda_1 t}} - \varepsilon\right]\right\} \\ &\quad = \frac{e^{\gamma t}\left[(\gamma - \gamma\varepsilon) + \alpha\left(-2\gamma\varepsilon + \gamma - \gamma\varepsilon\alpha e^{-\lambda_1 t} + \lambda_1\right)e^{-\lambda_1 t}\right]}{\left(1+\alpha e^{-\lambda_1 t}\right)^2} \\ &\quad \geqslant \frac{e^{\gamma t}\left[(\gamma - \gamma\varepsilon) + \alpha\left(-2\gamma\varepsilon + \gamma - \gamma\varepsilon\alpha + \lambda_1\right)e^{-\lambda_1 t}\right]}{\left(1+\alpha e^{-\lambda_1 t}\right)^2}.\end{aligned}$$

For $t < 0$, $\phi(t) = \varepsilon e^{\lambda_3 t}$, $e^{\lambda_3 t} \leqslant 1$ and $e^{(\lambda_3-\lambda_1)t} \leqslant 1$, a direct calculation yields

$$\frac{d}{dt}\left\{e^{\gamma t}\left[\frac{1}{1+\alpha e^{-\lambda_1 t}} - \varepsilon e^{\lambda_3 t}\right]\right\}$$

$$
\begin{aligned}
&= \frac{e^{\gamma t}\left[\gamma-(\gamma+\lambda_3)\,\varepsilon e^{\lambda_3 t}\right]}{\left(1+\alpha e^{-\lambda_1 t}\right)^2} \\
&\quad + \frac{\alpha e^{(\gamma-\lambda_1)t}\left[-2\gamma\varepsilon e^{\lambda_3 t}+\gamma-\gamma\varepsilon\alpha e^{(\lambda_3-\lambda_1)t}-2\lambda_3\varepsilon e^{\lambda_3 t}-\lambda_3\varepsilon\alpha e^{(\lambda_3-\lambda_1)t}+\lambda_1\right]}{\left(1+\alpha e^{-\lambda_1 t}\right)^2} \\
&\geqslant \frac{e^{\gamma t}\left[(\gamma-\gamma\varepsilon-\lambda_3\varepsilon)+\alpha\,(-2\gamma\varepsilon+\gamma-\gamma\varepsilon\alpha-2\lambda_3\varepsilon-\lambda_3\varepsilon\alpha+\lambda_1)\,e^{-\lambda_1 t}\right]}{\left(1+\alpha e^{-\lambda_1 t}\right)^2}.
\end{aligned}
$$

If let $\varepsilon \to 0$ and $\alpha \to 0$, then we have

$$\frac{d}{dt}\left\{e^{\gamma t}\left[\psi(t)-\phi(t)\right]\right\} \geqslant 0.$$

Second, we show that $e^{-\gamma t}\left[\psi(t)-\phi(t)\right]$ is decreasing in $t \in \mathbb{R}$.
For $t > 0$, $\phi(t) = \varepsilon$ and $e^{-\lambda_1 t} \leqslant 1$, by a direct calculation we have

$$
\begin{aligned}
&\frac{d}{dt}\left\{e^{-\gamma t}\left[\frac{1}{1+\alpha e^{-\lambda_1 t}}-\varepsilon\right]\right\} \\
&\quad = \frac{e^{-\gamma t}\left[(-\gamma+\gamma\varepsilon)+\alpha\left(2\gamma\varepsilon-\gamma+\gamma\varepsilon\alpha e^{-\lambda_1 t}+\lambda_1\right)e^{-\lambda_1 t}\right]}{\left(1+\alpha e^{-\lambda_1 t}\right)^2} \\
&\quad \leqslant \frac{e^{\gamma t}\left[(-\gamma+\gamma\varepsilon)+\alpha\,(2\gamma\varepsilon-\gamma+\gamma\varepsilon\alpha+\lambda_1)\,e^{-\lambda_1 t}\right]}{\left(1+\alpha e^{-\lambda_1 t}\right)^2}.
\end{aligned}
$$

For $t < 0$, $\phi(t) = \varepsilon e^{\lambda_3 t}$, $e^{\lambda_3 t} \leqslant 1$ and $e^{(\lambda_3-\lambda_1)t} \leqslant 1$, a direct calculation implies that

$$
\begin{aligned}
&\frac{d}{dt}\left\{e^{-\gamma t}\left[\frac{1}{1+\alpha e^{-\lambda_1 t}}-\varepsilon e^{\lambda_3 t}\right]\right\} \\
&= \frac{e^{\gamma t}\left[-\gamma+(\gamma-\lambda_3)\,\varepsilon e^{\lambda_3 t}\right]}{\left(1+\alpha e^{-\lambda_1 t}\right)^2} \\
&\quad + \frac{\alpha e^{-(\gamma+\lambda_1)t}\left(2\gamma\varepsilon e^{\lambda_3 t}-\gamma+\gamma\varepsilon\alpha e^{(\lambda_3-\lambda_1)t}-2\lambda_3\varepsilon e^{\lambda_3 t}-\lambda_3\varepsilon\alpha e^{(\lambda_3-\lambda_1)t}+\lambda_1\right)}{\left(1+\alpha e^{-\lambda_1 t}\right)^2} \\
&\leqslant \frac{e^{\gamma t}\left[(-\gamma+\gamma\varepsilon)+\alpha\,(2\gamma\varepsilon-\gamma+\gamma\varepsilon\alpha+\lambda_1)\,e^{-\lambda_1 t}\right]}{\left(1+\alpha e^{-\lambda_1 t}\right)^2}.
\end{aligned}
$$

In view of $\gamma > \lambda_1$, $\varepsilon \to 0$ and $\alpha \to 0$, then we have

$$\frac{d}{dt}\left\{e^{\gamma t}\left[\psi(t) - \phi(t)\right]\right\} \leqslant 0.$$

(ii) First, we show that $e^{\gamma t}\left[\psi(t+s) - \psi(t)\right]$ is increasing in $t \in \mathbb{R}$ for every $s > 0$. Fix $s > 0$. Since $e^{-\lambda_1 s} < 1$, by a direct calculation, it follows that

$$\begin{aligned}
&\frac{d}{dt}\left\{e^{\gamma t}\left[\psi(t+s) - \psi(t)\right]\right\}\\
&\quad= \frac{d}{dt}\left\{e^{\gamma t}\left[\frac{1}{1+\alpha e^{-\lambda_1(t+s)}} - \frac{1}{1+\alpha e^{-\lambda_1 t}}\right]\right\}\\
&\quad= \frac{\alpha(\gamma-\lambda_1)e^{(\gamma-\lambda_1)t}\left(1-e^{-\lambda_1 s}\right)\left(1+\alpha e^{-\lambda_1 s}e^{-\lambda_1 t}\right)\left(1+\alpha e^{-\lambda_1 t}\right)}{\left(1+\alpha e^{-\lambda_1(t+s)}\right)^2\left(1+\alpha e^{-\lambda_1 t}\right)^2}\\
&\qquad+\frac{\alpha^2\lambda_1 e^{-\lambda_1 t}e^{(\gamma-\lambda_1)t}\left(1-e^{-\lambda_1 s}\right)\left(1+e^{-\lambda_1 s}+2\alpha e^{-\lambda_1 s}e^{-\lambda_1 t}\right)}{\left(1+\alpha e^{-\lambda_1(t+s)}\right)^2\left(1+\alpha e^{-\lambda_1 t}\right)^2}\\
&\quad= \alpha e^{(\gamma-\lambda_1)t}\left(1-e^{-\lambda_1 s}\right)\\
&\qquad\times\frac{\left[(\gamma-\lambda_1)+\gamma\alpha e^{-\lambda_1 t}\left(1+e^{-\lambda_1 s}\right)+(\gamma+\lambda_1)\alpha^2 e^{-\lambda_1 s}e^{-2\lambda_1 t}\right]}{\left(1+\alpha e^{-\lambda_1(t+s)}\right)^2\left(1+\alpha e^{-\lambda_1 t}\right)^2}.
\end{aligned}$$

