Applied Mathematical Modelling 46 (2017) 203–217

Contents lists available at ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

A free boundary problem for *Aedes aegypti* mosquito invasion

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ARTICLE INFO

Article history:
Received 26 August 2016
Revised 8 January 2017
Accepted 17 January 2017
Available online 24 January 2017

MSC:
35B35
35K60
92B05

Keywords:
Mosquito invasion
Advection–reaction–diffusion model
Free boundary
Travelling wave

ABSTRACT

An advection–reaction–diffusion model with free boundary is proposed to investigate the invasive process of *Aedes aegypti* mosquitoes. By analyzing the free boundary problem, we show that there are two main scenarios of invasive regime: vanishing regime or spreading regime, depending on a threshold in terms of model parameters. Once the mortality rate of the mosquito becomes large with a small specific rate of maturation, the invasive mosquito will go extinct. By introducing the definition of asymptotic spreading speed to describe the spreading front, we provide an estimate to show that the boundary moving speed cannot be faster than the minimal traveling wave speed. By numerical simulations, we consider that the mosquitoes invasive ability and wind driven advection effect on the boundary moving speed. The greater the mosquito invasive ability or advection, the larger the boundary moving speed. Our results indicate that the mosquitoes asymptotic spreading speed can be controlled by modulating the invasive ability of winged mosquitoes.

1. Introduction

Invasions by insect vectors such as mosquitoes of human diseases have profound effects on global public health (Lounibos [1]). *Aedes aegypti* mosquito is an invasive domestic species with tropical and subtropical worldwide distribution and is an insect closely associated with humans and their dwellings. *A aegypti* mosquito is the primary transmitter of dengue (Carbajo et al. [2], Cummings et al. [3], Gubler [4], WHO [5,6]), chikungunya (Fischer and Staples [7], WHO [8]), and yellow fever (Cauchemez et al. [9], WHO [10]). Moreover, the epidemics of Zika virus in Latin America and the Caribbeans in 2016 had raised international public health concerns and it has been reported that *A. aegypti* mosquitoes transmit the Zika virus (Cao-Lormeau [11], Fauci and Morens [12], Gao et al. [13], Hayes [14], Hennessey et al. [15]).

It is known that temperature, humidity and rainfall impact adult *A. aegypti* survival and availability of oviposition sites. *A. aegypti* control programs aim to reduce the population density of adult mosquitoes below a critical threshold so that epidemics of the *A. aegypti* borne diseases are unlikely to occur. Vector population suppression programs usually involve the insecticide treatment of larval habitats. During outbreaks of mosquito-borne diseases, spraying insecticides is usually used as an emergency control measure to reduce the adult *A. aegypti* population. The influence of dispersal patterns of adult *A. aegypti* mosquitoes is critical to the long-term success of vector population suppression. Understanding the dispersal and invasive behavior of *A. aegypti* mosquitoes is essential in implementing vector control strategies.

The population structure and dynamics of *A. aegypti* mosquitoes are complex and influenced by environmental and geographical factors. Several interesting experimental and field studies have been performed since the beginning of the twen-

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http://dx.doi.org/10.1016/j.apm.2017.01.050

tieth century in order to study the dispersal of *A. aegypti*. These studies seem to show different results because the experimental conditions during the experiments and observational studies influence dramatically the dispersal pattern of the mosquitoes. In order to study the dispersal dynamics of *A. aegypti*, Takahashi et al. [16] developed an advection–reaction–diffusion model that accounts for the effect of wind and predicts the existence of stable traveling waves in several situations. Although Takahashi et al. gave an estimation of the speed of traveling waves of *A. aegypti*, it is the asymptotic wave speed that usually gives an approximation of the progressive spreading speed of *A. aegypti*, and it does not really show the spread of *A. aegypti* in the early stage of spatial expanding to larger areas. To describe the spatial spreading of *A. aegypti*, it is necessary to consider a free boundary. At the boundary front of a source, *A. aegypti* expands and pushes forward to cause further spreading till it extends to the whole area.

Free boundary problems have been used to study biological invasions. Du and Lin [17] proposed a biological invasive model to investigate how one species spreads into a new environment via a free boundary problem. Recently, Lin and Zhu [18] applied a free boundary problem to model the spatial spreading of West Nile virus in vector mosquitoes and host birds in North America and showed the spreading or vanishing of the boundary depends on the basic reproduction number. For a SIS model with spatial heterogeneity of environment, Ge et al. [19] proposed to use the basic reproduction numbers to dominate the spreading of infectious disease. In some other studies [20–22], researchers have given conditions for the spreading front expanding or vanishing in various advection–reaction–diffusion models. Other related mathematical results provided theoretical frame work for free boundary problems, see Guo and Wu [23], Lin [24], Wang and Zhao [25] and references therein.

In this paper, based on the modeling studies of *A. aegypti* in [16], we propose and study an advection–reaction–diffusion model with free boundary to explore the temporal-spatial transmission of the *A. aegypti* mosquitoes, where the population of the vector mosquitoes is described by a system for the two life stages: the winged form (mature female mosquitoes) and an aquatic population (eggs, larvae and pupae), the expanding front is expressed by a free boundary which models the spatial expanding of the source area. The female mosquitoes are initially located at a habitat, then spread to other places owing to their dispersal ability. In our model the dispersal ability includes the long distance dispersal as well as the short distance dispersal. The long distance dispersal of *A. aegypti* is caused by wind, while the short dispersal thanks to the random walk of each individual mosquito. From the viewpoint of mathematical modeling, the long distance dispersal is described by the advection while the short distance dispersal is described the classical Laplacian diffusion. The availability of such a model enables us to apply the free boundary theory to show the existence and uniqueness of the global solution to an advection–reaction–diffusion model. Moreover we estimate the boundedness of moving speed of the free boundary.

The rest of the paper is organized as follows. In Section 2 we present the advection–reaction–diffusion model with free boundary. The existence and uniqueness of solutions to the free boundary problem is considered in Section 3. Section 4 addresses the issue of boundary vanishing and Section 5 deals with the asymptotic spreading speed. Numerical simulations are presented in Section 6. The article ends with a brief discussion in Section 7.