Since $\gamma > \lambda_1$, we can see that

$$\frac{d}{dt}\left\{e^{\gamma t}\left[\psi(t+s) - \psi(t)\right]\right\} > 0.$$

Second, we show that $e^{-\gamma t}\left[\psi(t+s) - \psi(t)\right]$ is decreasing in $t \in \mathbb{R}$ for every $s > 0$. Fix $s > 0$. Since $e^{-\lambda_1 s} < 1$, we have

$$\begin{aligned}
&\frac{d}{dt}\left\{e^{-\gamma t}\left[\psi(t+s) - \psi(t)\right]\right\}\\
&\quad= \frac{d}{dt}\left\{e^{-\gamma t}\left[\frac{1}{1+\alpha e^{-\lambda_1(t+s)}} - \frac{1}{1+\alpha e^{-\lambda_1 t}}\right]\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{-\alpha\left(\gamma+\lambda_1\right) e^{-\left(\gamma+\lambda_1\right) t}\left(1-e^{-\lambda_1 s}\right)\left(1+\alpha e^{-\lambda_1 s} e^{-\lambda_1 t}\right)\left(1+\alpha e^{-\lambda_1 t}\right)}{\left(1+\alpha e^{-\lambda_1(t+s)}\right)^2\left(1+\alpha e^{-\lambda_1 t}\right)^2} \\
&\quad+\frac{\alpha^2 \lambda_1 e^{-\lambda_1 t} e^{-\left(\gamma+\lambda_1\right) t}\left(1-e^{-\lambda_1 s}\right)\left(1+e^{-\lambda_1 s}+2 \alpha e^{-\lambda_1 s} e^{-\lambda_1 t}\right)}{\left(1+\alpha e^{-\lambda_1(t+s)}\right)^2\left(1+\alpha e^{-\lambda_1 t}\right)^2} \\
&= -\alpha e^{-\left(\gamma+\lambda_1\right) t}\left(1-e^{-\lambda_1 s}\right) \\
&\quad\times \frac{\left[\left(\gamma+\lambda_1\right)+\gamma \alpha e^{-\lambda_1 t}\left(1+e^{-\lambda_1 s}\right)+\left(\gamma-\lambda_1\right) \alpha^2 e^{-\lambda_1 s} e^{-2 \lambda_1 t}\right]}{\left(1+\alpha e^{-\lambda_1(t+s)}\right)^2\left(1+\alpha e^{-\lambda_1 t}\right)^2}.
\end{aligned}
$$

Taking into account that $\gamma > \lambda_1$, we have

$$
\frac{d}{dt}\left\{e^{-\gamma t}\left[\psi(t+s)-\psi(t)\right]\right\} < 0.
$$

The proof is complete. □

Lemmas 5.1 and 5.2 imply that $\psi(t) \in \Gamma$, $\psi(t) \in \Gamma^*$, $\psi(t) \in \Gamma^{**}$ and $0 < \sup_{t\in\mathbb{R}} \phi(t) = \varepsilon$.

Lemma 5.3. $\phi(t)$ *defined by* (5.4) *is a lower solution of* (5.2).

Proof. For $t > 0$, $\phi(t) = \varepsilon$, $\phi'(t) = 0$, $\phi''(t) = 0$, we have

$$
\begin{aligned}
&-\phi''(t) + c\phi'(t) - f(\phi(t), (g*\phi)(t)) \\
&\quad= -\phi''(t) + c\phi'(t) - \phi(t)\left[1 + a\phi(t) - (1+a)(g*\phi)(t)\right] \\
&\quad= -\phi(t) - a\phi^2(t) + (1+a)\phi(t)(g*\phi)(t) \\
&\quad\leqslant -\varepsilon - a\varepsilon^2 + (1+a)\varepsilon^2 = -\varepsilon(1-\varepsilon) < 0.
\end{aligned}
$$

For $t < 0$, $\phi(t) = \varepsilon e^{\lambda_3 t}$, $\phi'(t) = \varepsilon\lambda_3 e^{\lambda_3 t}$, $\phi''(t) = \varepsilon\lambda_3^2 e^{\lambda_3 t}$, we have

$$
\begin{aligned}
&-\phi''(t) + c\phi'(t) - f(\phi(t), (g*\phi)(t)) \\
&\quad= -\phi''(t) + c\phi'(t) - \phi(t)\left[1 + a\phi(t) - (1+a)(g*\phi)(t)\right] \\
&\quad= -\phi''(t) + c\phi'(t) - \phi(t) - a\phi^2(t) + (1+a)\phi(t)(g*\phi)(t) \\
&\quad\leqslant -\phi''(t) + c\phi'(t) - \phi(t) - a\phi^2(t) + (1+a)\phi(t) \\
&\quad\leqslant -\phi''(t) + c\phi'(t) + a\phi(t) = \varepsilon\left(-\lambda_3^2 + c\lambda_3 + a\right)e^{\lambda_3 t} = 0.
\end{aligned}
$$

This implies that $\phi(t)$ is a lower solution of (5.2). □

Now we show that $\psi(t)$ is an upper solution of (5.2) by choosing different kernel functions g. Here we consider three cases:

(i) $g(t,x) = \frac{1}{\tau}e^{-\frac{t}{\tau}}\delta(x), \tau > 0$;

(ii) $g(t,x) = \delta(t)\frac{1}{\sqrt{4\pi\rho}}e^{-\frac{x^2}{4\rho}}, \rho > 0$;

(iii) $g(t,x) = \frac{1}{\tau}e^{-\frac{t}{\tau}}\frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}, \tau > 0$.

5.1. The case $g(t,x) = \frac{1}{\tau}e^{-\frac{t}{\tau}}\delta(x), \tau > 0$

In this case, Eq. (5.1) becomes a reaction–diffusion model with temporal delay. Obviously, $g(t,x) = \frac{1}{\tau}e^{-\frac{t}{\tau}}\delta(x)$ satisfies (H_0). Then

$$(g * \varphi)(t) = \int_0^{+\infty} \frac{1}{\tau}e^{-\frac{\xi}{\tau}}\varphi(t - c\xi)\,d\xi$$

and

$$f(\varphi(t), (g*\varphi)(t)) = \varphi(t)\left[1 + a\varphi(t) - (1+a)\int_0^{+\infty}\frac{1}{\tau}e^{-\frac{\xi}{\tau}}\varphi(t-c\xi)\,d\xi\right].$$

Lemma 5.4. *For sufficiently small* $\tau > 0$, $f(\varphi(t), (g*\varphi)(t))$ *satisfies* (H_1^*).

Proof. Let $\varphi_1, \varphi_2 \in C(\mathbb{R}, \mathbb{R})$ with $0 \leqslant \varphi_1(t) \leqslant \varphi_2(t) \leqslant K = 1$ so that $e^{\gamma t}[\varphi_2(t) - \varphi_1(t)]$ is increasing in $t \in \mathbb{R}$. It is easy to see that for any $s \in \mathbb{R}$, $e^{\gamma t}[\varphi_2(s+t) - \varphi_1(s+t)]$ is increasing in $t \in \mathbb{R}$, then for $\gamma > 3a + 2$ and sufficiently small $\tau > 0$ satisfying $1 - \gamma c\tau \geqslant 1/2$,

$$\begin{aligned}
&f\left(\varphi_2(t), \left(g*\varphi_2\right)(t)\right) - f\left(\varphi_1(t), \left(g*\varphi_1\right)(t)\right)\\
&\quad= \varphi_2(t)\left[1 + a\varphi_2(t) - (1+a)\left(g*\varphi_2\right)(t)\right]\\
&\qquad - \varphi_1(t)\left[1 + a\varphi_1(t) - (1+a)\left(g*\varphi_1\right)(t)\right]\\
&\quad= \left[\varphi_2(t) - \varphi_1(t)\right] + a\left[\varphi_2^2(t) - \varphi_1^2(t)\right]\\
&\qquad - (1+a)\left[\varphi_2(t)\left(g*\varphi_2\right)(t) - \varphi_1(t)\left(g*\varphi_1\right)(t)\right]\\
&\quad= \left[\varphi_2(t) - \varphi_1(t)\right]\left[1 + a\varphi_2(t) + a\varphi_1(t)\right]\\
&\qquad - (1+a)\left[\varphi_2(t) - \varphi_1(t)\right]\left(g*\varphi_2\right)(t)\\
&\qquad - (1+a)\varphi_1(t)\left[\left(g*\varphi_2\right)(t) - \left(g*\varphi_1\right)(t)\right]\\
&\quad\geqslant -a\left[\varphi_2(t) - \varphi_1(t)\right] - (1+a)\left[\left(g*\varphi_2\right)(t) - \left(g*\varphi_1\right)(t)\right]
\end{aligned}$$

$$
\begin{aligned}
&= -a\left[\varphi_2(t) - \varphi_1(t)\right] - (1+a)\int_0^{+\infty} \frac{1}{\tau} e^{-\frac{\xi}{\tau}} \left[\varphi_2(t-c\xi) - \varphi_1(t-c\xi)\right] d\xi \\
&= -a\left[\varphi_2(t) - \varphi_1(t)\right] \\
&\quad - (1+a)\int_0^{+\infty} \frac{1}{\tau} e^{-\frac{\xi}{\tau}} e^{\gamma c\xi} \left\{ e^{-\gamma c\xi} \left[\varphi_2(t-c\xi) - \varphi_1(t-c\xi)\right]\right\} d\xi \\
&\geqslant -a\left[\varphi_2(t) - \varphi_1(t)\right] - (1+a)\int_0^{+\infty} \frac{1}{\tau} e^{-\frac{\xi}{\tau}} e^{\gamma c\xi} \left[\varphi_2(t) - \varphi_1(t)\right] d\xi \\
&= -\left[a + (1+a)\int_0^{+\infty} \frac{1}{\tau} e^{-\frac{\xi}{\tau}} e^{\gamma c\xi} d\xi\right] \left[\varphi_2(t) - \varphi_1(t)\right] \\
&= -\left(a + \frac{1+a}{1-\gamma c\tau}\right)\left[\varphi_2(t) - \varphi_1(t)\right] \\
&\geqslant -(3a+2)\left[\varphi_2(t) - \varphi_1(t)\right] \geqslant -\gamma\left[\varphi_2(t) - \varphi_1(t)\right].
\end{aligned}
$$

The proof is complete. □

Lemma 5.5. *For sufficiently small* $\tau > 0$, $\psi(t)$ *defined by* (5.4) *is an upper solution of* (5.2).

Proof. Since

$$
\psi'(t) = \frac{\alpha\lambda_1 e^{-\lambda_1 t}}{\left(1+\alpha e^{-\lambda_1 t}\right)^2}, \quad \psi''(t) = \frac{-\alpha\lambda_1^2 e^{-\lambda_1 t} + \alpha^2\lambda_1^2 e^{-2\lambda_1 t}}{\left(1+\alpha e^{-\lambda_1 t}\right)^3},
$$

and for $\tau > 0$ such that $1 - 2c\lambda_1\tau > 0$,

$$
\begin{aligned}
(g*\psi)(t) &= \int_0^{\infty} \frac{1}{\tau} e^{-\frac{\xi}{\tau}} \frac{1}{1+\alpha e^{-\lambda_1(t-c\xi)}} d\xi = -\int_0^{\infty} \frac{1}{1+\alpha e^{-\lambda_1(t-c\xi)}} de^{-\frac{\xi}{\tau}} \\
&= \left[-\frac{e^{-\frac{\xi}{\tau}}}{1+\alpha e^{-\lambda_1(t-c\xi)}}\right]_0^{+\infty} + \int_0^{\infty} e^{-\frac{\xi}{\tau}} d\left(\frac{1}{1+\alpha e^{-\lambda_1(t-c\xi)}}\right) \\
&= \frac{1}{1+\alpha e^{-\lambda_1 t}} - c\alpha\lambda_1 e^{-\lambda_1 t} \int_0^{\infty} \frac{e^{\left(c\lambda_1 - \frac{1}{\tau}\right)\xi}}{\left[1+\alpha e^{-\lambda_1(t-c\xi)}\right]^2} d\xi \\
&= \frac{1}{1+\alpha e^{-\lambda_1 t}} + \frac{c\alpha\lambda_1\tau e^{-\lambda_1 t}}{1-c\lambda_1\tau} \cdot \left[\frac{e^{\left(c\lambda_1 - \frac{1}{\tau}\right)\xi}}{\left[1+\alpha e^{-\lambda_1(t-c\xi)}\right]^2}\right]_0^{+\infty}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{c\alpha\lambda_1\tau e^{-\lambda_1 t}}{1-c\lambda_1\tau}\int_0^\infty e^{\left(c\lambda_1-\frac{1}{\tau}\right)\xi}d\left(\frac{1}{\left[1+\alpha e^{-\lambda_1(t-c\xi)}\right]^2}\right)\\
&= \frac{1}{1+\alpha e^{-\lambda_1 t}}-\frac{c\alpha\lambda_1\tau e^{-\lambda_1 t}}{(1-c\lambda_1\tau)\left(1+\alpha e^{-\lambda_1 t}\right)^2}\\
&\quad+\frac{2c^2\alpha^2\lambda_1^2\tau e^{-2\lambda_1 t}}{1-c\lambda_1\tau}\int_0^\infty \frac{e^{\left(2c\lambda_1-\frac{1}{\tau}\right)\xi}}{\left[1+\alpha e^{-\lambda_1(t-c\xi)}\right]^3}\,d\xi\\
&\geqslant \frac{1}{1+\alpha e^{-\lambda_1 t}}-\frac{c\alpha\lambda_1\tau e^{-\lambda_1 t}}{(1-c\lambda_1\tau)\left(1+\alpha e^{-\lambda_1 t}\right)^2},
\end{aligned}
$$

we have

$$
\begin{aligned}
&-\psi''(t)+c\psi'(t)-f\left(\psi(t),(g*\psi)(t)\right)\\
&\quad=-\psi''(t)+c\psi'(t)-\psi(t)\left[1+a\psi(t)-(1+a)(g*\psi)(t)\right]\\
&\quad\geqslant-\psi''(t)+c\psi'(t)-(1+a)\psi(t)+(1+a)\psi(t)(g*\psi)(t)\\
&\quad\geqslant-\frac{-\alpha\lambda_1^2e^{-\lambda_1 t}+\alpha^2\lambda_1^2e^{-2\lambda_1 t}}{\left(1+\alpha e^{-\lambda_1 t}\right)^3}+\frac{c\alpha\lambda_1 e^{-\lambda_1 t}}{\left(1+\alpha e^{-\lambda_1 t}\right)^2}-\frac{1+a}{1+\alpha e^{-\lambda_1 t}}\\
&\qquad+\frac{1+a}{\left(1+\alpha e^{-\lambda_1 t}\right)^2}-\frac{(1+a)c\alpha\lambda_1\tau e^{-\lambda_1 t}}{(1-c\lambda_1\tau)\left(1+\alpha e^{-\lambda_1 t}\right)^3}.
\end{aligned}
$$