2. Model derivation

Takahashi et al. [16] proposed an advection–reaction–diffusion model to describe the spatial dispersal dynamics of *A. aegypti* mosquitoes. Since mosquito invasion is an asymptotic process, the habitation of mosquitoes will change with time. In order to describe the dynamics of habitation, we assume that the habitation has a moving free boundary and formulate the free boundary problem as follows:

$$\begin{cases} M_t = \tilde{D}M_{xx} - \tilde{\nu}M_x + \tilde{\gamma}A(1-\frac{M}{\tilde{k}_1}) - \tilde{\mu}_1 M, & t>0,\ 0<x<\tilde{h}(t), \\ M(t,x)=0, & t>0,\ x\geq \tilde{h}(t), \\ A_t = \tilde{r}(1-\frac{A}{\tilde{k}_2})M - (\tilde{\mu}_2+\tilde{\gamma})A, & t>0,\ x>0, \\ M_x(t,0)=0,\ A_x(t,0)=0, & t>0, \\ M(t,\tilde{h}(t))=0,\ \tilde{h}'(t) = -\tilde{\mu}M_x(t,\tilde{h}(t)), & t>0, \\ \tilde{h}(0)=\tilde{h}_0,\ M(0,x)=M_0(x), & x\in[0,\tilde{h}_0], \\ A(0,x)=A_0(x), & x\in[0,\infty). \end{cases} \tag{2.1}$$

Here $M(t, x)$ is the spatial density of the winged mosquitoes at time t and space location x. Likewise $A(t, x)$ is the density of the aquatic mosquitoes. The winged mosquitoes are initially limited to a specific part of the domain $[0, \tilde{h}_0]$. To be more specific, we only consider the one-dimensional case. We assume that the aquatic mosquitoes migrate in the habitat $[0, \infty)$. The habitat of winged mosquitoes is affected by random diffusion and wind, whose boundary is described by $x = \tilde{h}(t)$. Outside the habitat, there is no winged mosquitoes. $M_x(t,0) = A_x(t,0) = 0$ indicate that on the left fixed boundary, there is no flux of winged mosquitoes and aquatic mosquitoes.

The biological meanings of the parameters are described as follows: $\tilde{D}$ is the diffusion rate, which is considered as the result of a random (and local) flying movement. $\tilde{\nu}$ is the advection rate which is caused by the wind blowing the winged mosquitoes. $\tilde{\gamma}$ is the specific rate of maturation of the aquatic form into winged female mosquitoes, and $\tilde{k}_1$ is the carrying capacity of winged mosquitoes. Likewise $\tilde{r}$ is the oviposition of winged female mosquitoes, and $\tilde{k}_2$ is the carrying capacity of aquatic mosquitoes. $\tilde{\mu}_1$ and $\tilde{\mu}_2$ are, respectively, the mortality rate of the winged mosquitoes and the aquatic forms.

In order to describe the moving of the boundary of winged mosquitoes, the free boundary satisfies the Stefan condition $\tilde{h}'(t) = -\tilde{\mu} M_x(t, \tilde{h}(t))$. $\tilde{\mu}$ is the winged mosquitoes' invasive ability on the free boundary, which can be introduced by using the Fickian conservation law. In view of the winged mosquitoes' random diffusion and the advection, the number of winged mosquitoes is transported at a flux $J = -\tilde{D}M_x + \tilde{\nu}M$ across the boundary $x = \tilde{h}(t)$. From time t to time $t + \Delta t$, the boundary moves from $\tilde{h}(t)$ to $\tilde{h}(t+\Delta t)$. We denote a function f by the total number of moving winged mosquitoes across the boundary. Then during the time interval $[t, t+\Delta t]$, the number of crossing boundary winged mosquitoes is

$$J\Delta t = (-\tilde{D}M_x + \tilde{\nu}M)\Delta t = f(\tilde{h}(t+\Delta t) - \tilde{h}(t)).$$

From the biological viewpoint, f is increasing because the total population number is increasing with respect to the length of habitat. Since $f(0) = 0$, the Taylor expansion of the function f gives

$$f(\tilde{h}(t+\Delta t) - \tilde{h}(t)) = f'(0)(\tilde{h}(t+\Delta t) - \tilde{h}(t)) + \frac{f''(0)}{2}(\tilde{h}(t+\Delta t) - \tilde{h}(t))^2 + \cdots .$$

Therefore, the flux is

$$-\tilde{D}M_x + \tilde{\nu}M = f'(0)\frac{\tilde{h}(t+\Delta t) - \tilde{h}(t)}{\Delta t} + \frac{f''(0)}{2}\frac{(\tilde{h}(t+\Delta t) - \tilde{h}(t))^2}{\Delta t} + \cdots .$$

Letting $\Delta t \to 0$, the flux becomes

$$-\tilde{D}M_x + \tilde{\nu}M = f'(0)\tilde{h}'(t).$$

Now denoting

$$\tilde{\mu} = \frac{\tilde{D}}{f'(0)} \text{ and } \tilde{\rho} = \frac{\tilde{\nu}}{f'(0)},$$

in view of on the boundary the spatial density of the winged mosquitoes $M(t, \tilde{h}(t)) = 0$, we obtain $\tilde{\rho}M(t, \tilde{h}(t)) = 0$. Hence we derive the boundary Stefan condition. In order to minimize the number of parameters involved in the model, we introduce the dimensionless variables. Set

$$u = \frac{1}{\tilde{k}_1}M,\ v = \frac{1}{\tilde{k}_2}A,\ \bar{t} = \tilde{r}t,\ \bar{x} = \sqrt{\frac{\tilde{r}}{\tilde{D}}}x. \tag{2.2}$$

Then the free boundary becomes $\sqrt{\frac{\tilde{r}}{\tilde{D}}}\tilde{h}(\frac{\bar{t}}{\tilde{r}})$. Denote it by $h(\bar{t}) \equiv \sqrt{\frac{\tilde{r}}{\tilde{D}}}\tilde{h}(\frac{\bar{t}}{\tilde{r}})$. For the sake of simplicity, we omit the caps of t and x. The problem (2.1) becomes

$$\begin{cases} u_t = u_{xx} - \nu u_x + \frac{\gamma}{k}v(1-u) - \mu_1 u, & t > 0,\ 0 < x < h(t), \\ u(t,x) = 0, & t > 0,\ x \ge h(t), \\ v_t = k(1-v)u - (\mu_2 + \gamma)v, & t > 0,\ x > 0, \\ u_x(t,0) = 0,\ v_x(t,0) = 0, & t > 0, \\ u(t,h(t)) = 0,\ h'(t) = -\mu u_x(t,h(t)), & t > 0, \\ h(0) = h_0,\ u(0,x) = u_0(x), & x \in [0,h_0], \\ v(0,x) = v_0(x), & x \in [0,\infty), \end{cases} \tag{2.3}$$

where $\mu_1 = \frac{\tilde{\mu}_1}{\tilde{r}}$, $\mu_2 = \frac{\tilde{\mu}_2}{\tilde{r}}$, $\gamma = \frac{\tilde{\gamma}}{\tilde{r}}$, $\nu = \frac{\tilde{\nu}}{\sqrt{\tilde{r}\tilde{D}}}$, $k = \frac{\tilde{k}_1}{\tilde{k}_2}$, $\mu = \frac{\tilde{k}_1\tilde{\mu}}{\sqrt{\tilde{r}\tilde{D}}}$, $h_0 = \sqrt{\frac{\tilde{r}}{\tilde{D}}}\tilde{h}_0$, $u_0(\bar{x}) = \frac{1}{\tilde{k}_1}M_0(\sqrt{\frac{\tilde{D}}{\tilde{r}}}x)$, and $v_0(\bar{x}) = \frac{1}{\tilde{k}_2}A_0(\sqrt{\frac{\tilde{r}}{\tilde{D}}}x)$.

Moreover, the initial conditions u_0 and v_0 are bounded functions with compact support, which describe a restricted distribution of population. We assume that u_0 and v_0 satisfy

$$\begin{aligned} &u_0 \in C^2([0,h_0]),\ \|u_0\|_{L^\infty([0,h_0])} \le 1,\ u_0'(0) = u_0(h_0) = 0, u_0 > 0, \\ &v_0 \in C^2([0,\infty)),\ \|v_0\|_{L^\infty([0,\infty))} \le 1,\ v_0'(0) = 0, v_0 > 0. \end{aligned} \tag{2.4}$$

In fact, $\|u_0\|_{L^\infty} \le 1$ corresponds with the biological meaning. According to the dimensionless transform $u = \frac{1}{\tilde{k}_1}M$, the density of the winged mosquito is less than the carrying capacity.

For the free boundary problem (2.3), as the same method in Guo and Wu [23], we define $\lim_{t\to\infty}\frac{h(t)}{t}$ as the asymptotic spreading speed. The study of asymptotic spreading speed plays an important role in invasive ecology since it can be used to predict the mean spreading rate of species. For some biological invasive mathematical models, Petrovskii et al. [26] gave a classification of invasive regime in terms of the invasive population's spatial distribution. When the invasive population tends to 0, the biological invasion is an extinct regime. Otherwise, the biological invasion is a geographical spread regime. Our main aim is to consider both the dynamics of the free boundary and the distribution of the winged mosquito population. When the winged mosquito population tends to 0 and the free boundary is bounded, the biological invasion is called a *vanishing regime*. When the winged mosquitoes exist and the free boundary tends to infinity, the biological invasion is called a *spreading regime*.

3. Existence and uniqueness

In this section, we first present the following local existence and uniqueness result by using the contraction mapping theorem. The proof can be done by modifying the argument of Wang and Zhao [25]. So we omit the details.

Theorem 3.1. *For any given* (u_0, v_0) *satisfying (2.4) and any* $\alpha \in (0, 1)$, *there is a* $T > 0$ *such that problem (2.3) admits a unique bounded solution*

$$(u, v, h) \in C^{\frac{1+\alpha}{2}, 1+\alpha}(\bar{D}_T) \times C^{\frac{1+\alpha}{2}, 1+\alpha}([0, T] \times [0, \infty)) \times C^{1+\frac{\alpha}{2}}([0, T]);$$

Moreover,

$$\|u\|_{C^{\frac{1+\alpha}{2}, 1+\alpha}(\bar{D}_T)} + ||h||_{C^{1+\frac{\alpha}{2}}([0,T])} \leq C, \tag{3.1}$$

where

$$D_T = \{(t, x) \in \mathbb{R}^2 : t \in (0, T], x \in (0, h(t))\}.$$

Here positive constants C and T depend only on $h_0, \alpha, \|u_0\|_{C^2([0,h_0])}$, *and* $\|v_0\|_{C^2([0,\infty))}$.

To show that the local solution obtained in Theorem 3.1 can be extended to all $t > 0$, we need the following estimate.

Lemma 3.1. *Let* (u, v, h) *be a solution to problem (2.3) defined for* $t \in (0, T]$ *for some* $T \in (0, +\infty]$. *Then there exist constants* M_1 *and* M_2 *independent of T such that*

$$\begin{aligned} &0 < u(t, x) \leq M_1 \ \text{ for } 0 \leq t \leq T, \ 0 \leq x \leq h(t). \\ &\ \ 0 < v(t, x) \leq M_2 \ \text{ for } 0 \leq t \leq T, \ 0 \leq x < \infty. \end{aligned} \tag{3.2}$$

Proof. Since $h(t)$ is fixed, we now consider the following fixed parabolic boundary problem:

$$\begin{cases} u_t = u_{xx} - \nu u_x + \frac{\gamma}{k} v(1 - u) - \mu_1 u, & 0 < t \leq T, \ 0 < x < h(t), \\ u(t, x) = 0, & 0 < t \leq T, \ x \geq h(t), \\ v_t = k(1 - v)u - (\mu_2 + \gamma)v, & 0 < t \leq T, \ 0 < x < h(t), \\ u_x(t, 0) = 0, \ v_x(t, 0) = 0, & 0 < t \leq T, \\ u(t, h(t)) = 0, \ v(t, h(t)) = v_0(h_0)e^{-(\mu_2+\gamma)t}, & 0 < t \leq T, \\ h(0) = h_0, \ u(0, x) = u_0(x), & x \in [0, h_0], \\ v(0, x) = v_0(x), & x \in [0, \infty). \end{cases} \tag{3.3}$$

Applying the strong maximum principle to (3.3), we obtain

$$(\overline{u}, \overline{v}) > (0, 0) \text{ for } (t, x) \in ([0, T] \times [0, h(t)]. \tag{3.4}$$

We apply the upper and lower solutions theorem of Pao [27] to show the global boundedness of the solution. We use the same notation of upper and lower solutions as that in [27] and define $(\overline{u}, \overline{v})$ as a upper solution of system (3.3) provided that

$$\begin{cases} \overline{u}_t = \overline{u}_{xx} - \nu \overline{u}_x + \frac{\gamma}{k} \overline{v}(1 - \overline{u}) - \mu_1 \overline{u}, & 0 < t \leq T, \ 0 < x < h(t), \\ \overline{u}(t, x) \geq 0, & 0 < t \leq T, \ x \geq h(t), \\ \overline{v}_t = k(1 - \overline{v})\overline{u} - (\mu_2 + \gamma)\overline{v}, & 0 < t \leq T, \ 0 < x < h(t), \\ \overline{u}_x(t, 0) \leq 0, \ \overline{v}_x(t, 0) \leq 0, & 0 < t \leq T, \\ \overline{u}(t, h(t)) \geq 0, \ \overline{v}(t, h(t)) \geq v_0(h_0)e^{-(\mu_2+\gamma)t}, & 0 < t \leq T, \\ \overline{u}(0, x) \geq u_0(x), & x \in [0, h_0], \\ \overline{v}(0, x) \geq v_0(x), & x \in [0, \infty). \end{cases} \tag{3.5}$$

Otherwise, if $(\underline{u}, \underline{v})$ satisfies the inverse inequalities of (3.5), then it is called a lower solution of system (3.3).