Let the right part of the last inequality be $C/\left(1+\alpha e^{-\lambda_1 t}\right)^3$. Then

$$
\begin{aligned}
C&=\alpha\lambda_1^2e^{-\lambda_1 t}-\alpha^2\lambda_1^2e^{-2\lambda_1 t}+c\alpha\lambda_1e^{-\lambda_1 t}+c\alpha^2\lambda_1e^{-2\lambda_1 t}\\
&\quad-2(1+a)\alpha e^{-\lambda_1 t}-(1+a)\alpha^2e^{-2\lambda_1 t}+(1+a)\alpha e^{-\lambda_1 t}\\
&\quad-\frac{(1+a)c\alpha\lambda_1\tau}{1-c\lambda_1\tau}e^{-\lambda_1 t}\\
&=\alpha^2\left[-\lambda_1^2+c\lambda_1-(1+a)\right]e^{-2\lambda_1 t}\\
&\quad+\alpha\left[\lambda_1^2+c\lambda_1-(1+a)-\frac{(1+a)c\lambda_1\tau}{1-c\lambda_1\tau}\right]e^{-\lambda_1 t}\\
&=\alpha\left[2\lambda_1^2-\frac{(1+a)c\lambda_1\tau}{1-c\lambda_1\tau}\right]e^{-\lambda_1 t}.
\end{aligned}
$$

Obviously, for sufficiently small τ,

$$2\lambda_1^2 - \frac{(1+a)\,c\lambda_1\tau}{1-c\lambda_1\tau} > 0,$$

which implies that $C > 0$. Thus

$$-\psi''(t) + c\psi'(t) - f(\psi(t), (g*\psi)(t)) \geqslant 0.$$

The proof is complete. $\square$

Now by Theorem 4.8(ii), the following result is true.

Theorem 5.6. *For any* $c \geqslant 2\sqrt{1+a}$, *there exists* $\tau^*(c) > 0$ *such that for any* $\tau < \tau^*(c)$, *system* (5.1) *has a travelling wave front which satisfies* (5.3).

5.2. The case $g(t,x) = \delta(t)\frac{1}{\sqrt{4\pi\rho}}e^{-\frac{x^2}{4\rho}}, \rho > 0$

In this case, Eq. (5.1) becomes a reaction–diffusion model with a nonlocal (spatial) delay. It is easy to see that the kernel g satisfies (H_0),

$$(g*\varphi)(t) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi\rho}}e^{-\frac{y^2}{4\rho}}\varphi(t-y)\,dy$$

and

$$f(\varphi(t), (g*\varphi)(t)) = \varphi(t)\left[1 + a\varphi(t) - (1+a)\int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi\rho}}e^{-\frac{y^2}{4\rho}}\varphi(t-y)\,dy\right].$$

Lemma 5.7. *Assume that* $\gamma > a + 2(1+a)e^{\rho\gamma^2}$. *Then* $f(\varphi(t), (g*\varphi)(t))$ *satisfies* (H_1^{**}).

Proof. Assume that $\varphi_1, \varphi_2 \in C(\mathbb{R}, \mathbb{R})$ and satisfy that $0 \leqslant \varphi_1(t) \leqslant \varphi_2(t) \leqslant K = 1$, $e^{\gamma t}[\varphi_2(t) - \varphi_1(t)]$ is increasing in $t \in \mathbb{R}$ and $e^{-\gamma t}[\varphi_2(t) - \varphi_1(t)]$ is decreasing in $t \in \mathbb{R}$. Then

$$\begin{aligned}
&f\left(\varphi_2(t), \left(g*\varphi_2\right)(t)\right) - f\left(\varphi_1(t), \left(g*\varphi_1\right)(t)\right) \\
&\quad\geqslant -a\left[\varphi_2(t) - \varphi_1(t)\right] - (1+a)\left[\left(g*\varphi_2\right)(t) - \left(g*\varphi_1\right)(t)\right] \\
&\quad= -a\left[\varphi_2(t) - \varphi_1(t)\right] \\
&\qquad - (1+a)\int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi\rho}}e^{-\frac{y^2}{4\rho}}\left[\varphi_2(t-y) - \varphi_1(t-y)\right]dy
\end{aligned}$$

$$
\begin{aligned}
&= -a\left[\varphi_2(t)-\varphi_1(t)\right] \\
&\quad -(1+a)\int_0^{+\infty}\frac{1}{\sqrt{4\pi\rho}}e^{-\frac{y^2}{4\rho}}e^{\gamma y}\left\{e^{-\gamma y}\left[\varphi_2(t-y)-\varphi_1(t-y)\right]\right\}dy \\
&\quad -(1+a)\int_{-\infty}^{0}\frac{1}{\sqrt{4\pi\rho}}e^{-\frac{y^2}{4\rho}}e^{-\gamma y}\left\{e^{\gamma y}\left[\varphi_2(t-y)-\varphi_1(t-y)\right]\right\}dy \\
&\geqslant -a\left[\varphi_2(t)-\varphi_1(t)\right]-(1+a)\int_0^{+\infty}\frac{1}{\sqrt{4\pi\rho}}e^{-\frac{y^2}{4\rho}}e^{\gamma y}\left[\varphi_2(t)-\varphi_1(t)\right]dy \\
&\quad -(1+a)\int_{-\infty}^{0}\frac{1}{\sqrt{4\pi\rho}}e^{-\frac{y^2}{4\rho}}e^{-\gamma y}\left[\varphi_2(t)-\varphi_1(t)\right]dy \\
&= -a\left[\varphi_2(t)-\varphi_1(t)\right]-(1+a)\int_0^{+\infty}\frac{1}{\sqrt{4\pi\rho}}e^{-\frac{(y-2\gamma\rho)^2}{4\rho}}e^{\rho\gamma^2}\left[\varphi_2(t)-\varphi_1(t)\right]dy \\
&\quad -(1+a)\int_{-\infty}^{0}\frac{1}{\sqrt{4\pi\rho}}e^{-\frac{(y+2\gamma\rho)^2}{4\rho}}e^{\rho\gamma^2}\left[\varphi_2(t)-\varphi_1(t)\right]dy \\
&\geqslant -a\left[\varphi_2(t)-\varphi_1(t)\right]-(1+a)e^{\rho\gamma^2}\left[\varphi_2(t)-\varphi_1(t)\right]\int_{-\infty}^{+\infty}\frac{1}{\sqrt{4\pi\rho}}e^{-\frac{(y-2\gamma\rho)^2}{4\rho}}dy \\
&\quad -(1+a)e^{\rho\gamma^2}\left[\varphi_2(t)-\varphi_1(t)\right]\int_{-\infty}^{+\infty}\frac{1}{\sqrt{4\pi\rho}}e^{-\frac{(y+2\gamma\rho)^2}{4\rho}}dy \\
&= -\left[a+2(1+a)e^{\rho\gamma^2}\right]\left[\varphi_2(t)-\varphi_1(t)\right]\geqslant -\gamma\left[\varphi_2(t)-\varphi_1(t)\right],
\end{aligned}
$$

The proof is complete. □

Lemma 5.8. *For sufficiently small* $\rho > 0$, $\psi(t)$ *defined by* (5.4) *is an upper solution of* (5.2).