Next, we seek the upper and lower solutions of system (3.3) by constructing the proper ordinary differential system. Let $(\overline{u}, \overline{v})$ be a solution of the following system:

$$\begin{cases} \overline{u}' = \frac{\gamma}{k} \overline{v}(1 - \overline{u}) - \mu_1 \overline{u}, & 0 < t \leq T, \\ \overline{v}' = k(1 - \overline{v})\overline{u} - (\mu_2 + \gamma)\overline{v}, & 0 < t \leq T, \\ \overline{u}|_{t=0} = \sup_{x \in [0,h_0]} u_0(x), \ \overline{v}|_{t=0} = \sup_{x \in [0,\infty)} v_0(x). \end{cases} \tag{3.6}$$

Then on the boundary of problem (2.3), we have $\overline{u}(t, h(t)) > u(t, h(t)) = 0$ and $\overline{v}(t, h(t)) \geq \overline{v}(0)e^{-(\mu_2+\gamma)t} \geq v_0(h_0)e^{-(\mu_2+\gamma)t} = v(t, h(t))$ for $0 < t \leq T$, and $\overline{u}_x(t, 0) = 0 \geq u_x(t, 0)$ for $0 < t \leq T$. On the other hand, we have $\overline{v}_x(t, 0) = 0 \geq v_x(t, 0)$ for $0 < t \leq T$. Moreover, for the initial conditions, it is easy to see that $\overline{u}|_{t=0} \geq u_0(x)$ for $0 < x < h_0$, and $\bar{v}|_{t=0} \geq v_0(x)$ for $0 < x < \infty$. Then $(\bar{u}, \bar{v})$ is an upper solution to system (3.3). Applying the upper and lower solutions theorem (Theorem 2.1 of Pao [27]), we have

$$\bar{u} \geq u, \ \ \bar{v} \geq v \text{ for } (t,x) \in [0,T] \times [0,h(t)]. \tag{3.7}$$

The solution of the ordinary differential equation (3.6) possesses the upper boundedness:

$$\begin{aligned} &\bar{u} \leq \max\{\sup_{x\in[0,h_0]} u_0(x), 1\} \text{ for } (t,x) \in ([0,T] \times [0,h(t)], \\ &\bar{v} \leq \max\{\sup_{x\in[0,h_0]} v_0(x), 1\} \text{ for } (t,x) \in [0,T] \times [0,h(t)]. \end{aligned} \tag{3.8}$$

Combining (3.4), (3.7) and (3.8), we obtain

$$\begin{aligned} &0 \leq u \leq \max\{\sup_{x\in[0,h_0]} u_0(x), 1\} \text{ for } (t,x) \in [0,T] \times [0,h(t)], \\ &0 \leq v \leq \max\{\sup_{x\in[0,h_0]} v_0(x), 1\} \text{ for } (t,x) \in [0,T] \times [0,h(t)]. \end{aligned} \tag{3.9}$$

For $(t, x) \in (0, T] \times [h(t), \infty)$, problem (2.3) implies that v satisfies

$$v_t = -(\mu_2 + \gamma)v. \tag{3.10}$$

Thus v is monotonically decreasing for $(t, x) \in [0, T] \times [h(t), \infty)$. We have

$$0 < v \leq \sup_{x\in[0,h_0]} v_0(x) \text{ for } (t,x) \in (0,T] \times [h(t),\infty). \tag{3.11}$$

Combining (3.9) and (3.11), we have

$$\begin{aligned} &0 \leq u \leq \max\{\sup_{x\in[0,h_0]} u_0(x), 1\} \text{ for } (t,x) \in [0,T] \times [0,h(t)], \\ &0 \leq v \leq \max\{\sup_{x\in[0,h_0]} v_0(x), 1\} \text{ for } (t,x) \in [0,T] \times [0,\infty). \end{aligned} \tag{3.12}$$

Using a similar argument, we choose ($\underline{u}$, $\underline{v}$) as a solution of the following system:

$$\begin{cases} \underline{u}' = \frac{\gamma}{k}\underline{v}(1-\underline{u}) - \mu_1\underline{u}, & 0 < t \leq T, \\ \underline{v}' = k(1-\bar{v})\underline{u} - (\mu_2+\gamma)\underline{v}, & 0 < t \leq T, \\ \underline{u}|_{t=0} = \inf_{x\in[0,h_0]} u_0(x), \ \ \underline{v}|_{t=0} = \inf_{x\in[0,\infty)} v_0(x). \end{cases} \tag{3.13}$$

Then ($\underline{u}$, $\underline{v}$) is a lower solution to system (3.3). Applying the upper and lower solutions theorem of Pao [27] one more time, we have

$$0 < \underline{u} \leq u, \ \ 0 < \underline{v} \leq v \text{ for } (t,x) \in [0,T] \times [0,h(t)]. \tag{3.14}$$

In view of (3.11) and (3.14), we have

$$\begin{aligned} &0 < u \leq \max\{\sup_{x\in[0,h_0]} u_0(x), 1\} \text{ for } (t,x) \in [0,T] \times [0,h(t)], \\ &0 < v \leq \max\{\sup_{x\in[0,h_0]} v_0(x), 1\} \text{ for } (t,x) \in [0,T] \times [0,\infty). \end{aligned} \tag{3.15}$$

By choosing

$$M_1 = \max\{\sup_{x\in[0,h_0]} u_0(x), 1\}, \ \ M_2 = \max\{\sup_{x\in[0,\infty)} v_0(x), 1\}, \tag{3.16}$$

we can see that (3.2) holds. The proof is complete. □

Lemma 3.2 (Hopf Lemma)**.** *Let (u, v, h) be a solution to problem (2.3) defined for $t \in (0, T]$ for some $T \in (0, +\infty]$. Then*

$$u_x(t,h(t)) < 0 \ \text{ for } \ t \in (0,T].$$

Proof. For any $T_0 \in (0, T]$, we denote $Q_{T_0} := \{0 < t \leq T_0, \ 0 < x < h(t)\}$. We plot a line $x = h(T_0) - \beta(T_0 - t)$ across the boundary point $(T_0, h(T_0))$, where the constant $\beta > 0$ is chosen as follows: when $h'(T_0) \leq 0$, β is an arbitrary positive constant; when $h'(T_0) > 0$, $\beta = 2h'(T_0)$. Thus for sufficiently small $\delta > 0$, it is easy to see that the triangular area $\overline{\Delta}_{T_0} = \{T_0 - \frac{\delta}{\beta} \leq t \leq T_0, \ h(T_0) - \delta \leq x \leq h(T_0) - \beta(T_0 - t)\}$ belongs to the interior of Q_{T_0} except the boundary point $(T_0, h(T_0))$.

In $\overline{\Delta}_{T_0}$, we consider an auxiliary function $w(t,x) = e^{-\alpha x} - e^{-\alpha[h(T_0)-\beta(T_0-t)]}$. Using the boundedness of u and v, we have

$$u_t - u_{xx} + \nu u_x = \frac{\gamma}{k}v - (\frac{\gamma}{k}v + \mu_1)u \geq -(\frac{\gamma M_2}{k} + \mu_1)u. \tag{3.17}$$

Define a linear operator

$$Lu := u_t - u_{xx} + \nu u_x + \left(\frac{\gamma M_2}{k} + \mu_1\right)u. \tag{3.18}$$

Substituting the auxiliary function $w(t, x)$ into (3.18), we have

$$Lw = \alpha\beta e^{-\alpha[h(T_0)-\beta(T_0-t)]} - (\alpha^2 + \alpha\nu)e^{-\alpha x} + \left(\frac{\gamma M_2}{k} + \mu_1\right)(e^{-\alpha x} - e^{-\alpha[h(T_0)-\beta(T_0-t)]}). \tag{3.19}$$

As long as $\alpha > \max\{\beta, \ \frac{\gamma M_2 + k\mu_1}{k\nu}\}$, we have $Lw < 0$. Set $z(t,x) = u(t,x) - u(T_0, h(T_0)) - \varepsilon w(t,x)$, where ε is a sufficiently small positive constant such that $w(t, h(T_0) - \delta)$ satisfies

$$\varepsilon \max_{t\in[\frac{T_0-\delta}{\beta}, T_0]} |w(t, h(T_0) - \delta)| < \min_{t\in[\frac{T_0-\delta}{\beta}, T_0]} [u(t, h(T_0) - \delta) - u(T_0, h(T_0))]. \tag{3.20}$$

In fact, in view of $u(t, x) > 0$, the right hand of (3.20) has a positive lower bound. Thus there exists a positive ε such that on the boundary of $\overline{\Delta}_{T_0}$, it follows from (3.20) that:

$$\begin{aligned} z|_{x=h(T_0)-\beta(T_0-t)} &= u(t, h(T_0) - \beta(T_0 - t)) - u(T_0, h(T_0)) > 0, \\ z|_{x=h(T_0)-\delta} &= u(t, h(T_0) - \delta) - u(T_0, h(T_0)) - \varepsilon w(t, h(T_0) - \delta) > 0, \\ z(T_0, h(T_0)) &= 0. \end{aligned} \tag{3.21}$$

In the interior of $\overline{\Delta}_{T_0}$, since

$$Lz \geq -\varepsilon Lw > 0,$$

we have

$$\begin{aligned} z|_{x=h(T_0)-\beta(T_0-t)} &= u(t, h(T_0) - \beta(T_0 - t)) - u(T_0, h(T_0)) > 0, \\ z|_{x=h(T_0)-\delta} &= u(t, h(T_0) - \delta) - u(T_0, h(T_0)) - \varepsilon w(t, h(T_0) - \delta) > 0, \\ z(T_0, h(T_0)) &= 0. \end{aligned} \tag{3.22}$$

then $z(t, x)$ attains its minimum nonnegative value on $\overline{\Delta}_{T_0}$ at $(T_0, h(T_0))$. Therefore,

$$0 \leq \frac{\partial z(T_0, h(T_0))}{\partial x} = \frac{\partial u(T_0, h(T_0))}{\partial x} - \varepsilon\frac{\partial w(T_0, h(T_0))}{\partial x}.$$

Hence,

$$\frac{\partial u(T_0, h(T_0))}{\partial x} \geq \varepsilon\frac{\partial w(T_0, h(T_0))}{\partial x} = -\varepsilon\alpha e^{-\alpha h(T_0)} < 0. \tag{3.23}$$

By the arbitrariness of T_0, we have $u_x(t, h(t)) < 0$ for all $t \in (0, T]$. The proof is complete. □

The next lemma shows that the free boundary for problem (2.3) is strictly monotone increasing.

Lemma 3.3. *Let (u, v, h) be a solution to problem (2.3) defined for $t \in (0, T]$ for some $T \in (0, +\infty]$. Then there exists a constant M_3 independent of T such that*

$$0 < h'(t) \leq M_3 \ \text{ for } \ t \in (0, T].$$

Proof. Using the Hopf Lemma (Lemma 3.2), we have $h'(t) > 0$ for $t \in (0, T]$. Next we show that $h'(t) \leq M_3$ for all $t \in (0, T]$ and some M_3 independent of T. As in Lin [24], we define

$$\Omega =: \{(t, x): \ 0 < t \leq T, \ h(t) - \frac{1}{M} < x < h(t)\},$$

and construct an auxiliary function

$$w(t, x) := M_1[2M(h(t) - x) - M^2(h(t) - x)^2].$$

We will choose M so that $w(t, x) \geq u(t, x)$ holds over Ω.

By the boundedness of solutions (Lemma 3.1), for $(t, x) \in \Omega$ we have

$$\begin{aligned} &w_t = 2M_1Mh'(t)(1 - M(h(t) - x)) \geq 0, \\ &w_x = 2M_1M^2(h(t) - x) - 2M_1M > -2M_1M, \\ &-w_{xx} = 2M_1M^2, \\ &\frac{\gamma}{k}v(1 - u) - \mu_1 u \leq \frac{\gamma}{k}M_2(1 + M_1) + \mu_1 M_1. \end{aligned}$$

Thus,

$$w_t - w_{xx} + \nu w_x \geq 2M_1M(M-\nu) \geq \frac{\gamma}{k}M_2(1+M_1) + \mu_1 M_1 \text{ for } (t,x) \in \Omega, \tag{3.24}$$

if

$$M \geq \nu + \sqrt{\frac{1}{2M_1}(\frac{\gamma}{k}M_2(1+M_1)+\mu_1M_1)}. \tag{3.25}$$

On the other hand, we have

$$\begin{aligned} &w\Big(t, h(t) - \frac{1}{M}\Big) = M_1 \geq u(t, h(t) - \frac{1}{M}) \text{ for } t \in (0,T), \\ &w(t,h(t)) = 0 = u(t,h(t)) \text{ for } t \in (0,T). \end{aligned} \tag{3.26}$$

Hence, if we can choose M such that

$$u_0(x) \leq w(0,x) \text{ for } x \in [h_0 - \frac{1}{M}, h_0], \tag{3.27}$$

in view of (3.24), (3.26), and (3.27), we can apply the maximum principle to $w-u$ over Ω to deduce that $u(t, x) \leq w(t, x)$ for $(t, x) \in \Omega$. Then $u_x(t,h(t)) \geq w_x(t,h(t)) = -2MM_1$. It would then follow that

$$h'(t) = -\mu u_x(t,h(t)) \leq 2MM_1\mu := M_3. \tag{3.28}$$

To complete the proof, we only have to find some M independent of T such that (3.27) holds. We have

$$w_x(0,x) = -2M_1M(1-M(h_0-x)) \leq -M_1M \text{ for } x \in \Big[h_0 - \frac{1}{2M}, h_0\Big].$$

In view of (3.25), choosing

$$M := \max\left\{\nu + \sqrt{\frac{1}{2M_1}(\frac{\gamma}{k}M_2(1+M_1)+\mu_1M_1)},\ \frac{4\|u_0\|_{C^1([0,h_0])}}{3M_1}\right\}, \tag{3.29}$$

we have

$$w_x(0,x) \leq -MM_1 \leq -\frac{4}{3}||u_0||_{C^1}([0,h_0]) \leq u_0'(x) \text{ for } r \in [h_0-(2M)^{-1}, h_0].$$

Since $w(0,h_0) = u_0(h_0) = 0,$ the above inequality implies that

$$w(0,x) \geq u_0(x) \text{ for } x \in [h_0 - \frac{1}{2M}, h_0].$$

Moreover, for $x \in [h_0 - \frac{1}{M}, h_0 - \frac{1}{2M}],$ we have

$$w(0,x) \geq \frac{3}{4}M_1,\ u_0(x) \leq \frac{1}{M}\|u_0\|_{C^1([0,h_0])} \leq \frac{3}{4}M_1.$$