Proof. Let

$$F(y,\rho)=\int_{-\infty}^{y}\frac{1}{\sqrt{4\pi\rho}}e^{-\frac{\xi^2}{4\rho}}\,d\xi.$$

Then

$$\frac{\partial}{\partial y}F(y,\rho)=\frac{1}{\sqrt{4\pi\rho}}e^{-\frac{y^2}{4\rho}},\quad F(-\infty,\rho)=0,\ F(0,\rho)=\frac{1}{2}.$$

Since $\lim_{y\to-\infty} e^{-\lambda_1 y} F(y,\rho) = 0$, we have

$$\begin{aligned}
&\int_{-\infty}^{0} e^{-\lambda_1 y} F(y,\rho)\,dy \\
&\quad = -\frac{1}{\lambda_1}\int_{-\infty}^{0} F(y,\rho)\,de^{-\lambda_1 y} \\
&\quad = -\frac{1}{\lambda_1}\left[e^{-\lambda_1 y}F(y,\rho)\right]_{-\infty}^{0} + \frac{1}{\lambda_1}\int_{-\infty}^{0}\frac{1}{\sqrt{4\pi\rho}}e^{-\frac{y^2}{4\rho}-\lambda_1 y}\,dy \\
&\quad = -\frac{1}{2\lambda_1} + \frac{1}{\lambda_1}e^{\rho\lambda_1^2}\int_{-\infty}^{0}\frac{1}{\sqrt{4\pi\rho}}e^{-\frac{(y+2\rho\lambda_1)^2}{4\rho}}\,dy \\
&\quad = -\frac{1}{2\lambda_1} + \frac{1}{\lambda_1}e^{\rho\lambda_1^2}\int_{-\infty}^{2\rho\lambda_1}\frac{1}{\sqrt{4\pi\rho}}e^{-\frac{y^2}{4\rho}}\,dy \\
&\quad = -\frac{1}{2\lambda_1} + \frac{1}{2\lambda_1}e^{\rho\lambda_1^2} + \frac{1}{\lambda_1}e^{\rho\lambda_1^2}\int_{0}^{2\rho\lambda_1}\frac{1}{\sqrt{4\pi\rho}}e^{-\frac{y^2}{4\rho}}\,dy \\
&\quad = -\frac{1}{2\lambda_1} + \frac{1}{2\lambda_1}e^{\rho\lambda_1^2} + \frac{1}{\lambda_1\sqrt{\pi}}e^{\rho\lambda_1^2}\int_{0}^{\sqrt{\rho}\lambda_1} e^{-y^2}\,dy.
\end{aligned}$$

Now define

$$G_-(y,\rho) = \int_{-\infty}^{y} e^{-\lambda_1 \xi} F(\xi,\rho)\,d\xi.$$

Then

$$\begin{aligned}
G_-(0,\rho) &= -\frac{1}{2\lambda_1} + \frac{1}{2\lambda_1}e^{\rho\lambda_1^2} + \frac{1}{\lambda_1\sqrt{\pi}}e^{\rho\lambda_1^2}\int_{0}^{\sqrt{\rho}\lambda_1} e^{-y^2}\,dy, \\
G_-(-\infty,\rho) &= 0.
\end{aligned}$$

Similarly, we can define

$$G_+(y,\rho) = \int_{-\infty}^{y} e^{\lambda_1 \xi} F(\xi,\rho)\,d\xi$$

and obtain

$$\begin{aligned}
G_+(0,\rho) &= \frac{1}{2\lambda_1} - \frac{1}{2\lambda_1}e^{\rho\lambda_1^2} + \frac{1}{\lambda_1\sqrt{\pi}}e^{\rho\lambda_1^2}\int_{0}^{\sqrt{\rho}\lambda_1} e^{-y^2}\,dy, \\
G_+(-\infty,\rho) &= 0.
\end{aligned}$$

Furthermore,

$$\int_{-\infty}^{0} e^{-\lambda_1 y} G_-(y,\rho)\,dy < \infty \quad \text{and} \quad \int_{-\infty}^{0} e^{\lambda_1 y} G_+(y,\rho)\,dy < \infty.$$

Consequently, it follows that

$$\begin{aligned}
(g*\psi)(t) &= \int_0^\infty \frac{1}{\sqrt{4\pi\rho}} e^{-\frac{y^2}{4\rho}} \frac{1}{1+\alpha e^{-\lambda_1(t-y)}}\,dy \\
&\quad + \int_{-\infty}^0 \frac{1}{\sqrt{4\pi\rho}} e^{-\frac{y^2}{4\rho}} \frac{1}{1+\alpha e^{-\lambda_1(t-y)}}\,dy \\
&= \int_{-\infty}^0 \frac{1}{\sqrt{4\pi\rho}} e^{-\frac{y^2}{4\rho}} \frac{1}{1+\alpha e^{-\lambda_1(t+y)}}\,dy \\
&\quad + \int_{-\infty}^0 \frac{1}{\sqrt{4\pi\rho}} e^{-\frac{y^2}{4\rho}} \frac{1}{1+\alpha e^{-\lambda_1(t-y)}}\,dy \\
&= \int_{-\infty}^0 \frac{1}{1+\alpha e^{-\lambda_1(t+y)}}\,dF(y,\rho) + \int_{-\infty}^0 \frac{1}{1+\alpha e^{-\lambda_1(t-y)}}\,dF(y,\rho) \\
&= \left[\frac{F(y,\rho)}{1+\alpha e^{-\lambda_1(t+y)}}\right]_{-\infty}^0 - \alpha\lambda_1 e^{-\lambda_1 t} \int_{-\infty}^0 \frac{e^{-\lambda_1 y} F(y,\rho)}{\left[1+\alpha e^{-\lambda_1(t+y)}\right]^2}\,dy \\
&\quad + \left[\frac{F(y,\rho)}{1+\alpha e^{-\lambda_1(t-y)}}\right]_{-\infty}^0 + \alpha\lambda_1 e^{-\lambda_1 t} \int_{-\infty}^0 \frac{e^{\lambda_1 y} F(y,\rho)}{\left[1+\alpha e^{-\lambda_1(t-y)}\right]^2}\,dy \\
&= \frac{1}{2\left(1+\alpha e^{-\lambda_1 t}\right)} - \alpha\lambda_1 e^{-\lambda_1 t} \int_{-\infty}^0 \frac{1}{\left[1+\alpha e^{-\lambda_1(t+y)}\right]^2}\,dG_-(y,\rho) \\
&\quad + \frac{1}{2\left(1+\alpha e^{-\lambda_1 t}\right)} + \alpha\lambda_1 e^{-\lambda_1 t} \int_{-\infty}^0 \frac{1}{\left[1+\alpha e^{-\lambda_1(t-y)}\right]^2}\,dG_+(y,\rho) \\
&= \frac{1}{1+\alpha e^{-\lambda_1 t}} - \left[\alpha\lambda_1 e^{-\lambda_1 t} \frac{G_-(y,\rho)}{\left[1+\alpha e^{-\lambda_1(t+y)}\right]^2}\right]_{-\infty}^0 \\
&\quad + 2\alpha^2\lambda_1^2 e^{-2\lambda_1 t} \int_{-\infty}^0 \frac{e^{-\lambda_1 y} G_-(y,\rho)}{\left[1+\alpha e^{-\lambda_1(t+y)}\right]^3}\,dy \\
&\quad + \left[\alpha\lambda_1 e^{-\lambda_1 t} \frac{G_+(y,\rho)}{\left[1+\alpha e^{-\lambda_1(t+y)}\right]^2}\right]_{-\infty}^0
\end{aligned}$$