Therefore $u_0(x) \leq w(0, x)$ for $x \in [h_0 - \frac{1}{M}, h_0]$. This completes the proof. □

Theorem 3.2. *The solution of problem (2.3) exists and is unique for all $t \in (0, \infty)$.*

Proof. It follows from the uniqueness of solutions (Theorem 3.1) that there is a number T_{max} such that $[0, T_{max})$ is the maximal time interval in which the solution exists. Now we prove that $T_{max} = \infty$ by a contradiction argument. Assume that $T_{max} < \infty$. Then it follows from Lemmas 3.1 that there exist M_1, M_2 and M_3 independent of T_{max} such that for $t \in [0, T_{\max})$ and $x \in [0, h(t)]$,

$$\begin{aligned} &0 < u(t,x) \leq M_1 \ \text{ for } 0 \leq t \leq T,\ 0 \leq x \leq h(t). \\ &0 < v(t,x) \leq M_2 \ \text{ for } 0 \leq t \leq T,\ 0 \leq x < \infty. \\ &h_0 \leq h(t) \leq h_0 + M_3 t,\ 0 \leq h'(t) \leq M_3 \text{ for } t \in [0, T_{\max}). \end{aligned}$$

By the standard L^p estimates and the Sobolev embedding theorem, we can find a constant $C > 0$ depending only on $M_i (i = 1, 2)$ such that

$$\|u(t,\cdot)\|_{C^{1+\frac{\alpha}{2}}([0,h(t)])} \leq C$$

and v is continuous for $(t, x) \in [0, T_{max}) \times [0, \infty)$. It then follows from the proof of Theorem 3.1 that there exists a $\tau > 0$ depending only on C and $M_i(i=1,2)$ such that the solution of problem (2.3) with initial time $T_{max} - \frac{\tau}{2}$ can be extended uniquely to the time $T_{max} - \frac{\tau}{2} + \tau$. But this contradicts the assumption and hence the proof is complete. □

4. Vanishing regime

We next study whether the transmission of mosquito is geographical spreading or vanishing depends on a threshold in terms of model parameters

$$R_0 := \frac{\gamma}{\mu_1(\mu_2+\gamma)}, \tag{4.1}$$

which is derived following a standard technique used by van den Driessche and Watmough [28] in calculating the basic reproduction number.

It follows from Lemma 3.3 that $x = h(t)$ is monotonically increasing and, therefore, there exists $h_\infty \in (0, +\infty]$ such that $\lim_{t\to+\infty} h(t) = h_\infty$. When $h_\infty = \infty$, the free boundary will spread to the whole space, we call this case as *boundary spreading*. Otherwise, when $h_\infty < \infty$, the free boundary will limit to a region, we call this case as *boundary vanishing*.

Theorem 4.1. *If* $R_0 < 1$ *and* $\gamma = 0$, *then* $\lim_{t\to+\infty} ||u(\cdot,t)||_{C([0,h(t)])} = 0$ *and* $h_\infty < \infty$. *Moreover,* $\lim_{t\to+\infty} v(t,x) = 0$ *uniformly in any bounded subset of* $[0, \infty)$.

Proof. After using a similar argument as in Lemma 3.1, we have

$$\begin{aligned} &0 < u \le \overline{u} \text{ for } (t,x) \in [0,\infty)\times[0,h(t)],\\ &0 < v \le \overline{v} \text{ for } (t,x) \in [0,\infty)\times[0,\infty), \end{aligned} \tag{4.2}$$

where $(\overline{u},\overline{v})$ is the solution of the following system:

$$\begin{cases} \overline{u}' = \frac{\gamma}{k}\overline{v}(1-\overline{u}) - \mu_1\overline{u},\\ \overline{v}' = k(1-\overline{v})\overline{u} - (\mu_2+\gamma)\overline{v},\\ \overline{u}|_{t=0} = \sup_{x\in[0,h_0]} u_0(x),\ \ \overline{v}|_{t=0} = \sup_{x\in[0,\infty)} v_0(x). \end{cases} \tag{4.3}$$

Now, we consider the global stability of $(\overline{u},\overline{v})$ to system (4.3). We define the following Lyapunov function:

$$V(t) = \frac{k}{2\gamma}\overline{u}^2 + \frac{1}{2k}\overline{v}^2, \tag{4.4}$$

where $(\overline{u},\overline{v})$ is an arbitrary positive solution of system (4.3). It is easy to see that $V(t)$ is positive definite. We calculate the time derivative of $V(t)$ along the solutions of (4.3):

$$\begin{aligned} \frac{dV(t)}{dt} &= -\frac{k\mu_1}{\gamma}\overline{u}^2 + 2\overline{uv} - \frac{\mu_2+\gamma}{k}\overline{v}^2 - \overline{vu}^2 - \overline{uv}^2\\ &\le -\frac{k\mu_1}{\gamma}\overline{u}^2 + 2\overline{uv} - \frac{\mu_2+\gamma}{k}\overline{v}^2. \end{aligned} \tag{4.5}$$

Hence, for any $t > 0$, $R_0 < 1$ ensures that $\frac{dV(t)}{dt} \le 0$ for all $\overline{u},\overline{v} \ge 0$. Notice that $\frac{dV(t)}{dt} = 0$ if and only if $(\overline{u},\overline{v}) = (0,0)$. Thus, (0, 0) is globally asymptotically stable to system (4.3); that is,

$$\lim_{t\to\infty}\overline{u} = \lim_{t\to\infty}\overline{v} = 0. \tag{4.6}$$

In view of (4.2), we have $\lim_{t\to+\infty} ||u(\cdot,t)||_{C([0,h(t)])} = 0$. Moreover, $\lim_{t\to+\infty} v(t,x) = 0$ uniformly in any bounded subset of $[0, \infty)$.

Next we show that $h_\infty < +\infty$. Some direct calculation yields

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_0^{h(t)} u(t,x)\mathrm{d}x &= \int_0^{h(t)} u_t(t,x)\mathrm{d}x + h'(t)u(t,h(t))\\ &= \int_0^{h(t)} (u_{xx} - \nu u_x)\mathrm{d}x + \int_0^{h(t)} (\frac{\gamma}{k}v(1-u) - \mu_1 u)\mathrm{d}x, \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_0^{h(t)} v(t,x)\mathrm{d}x &= \int_0^{h(t)} v_t(t,x)\mathrm{d}x + h'(t)v(t,h(t))\\ &= \int_0^{h(t)} k(1-v)u - (\mu_2+\gamma)v\mathrm{d}x. \end{aligned} \tag{4.8}$$