$$
\begin{aligned}
&+2\alpha^2\lambda_1^2 e^{-2\lambda_1 t}\int_{-\infty}^{0}\frac{e^{\lambda_1 y}G_+(y,\rho)}{\left[1+\alpha e^{-\lambda_1(t-y)}\right]^3}\,dy\\
&\geqslant \frac{1}{1+\alpha e^{-\lambda_1 t}}+\frac{\alpha\lambda_1 e^{-\lambda_1 t}}{\left[1+\alpha e^{-\lambda_1 t}\right]^2}\left[G_+(0,\rho)-G_-(0,\rho)\right]\\
&= \frac{1}{1+\alpha e^{-\lambda_1 t}}+\frac{\alpha e^{-\lambda_1 t}\left(1-e^{\rho\lambda_1^2}\right)}{\left[1+\alpha e^{-\lambda_1 t}\right]^2}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&-\psi''(t)+c\psi'(t)-f(\psi(t),(g*\psi)(t))\\
&\quad\geqslant -\psi''(t)+c\psi'(t)-(1+a)\psi(t)+(1+a)\psi(t)(g*\psi)(t)\\
&\quad\geqslant -\frac{-\alpha\lambda_1^2 e^{-\lambda_1 t}+\alpha^2\lambda_1^2 e^{-2\lambda_1 t}}{\left(1+\alpha e^{-\lambda_1 t}\right)^3}+\frac{c\alpha\lambda_1 e^{-\lambda_1 t}}{\left(1+\alpha e^{-\lambda_1 t}\right)^2}-\frac{1+a}{1+\alpha e^{-\lambda_1 t}}\\
&\qquad+\frac{1+a}{\left(1+\alpha e^{-\lambda_1 t}\right)^2}+\frac{\alpha(1+a)e^{-\lambda_1 t}\left(1-e^{\rho\lambda_1^2}\right)}{\left(1+\alpha e^{-\lambda_1 t}\right)^3}\\
&\quad=\frac{-\alpha^2\left[\lambda_1^2-c\lambda_1+(1+a)\right]e^{-2\lambda_1 t}}{\left(1+\alpha e^{-\lambda_1 t}\right)^3}\\
&\qquad+\frac{\alpha\left[\lambda_1^2+c\lambda_1-(1+a)e^{\rho\lambda_1^2}\right]e^{-\lambda_1 t}}{\left(1+\alpha e^{-\lambda_1 t}\right)^3}\\
&\quad=\frac{\alpha\left[\lambda_1^2+c\lambda_1-(1+a)e^{\rho\lambda_1^2}\right]e^{-\lambda_1 t}}{\left(1+\alpha e^{-\lambda_1 t}\right)^3}\\
&\quad=\frac{\alpha\left[2\lambda_1^2+(1+a)\left(1-e^{\rho\lambda_1^2}\right)\right]e^{-\lambda_1 t}}{\left(1+\alpha e^{-\lambda_1 t}\right)^3}.
\end{aligned}
$$

Since

$$2\lambda_1^2>0 \quad\text{and}\quad \lim_{\rho\to 0}\left(1-e^{\rho\lambda_1^2}\right)=0$$

for sufficiently small $\rho>0$, we finally have

$$-\psi''(t)+c\psi'(t)-f(\psi(t),(g*\psi)(t))\geqslant 0.$$

This completes the proof. □

We remark that if ρ is sufficiently small, then we can choose a $\gamma > \lambda_1$ such that $\gamma > a + 2(1+a)e^{\gamma^2\rho} > 1$ and $\gamma > c+1$. Thus, by Theorem 4.8(iii), we have the following result.

Theorem 5.9. *For any* $c \geqslant 2\sqrt{1+a}$, *there exists* $\rho^*(c) > 0$ *such that for any* $\rho < \rho^*(c)$, *system* (5.1) *has a travelling wave front which satisfies* (5.3).

5.3. The case $g(t,x) = \frac{1}{\tau}e^{-\frac{t}{\tau}}\frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}, \tau > 0$

With this choice of kernel, Eq. (5.1) has a nonlocal spatio-temporal delay. We have

$$(g*\varphi)(t) = \int_0^{+\infty}\int_{-\infty}^{+\infty}\frac{1}{\tau}e^{-\frac{\theta}{\tau}}\frac{1}{\sqrt{4\pi\theta}}e^{-\frac{y^2}{4\theta}}\varphi(t-y-c\theta)\,dy\,d\theta$$

and

$$\begin{aligned} f(\varphi(t),(g*\varphi)(t)) &= \varphi(t) + a\varphi^2(t) - (1+a)\varphi(t) \\ &\quad \times \int_0^{+\infty}\int_{-\infty}^{+\infty}\frac{1}{\tau}e^{-\frac{\theta}{\tau}}\frac{1}{\sqrt{4\pi\theta}}e^{-\frac{y^2}{4\theta}}\varphi(t-y-c\theta)\,dy\,d\theta. \end{aligned}$$

We now show that the kernel g satisfies (H_0). Since

$$\lim_{t\to 0+} e^{-\frac{1}{8t}}\frac{1}{\sqrt{4\pi t}} = 0,$$

there exists $M > 0$ such that

$$e^{-\frac{1}{8t}}\frac{1}{\sqrt{4\pi t}} < M \quad \text{for} \quad t \in [0, A].$$

Furthermore, for $t \in [0, A]$

$$\begin{aligned} \int_1^{+\infty}\frac{1}{\tau}e^{-\frac{t}{\tau}}\frac{1}{\sqrt{4\pi t}}e^{-\frac{y^2}{4t}}dy &\leqslant \int_1^{+\infty}\frac{1}{\tau}e^{-\frac{t}{\tau}}\frac{1}{\sqrt{4\pi t}}e^{-\frac{1}{8t}}e^{-\frac{y^2}{8t}}dy \\ &\leqslant \frac{M}{\tau}\int_1^{+\infty}e^{-\frac{y^2}{8t}}dy \leqslant \frac{M}{\tau}\int_1^{+\infty}e^{-\frac{y^2}{8A}}dy, \end{aligned}$$

which implies that $g(t,x) = \frac{1}{\tau}e^{-\frac{t}{\tau}}\frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}$ satisfies (H_0).

Lemma 5.10. *For sufficiently small* $\tau > 0$, $f(\varphi(t),(g*\varphi)(t))$ *satisfies* (H_1^{**}).

Proof. Let $\varphi_1, \varphi_2 \in C(\mathbb{R}, \mathbb{R})$ such that $0 \leqslant \varphi_1(t) \leqslant \varphi_2(t) \leqslant K=1$, $e^{\gamma t}\left[\varphi_2(t) - \varphi_1(t)\right]$ is increasing in $t \in \mathbb{R}$ and $e^{-\gamma t}\left[\varphi_2(t) - \varphi_1(t)\right]$ is decreasing in $t \in \mathbb{R}$. Then