Choosing $m = \frac{\mu_1}{k}$ and using the fact that $R_0 < 1$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{h(t)}(u(t,x)+mv(t,x))\mathrm{d}x &= \int_0^{h(t)}(u_{xx}-\nu u_x)\mathrm{d}x\\ &\quad+\int_0^{h(t)}\Big(\frac{\gamma}{k}v-\frac{\gamma}{k}uv-\mu_1u+mku-mkuv-m(\mu_2+\gamma)v\Big)\mathrm{d}x\\ &\le \int_0^{h(t)}(u_{xx}-\nu u_x)\mathrm{d}x+\int_0^{h(t)}\frac{\gamma}{k}v(1-\frac{1}{R_0})\mathrm{d}x\\ &\le \int_0^{h(t)}(u_{xx}-\nu u_x)\mathrm{d}x,\\ &= u_x(t,h(t))-u_x(t,0)-\nu u(t,h(t))+\nu u(t,0),\\ &= -\frac{h'(t)}{\mu}+\nu u(t,0).\end{aligned} \tag{4.9}$$

Integrating (4.9) from 0 to t gives

$$\int_0^{h(t)}(u(t,x)+mv(t,x))\mathrm{d}x \le \int_0^{h(0)}(u_0+mv_0)\mathrm{d}x+\frac{h(0)}{\mu}-\frac{h(t)}{\mu}+\nu\int_0^t u(s,0)\mathrm{d}s. \tag{4.10}$$

It is followed from (4.10) that

$$\frac{h(t)}{\mu}\le \int_0^{h(0)}(u_0+mv_0)\mathrm{d}x+\frac{h(0)}{\mu}+\nu\int_0^t u(s,0)\mathrm{d}s. \tag{4.11}$$

Since that $\nu=0$, by letting $t\to\infty$, we have $h_\infty<\infty$. The proof is complete. □

In order to study the asymptotic behavior of solutions for problem (2.3), we give an estimate for boundary movement.

Theorem 4.2. *Let (u, v, h) be any solution of (2.3). If $R_0<1$ and $\nu=0$, then there exists a constant $K>0$ such that*

$$\|u(t,\cdot)\|_{C^1([0,h(t)])}<K,\quad \forall t\ge 1. \tag{4.12}$$

Moreover,

$$\lim_{t\to\infty}h'(t)=0. \tag{4.13}$$

Proof. Since $R_0<1$ and $\nu=0$, we have $h_\infty<\infty$. We now straighten the free boundary. Consider a new transformation

$$(t,x)\to(t,y)\ \text{with}\ x=h(t)y.$$

The above transformation changes the free boundary $x=h(t)$ to the line $y=1$. Now if we set

$$w(t,y)=u(t,h(t)y),\quad z(t,y)=v(t,h(t)y),$$

then w satisfies the following fixed boundary problem:

$$\begin{cases} w_t=\frac{1}{h(t)^2}w_{yy}-\frac{\nu}{h(t)}w_y+\frac{\gamma}{k}z(1-w)-\mu_1w, & t>0,\ 0<y<1,\\ w_y(t,0)=0,\ w(t,1)=0, & t>0,\\ w(0,y)=u_0(h_0y), & y\in[0,1].\end{cases} \tag{4.14}$$

By using the Schauder estimates, we obtain that there exists a positive constant K_0 such that

$$\|w\|_{C^{\frac{1+\alpha}{2},1+\alpha}([1,\infty)\times[0,1])}<K_0. \tag{4.15}$$

Since $u_x(t,x)=\frac{1}{h(t)}w_y(t,y)$, there exists a constant K such that (4.12) holds.

It remains to prove (4.13). Note $u_x(t,h(t))=\frac{1}{h(t)}w_y(t,1)$ and $h'(t)=-\mu u_x(t,h(t))$, by virtue of $0<h'(t)<M_3$ and $\|w_y(\cdot,y)\|_{C^{\frac{\alpha}{2}}([1,\infty))}<K_0$, we obtain

$$\|h'\|_{C^{\frac{\alpha}{2}}([1,\infty))}<L, \tag{4.16}$$

where L depends on K_0 and M_3. It follows from $h'(t)>0$, $h_\infty<\infty$ and (4.16) that (4.13) holds. The proof is complete. □

Remark 4.1. From Theorems 4.1 and 4.2, we can see that when $R_0<1$, the invasive mosquitoes will evolve to an immediate state in which the habitat extends to h_∞ while the terminal state of invasive mosquitoes is extinct. Here the conclusion that the terminal state is extinct coincides with the result in Takahashi et al. [16].

5. Spreading regime

From a biological point of view, we first establish the existence of a coexistent equilibrium. Consider the following Cauchy problem corresponding to (2.3):

$$\begin{cases} u_t = u_{xx} - \nu u_x + \frac{\gamma}{k} v(1-u) - \mu_1 u, & t > 0,\ -\infty < x < \infty, \\ v_t = k(1-v)u - (\mu_2+\gamma)v, & t > 0,\ -\infty < x < \infty, \\ u(0,x) = u_0(x),\ v(0,x) = v_0(x), & -\infty < x < \infty. \end{cases} \tag{5.1}$$

It is noted that the Cauchy problem (5.1) has a steady state. Moreover, after some directly computations, we have the following lemma:

Lemma 5.1. *If $R_0 > 1$, then problem (2.3) has a unique coexistent steady state $E^* = (u^*, v^*)$, where*

$$u^* = \frac{\gamma - \mu_1(\mu_2+\gamma)}{\mu_1 k + \gamma},\ \ v^* = \frac{k(\gamma - \mu_1\mu_2 - \mu_1\gamma)}{\gamma(k+\mu_2+\gamma)}. \tag{5.2}$$

In order to study the case that the threshold $R_0 > 1$, and for later applications, we need a comparison principle, which can be used to estimate $u(t, x)$, $v(t, x)$ and the free boundary $x = h(t)$. As in Du and Lin [17], the following comparison lemma can be obtained analogously.

Lemma 5.2. *Suppose that $T \in (0, \infty)$, $\overline{h} \in C^1([0,T])$, $\overline{v} \in C([0,T]\times[0,\infty)) \cap C^{1,2}((0,T]\times(0,\infty)) \le 1$, $\overline{u} \in C(\overline{D_T^*}) \cap C^{1,2}(D_T^*) \le 1$ with $D_T^* = \{(t,x) \in \mathbb{R}^2 :\ 0 < t \le T, 0 < x < \overline{h}(t)\}$, and*

$$\begin{cases} \overline{u}_t \ge \overline{u}_{xx} - \nu \overline{u}_x + \frac{\gamma}{k}\overline{v}(1-\overline{u}) - \mu_1 \overline{u}, & 0 < t \le T,\ 0 < x < \overline{h}(t), \\ \overline{u}(t,x) = 0, & 0 < t \le T,\ x = \overline{h}(t), \\ \overline{v}_t \ge k(1-\overline{v})\overline{u} - (\mu_2+\gamma)\overline{v}, & 0 < t \le T,\ x > 0, \\ \overline{u}_x(t,0) \le 0,\ \overline{v}_x(t,0) \le 0, & 0 < x < \overline{h}(t), \\ \overline{h}'(t) \ge -\mu \overline{u}_x(t, \overline{h}(t)), & 0 < t \le T, \\ \overline{h}(0) \ge h_0,\ \overline{u}(0,x) \ge u_0(x), & x \in [0, h_0], \\ \overline{v}(0,x) \ge \overline{v}_0(x), & x \in [0,\infty). \end{cases} \tag{5.3}$$