$$\begin{aligned}
&\left(g * \varphi_2\right)(t) - \left(g * \varphi_1\right)(t) \\
&= \int_0^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{\tau} e^{-\frac{\theta}{\tau}} \frac{1}{\sqrt{4\pi\theta}} e^{-\frac{y^2}{4\theta}} \left[\varphi_2(t - y - c\theta) - \varphi_1(t - y - c\theta)\right] dy\, d\theta \\
&= \int_0^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{\tau} e^{-\frac{\theta}{\tau}} \frac{1}{\sqrt{4\pi\theta}} e^{-\frac{y^2}{4\theta}} e^{\gamma c\theta} \left\{e^{-\gamma c\theta}\left[\varphi_2(t-y-c\theta) - \varphi_1(t-y-c\theta)\right]\right\} dy\, d\theta \\
&\leqslant \int_0^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{\tau} e^{-\frac{\theta}{\tau}} \frac{1}{\sqrt{4\pi\theta}} e^{-\frac{y^2}{4\theta}} e^{\gamma c\theta} \left[\varphi_2(t - y) - \varphi_1(t - y)\right] dy\, d\theta \\
&= \int_0^{+\infty} \int_0^{+\infty} \frac{1}{\tau} e^{-\frac{\theta}{\tau}} \frac{1}{\sqrt{4\pi\theta}} e^{-\frac{y^2}{4\theta}} e^{\gamma c\theta} e^{\gamma y} \left\{e^{-\gamma y}\left[\varphi_2(t - y) - \varphi_1(t - y)\right]\right\} dy\, d\theta \\
&\quad + \int_0^{+\infty} \int_{-\infty}^{0} \frac{1}{\tau} e^{-\frac{\theta}{\tau}} \frac{1}{\sqrt{4\pi\theta}} e^{-\frac{y^2}{4\theta}} e^{\gamma c\theta} e^{-\gamma y} \left\{e^{\gamma y}\left[\varphi_2(t - y) - \varphi_1(t - y)\right]\right\} dy\, d\theta \\
&\leqslant \int_0^{+\infty} \int_0^{+\infty} \frac{1}{\tau} e^{-\frac{\theta}{\tau}} \frac{1}{\sqrt{4\pi\theta}} e^{-\frac{y^2}{4\theta}} e^{\gamma c\theta} e^{\gamma y} \left[\varphi_2(t) - \varphi_1(t)\right] dy\, d\theta \\
&\quad + \int_0^{+\infty} \int_{-\infty}^{0} \frac{1}{\tau} e^{-\frac{\theta}{\tau}} \frac{1}{\sqrt{4\pi\theta}} e^{-\frac{y^2}{4\theta}} e^{\gamma c\theta} e^{-\gamma y} \left[\varphi_2(t) - \varphi_1(t)\right] dy\, d\theta \\
&= \int_0^{+\infty} \int_0^{+\infty} \frac{1}{\tau} e^{-\frac{\theta}{\tau}} \frac{1}{\sqrt{4\pi\theta}} e^{-\frac{(y-2\gamma\theta)^2}{4\theta}} e^{\gamma c\theta} e^{\gamma^2\theta} \left[\varphi_2(t) - \varphi_1(t)\right] dy\, d\theta \\
&\quad + \int_0^{+\infty} \int_{-\infty}^{0} \frac{1}{\tau} e^{-\frac{\theta}{\tau}} \frac{1}{\sqrt{4\pi\theta}} e^{-\frac{(y+2\gamma\theta)^2}{4\theta}} e^{\gamma c\theta} e^{\gamma^2\theta} \left[\varphi_2(t) - \varphi_1(t)\right] dy\, d\theta \\
&\leqslant 2\left[\varphi_2(t) - \varphi_1(t)\right] \int_0^{+\infty} \frac{1}{\tau} e^{-\frac{\theta}{\tau}} e^{\gamma c\theta} e^{\gamma^2\theta} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi\theta}} e^{-\frac{y^2}{4\theta}} dy\, d\theta \\
&= 2\left[\varphi_2(t) - \varphi_1(t)\right] \int_0^{+\infty} \frac{1}{\tau} e^{-\frac{\theta}{\tau}} e^{\gamma c\theta} e^{\gamma^2\theta} d\theta \\
&= \frac{2}{1 - c\tau\gamma - \tau\gamma^2} \left[\varphi_2(t) - \varphi_1(t)\right].
\end{aligned}$$

Since there exists $\gamma > \max\{c + 1, 5a + 4\}$ satisfying $1 - c\tau\gamma - \tau\gamma^2 \geqslant 1/2$ for sufficiently small $\tau > 0$, it follows that

$$\begin{aligned}
&f\left(\varphi_2(t), \left(g * \varphi_2\right)(t)\right) - f\left(\varphi_1(t), \left(g * \varphi_1\right)(t)\right) \\
&\quad \geqslant -a\left[\varphi_2(t) - \varphi_1(t)\right] - (1 + a)\left[\left(g * \varphi_2\right)(t) - \left(g * \varphi_1\right)(t)\right]
\end{aligned}$$

$$
\begin{aligned}
&\geqslant -a\left[\varphi_2(t)-\varphi_1(t)\right]-(1+a)\frac{2}{1-c\tau\gamma-\tau\gamma^2}\left[\varphi_2(t)-\varphi_1(t)\right]\\
&=-\left[a+\frac{2(1+a)}{1-c\tau\gamma-\tau\gamma^2}\right]\left[\varphi_2(t)-\varphi_1(t)\right]\\
&\geqslant -(5a+4)\left[\varphi_2(t)-\varphi_1(t)\right]\geqslant -\gamma\left[\varphi_2(t)-\varphi_1(t)\right].
\end{aligned}
$$

The proof is complete. □

Lemma 5.11. *For sufficiently small* $\tau>0$, $\psi(t)$ *defined by* (5.4) *is an upper solution of* (5.2).

Proof. Applying the estimates in Lemmas 5.5 and 5.8, for sufficiently small $\tau>0$, we have $1-2c\lambda_1\tau-\lambda_1^2\tau>0$ and

$$
\begin{aligned}
&(g*\psi)(t)\\
&\quad=\int_0^{+\infty}\frac{1}{\tau}e^{-\frac{\theta}{\tau}}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{4\pi\theta}}e^{-\frac{y^2}{4\theta}}\frac{1}{1+\alpha e^{-\lambda_1(t-y-c\theta)}}\,dy\,d\theta\\
&\quad\geqslant\int_0^{+\infty}\frac{1}{\tau}e^{-\frac{\theta}{\tau}}\left\{\frac{1}{1+\alpha e^{-\lambda_1(t-c\theta)}}+\frac{\alpha e^{-\lambda_1(t-c\theta)}\left(1-e^{\lambda_1^2\theta}\right)}{\left[1+\alpha e^{-\lambda_1(t-c\theta)}\right]^2}\right\}d\theta\\
&\quad\geqslant\frac{1}{1+\alpha e^{-\lambda_1 t}}-\frac{c\alpha\lambda_1\tau}{1-c\lambda_1\tau}\cdot\frac{e^{-\lambda_1 t}}{\left(1+\alpha e^{-\lambda_1 t}\right)^2}\\
&\qquad+\int_0^{+\infty}\frac{\frac{\alpha}{\tau}e^{-\lambda_1(t-c\theta)}e^{-\frac{\theta}{\tau}}}{\left[1+\alpha e^{-\lambda_1(t-c\theta)}\right]^2}\,d\theta-\int_0^{+\infty}\frac{\frac{\alpha}{\tau}e^{-\lambda_1(t-c\theta)}e^{\lambda_1^2\theta}e^{-\frac{\theta}{\tau}}}{\left[1+\alpha e^{-\lambda_1(t-c\theta)}\right]^2}\,d\theta\\
&\quad\geqslant\frac{1}{1+\alpha e^{-\lambda_1 t}}-\frac{c\alpha\lambda_1\tau}{1-c\lambda_1\tau}\cdot\frac{e^{-\lambda_1 t}}{\left(1+\alpha e^{-\lambda_1 t}\right)^2}\\
&\qquad+\frac{\alpha}{1-c\lambda_1\tau}\cdot\frac{e^{-\lambda_1 t}}{\left(1+\alpha e^{-\lambda_1 t}\right)^2}\\
&\qquad-\frac{2c\alpha^2\lambda_1 e^{-2\lambda_1 t}}{1-c\lambda_1\tau}\int_0^{+\infty}\frac{e^{\left(2c\lambda_1-\frac{1}{\tau}\right)\theta}}{\left[1+\alpha e^{-\lambda_1(t-c\theta)}\right]^3}\,d\theta\\
&\qquad-\frac{\alpha}{1-c\lambda_1\tau-\lambda_1^2\tau}\cdot\frac{e^{-\lambda_1 t}}{\left(1+\alpha e^{-\lambda_1 t}\right)^2}
\end{aligned}
$$

$$+\frac{2c\alpha^2\lambda_1 e^{-2\lambda_1 t}}{1-c\lambda_1\tau-\lambda_1^2\tau}\int_0^{+\infty}\frac{e^{\left(2c\lambda_1-\frac{1}{\tau}\right)\theta}e^{\lambda_1^2\theta}}{\left[1+\alpha e^{-\lambda_1(t-c\theta)}\right]^3}\,d\theta$$

$$\geqslant \frac{1}{1+\alpha e^{-\lambda_1 t}}-\frac{c\alpha\lambda_1\tau}{1-c\lambda_1\tau}\cdot\frac{e^{-\lambda_1 t}}{\left(1+\alpha e^{-\lambda_1 t}\right)^2}$$

$$-\frac{\alpha\lambda_1^2\tau}{(1-c\lambda_1\tau)\left(1-c\lambda_1\tau-\lambda_1^2\tau\right)}\cdot\frac{e^{-\lambda_1 t}}{\left(1+\alpha e^{-\lambda_1 t}\right)^2}.$$

Therefore, we have

$$
\begin{aligned}
&-\psi''(t)+c\psi'(t)-f(\psi(t),(g*\psi)(t))\\
&\geqslant -\psi''(t)+c\psi'(t)-(1+a)\psi(t)+(1+a)\psi(t)(g*\psi)(t)\\
&=-\frac{-\alpha\lambda_1^2 e^{-\lambda_1 t}+\alpha^2\lambda_1^2 e^{-2\lambda_1 t}}{\left(1+\alpha e^{-\lambda_1 t}\right)^3}+\frac{c\alpha\lambda_1 e^{-\lambda_1 t}}{\left(1+\alpha e^{-\lambda_1 t}\right)^2}-\frac{1+a}{1+\alpha e^{-\lambda_1 t}}\\
&\quad+\frac{1+a}{\left(1+\alpha e^{-\lambda_1 t}\right)^2}-\frac{c\alpha\lambda_1\tau(1+a)}{1-c\lambda_1\tau}\cdot\frac{e^{-\lambda_1 t}}{\left(1+\alpha e^{-\lambda_1 t}\right)^3}\\
&\quad-\frac{\alpha\lambda_1^2\tau(1+a)}{(1-c\lambda_1\tau)\left(1-c\lambda_1\tau-\lambda_1^2\tau\right)}\cdot\frac{e^{-\lambda_1 t}}{\left(1+\alpha e^{-\lambda_1 t}\right)^3}\\
&=\frac{-\alpha^2\left[\lambda_1^2-c\lambda_1+(1+a)\right]e^{-2\lambda_1 t}}{\left(1+\alpha e^{-\lambda_1 t}\right)^3}\\
&\quad+\frac{\alpha\left[\lambda_1^2+c\lambda_1-(1+a)-\frac{c\lambda_1\tau(1+a)}{1-c\lambda_1\tau}-\frac{\lambda_1^2\tau(1+a)}{(1-c\lambda_1\tau)\left(1-c\lambda_1\tau-\lambda_1^2\tau\right)}\right]e^{-\lambda_1 t}}{\left(1+\alpha e^{-\lambda_1 t}\right)^3}\\
&=\frac{\alpha\left[2\lambda_1^2-\frac{c\lambda_1\tau(1+a)}{1-c\lambda_1\tau}-\frac{\lambda_1^2\tau(1+a)}{(1-c\lambda_1\tau)\left(1-c\lambda_1\tau-\lambda_1^2\tau\right)}\right]e^{-\lambda_1 t}}{\left(1+\alpha e^{-\lambda_1 t}\right)^3}.
\end{aligned}
$$

Since

$$2\lambda_1^2>0,\ \lim_{\tau\to 0}\frac{c\lambda_1\tau(1+a)}{1-c\lambda_1\tau}=0\quad\text{and}\quad\lim_{\tau\to 0}\frac{\lambda_1^2\tau(1+a)}{(1-c\lambda_1\tau)\left(1-c\lambda_1\tau-\lambda_1^2\tau\right)}=0$$

for sufficiently small $\tau > 0$, it follows that

$$-\psi''(t) + c\psi'(t) - f(\psi(t), (g * \psi)(t)) \geqslant 0.$$

The proof is complete. □

By Theorem 4.8(iii), we have the following result on the existence of a travelling wave front for Eq. (5.1) with a spatio-temporal delay.

Theorem 5.12. *For any $c \geqslant 2\sqrt{1+a}$, there exists $\tau^*(c) > 0$ such that for any $\tau < \tau^*(c)$, system* (5.1) *has a travelling wave front which satisfies* (5.3).

Remark 5.13. We remark that if $a = 0$, then (5.1) reduces to the famous Fisher-KPP equation with spatio-temporal delay of the form

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + u(t,x)\left[1 - (g * u)(t,x)\right], \tag{5.5}$$

and our results still hold. This equation has been studied by many researchers, for example, see [2,9,33] and the references therein.

Remark 5.14. If $g(t,x) = \delta(t)\,\delta(x)$, then (5.5) is a local Fisher-KPP equation without time delay

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + u(t,x)\left[1 - u(t,x)\right].$$

It is well-known [4,8] that it has a travelling wave front for each wave speed $c \geqslant 2$.

Remark 5.15. If $g(t,x) = \delta(t-\tau)\,\delta(x)$, then (5.5) reduces to

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + u(t,x)\left[1 - u(t-\tau,x)\right],$$

which has been considered by Wu and Zou [33] who obtained that for any $c > 2$, there exists $\tau^*(c) > 0$ such that for any $\tau < \tau^*(c)$, Eq. (5.5) has a travelling wave front which satisfies (5.3). However, using our upper and lower solutions defined by (5.4) (our lower solution is different from that in [33]), it follows that for any $c \geqslant 2$, there exists $\tau^*(c) > 0$ such that for any $\tau < \tau^*(c)$, Eq. (5.5) has a travelling wave front which satisfies (5.3). Obviously, our result improves that in [33].

Remark 5.16. To the best of our knowledge, it seems that little has been done for the case $g(t,x) = p(t)\,\delta(x)$. However, by letting

$$p(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}}, \quad \tau > 0,$$

we can show that Eq. (5.5) has a travelling wave front for any $c \geqslant 2$. Similarly, if we let

$$p(t) = \frac{t}{\tau^2} e^{-\frac{t}{\tau}}, \quad \tau > 0,$$

then a similar result still holds.

Remark 5.17. If $g(t,x) = \delta(t)\,q(x)$, then (5.5) becomes

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + u(t,x)\left[1 - \int_{-\infty}^{\infty} q(x-y)u(t,y)\,dy\right],$$

which is a nonlocal Fisher-KPP equation without time delay and has been studied by Gourley [9] and Billingham [3]. By a further restriction for the kernel g, for example,

$$q(x) = \frac{1}{2\rho} e^{-\frac{|x|}{\rho}} \ (\rho > 0),$$

and applying the perturbation theory of ordinary differential equations, Gourley [9] showed that for sufficiently small $\rho > 0$ and any fixed $c \geqslant 2$, (5.5) has a travelling wave front satisfying (5.3). By a careful observation, we can find that by using our method, we can also show that for any $c \geqslant 2$, there exists $\tau^*(c) > 0$ such that for any $\tau < \tau^*(c)$, system (5.5) with a kernel satisfying the restriction in [9] has a travelling wave front which satisfies (5.3), as what we did in Case 2. Thus, we can improve the results in [9] by using our method.

Remark 5.18. Ashwin et al. [2] considered the same case as that we consider in Case 3. By using the geometric singular perturbation theory of Fenichel [7], they showed that for sufficiently small $\tau > 0$ and any fixed $c \geqslant 2$, (5.5) has a travelling wave front satisfying (5.3) for

$$g(t,x) = \frac{1}{\tau} e^{-\frac{t}{\tau}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}, \quad \tau > 0.$$

Undoubtedly, our Theorem 5.12 improves the results of Ashwin et al. [2]. In addition, our method depends on the kernel less than that in [2].

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