Then $(\overline{u}, \overline{v}, \overline{h})$ is called an upper solution of (2.3). Moreover, the solution (u, v, h) of the free boundary problem (2.3) satisfies

$$\begin{aligned} & v(t,x) \le \overline{v}(t,x),\ h(t) \le \overline{h}(t) \ \text{ for } t \in (0,T],\ x \in (0,\infty), \\ & u(t,x) \le \overline{u}(t,x) \ \text{ for } t \in (0,T],\ x \in (0,h(t)). \end{aligned} \tag{5.4}$$

In order to estimate the asymptotic spreading speed of the boundary, we consider traveling wave solutions of problem (2.3). A traveling wave solution of system (2.3) is a solution of the special form $U(z) = u(x-ct), V(z) = v(x-ct)$, where $c > 0$ is the wave speed. Substituting this special solution into problem (2.3), we then obtain the corresponding wave equations:

$$\begin{cases} U''(z) + (c-\nu)U'(z) + \frac{\gamma}{k}(1-U(z))V(z) - \mu_1 U(z) = 0, \\ V'(z) + \frac{k}{c}(1-V(z))U(z) - \frac{\mu_2+\gamma}{c}V(z) = 0, \\ U(-\infty) = u^*,\ V(-\infty) = v^*,\ U(\infty) = 0,\ V(\infty) = 0 \\ U'(z) < 0,\ V'(z) < 0. \end{cases} \tag{5.5}$$

As for (5.5), it is shown in Takahashi et al. [16] that when $R_0 > 1$ there exists a traveling wave connecting the extinct equilibrium and the coexistent equilibrium.

Lemma 5.3. *Suppose that $R_0 > 1$, then there exists a traveling wave connecting the extinct equilibrium and the coexistent equilibrium for (5.5). Moreover, the smallest travelling wave speed c_{min} is defined by the equation $P_0(\lambda, c_{min}) = 0$, where*

$$\lambda = \frac{1}{3}\left(\frac{\mu_2+\gamma}{c} + \nu - c - \sqrt{\left(\frac{\mu_2+\gamma}{c} + \nu - c\right)^2 + 3\left(\mu_1 - \frac{(\nu-c)(\mu_2+\gamma)}{c}\right)}\right),$$

$$P_0(\lambda, c) = -\lambda^3 + \left(\frac{\mu_2+\gamma}{c} + \nu - c\right)\lambda^2 + \left(\mu_1 - \frac{(\nu-c)(\mu_2+\gamma)}{c}\right)\lambda - \frac{\mu_1(\mu_2+\gamma)-\gamma}{c}.$$

Lemma 5.4 (Fluctuation Lemma [29])**.** *Let f: $[b, \infty) \to \mathbf{R}$ be bounded and differentiable. Then there exist sequences $\{s_k\}$ and $\{t_k\}$ such that when $k \to \infty$,*

$$f(s_k) \to f_\infty \equiv \limsup_{t\to\infty} f(t),\ \ f'(s_k) \to 0,$$

$$f(t_k) \to f^\infty \equiv \liminf_{t\to\infty} f(t), \;\; f'(t_k) \to 0.$$

Theorem 5.1. *Assume that* $R_0 > 1$. *Let* (u, v, h) *be a solution of (2.3) with* $h_\infty = \infty$. *Then the asymptotic spreading speed of spreading front satisfies*

$$\limsup_{t\to\infty} \frac{h(t)}{t} \le c_{min}. \tag{5.6}$$

Proof. We are going to construct a suitable upper solution to (2.3) and then apply Lemma 5.2. Using a similar argument of Guo and Wu [23], we choose sufficiently large M such that $MU(z) > ||u_0||_{L^\infty([0,h_0])}$ and $MV(z) > ||v_0||_{L^\infty([0,\infty))}$ for all $z \in [0, h_0]$. Next, fix $\sigma_0 > h_0$ depending on M and μ such that

$$U(\sigma_0) < \min_{x\in[0,h_0]}(U(x) - \frac{u_0(x)}{M}), \;\; V(\sigma_0) < \min_{x\in[0,h_0]}(V(x) - \frac{v_0(x)}{M}), \tag{5.7}$$

$$U(\sigma_0) \le 1 - \frac{1}{M}, \;\; V(\sigma_0) \le 1 - \frac{1}{M}. \tag{5.8}$$

According to Lemma 5.3, $U(z)$ and $V(z)$ are bounded differentiable decreasing functions. By the Fluctuation Lemma (Lemma 5.4), there exist a sequence $\{z_k\}$ such that $U'(z_k) \to 0$ as $z_k \to \infty$. When M is fixed, we can seek sufficiently large σ_0 such that

$$-\mu MU'(\sigma_0) < c_{min}, \;\; -\mu MV'(\sigma_0) < c_{min}. \tag{5.9}$$

Here when we choose sufficiently large σ_0, the inequalities (5.7) and (5.8) are still satisfied because $U(z)$ and $V(z)$ are decreasing with respect to z.

Now, set $\sigma(t) = \sigma_0 + c_{min}t$ and define

$$\bar{u}(t,x) = MU(x - c_{min}t) - MU(\sigma_0), \;\; \bar{v}(t,x) = MV(x - c_{min}t) - MV(\sigma_0).$$

It is obvious that

$$\bar{u}(t,\sigma(t)) = \bar{v}(t,\sigma(t)) = 0 \text{ for } t \in (0,\infty). \tag{5.10}$$

Since that $U'(z) < 0$ and $V'(z) < 0$, we have

$$\bar{u}_x(t,0) < 0, \;\; \bar{v}_x(t,0) < 0 \text{ for } t \in (0,\infty). \tag{5.11}$$

It is deduced from (5.7) that

$$\bar{u}(t,x) > u_0(x), \;\; \bar{v}(t,x) > v_0(x) \text{ for } x \in [0,h_0]. \tag{5.12}$$

By using (5.5) and (5.8), we have

$$\begin{aligned} &\bar{u}_t - \bar{u}_{xx} + \nu\bar{u}_x - \frac{\gamma}{k}\bar{v}(1-\bar{u}) + \mu_1\bar{u} \ge 0 \text{ for } (t,x) \in (0,\infty)\times(0,\sigma(t)),\\ &\bar{v}_t - k(1-\bar{v})\bar{u} + (\mu_2+\gamma)\bar{v} \ge 0 \text{ for } (t,x) \in (0,\infty)\times(0,\infty). \end{aligned} \tag{5.13}$$

Because of (5.9), we have

$$\sigma'(t) = c_{min} > -\mu MU'(\sigma_0) = -\mu\bar{u}_x(t,\sigma(t)). \tag{5.14}$$

It follows from (5.10)–(5.14) that $(\bar{u}(t,x), \bar{v}(t,x), \sigma(t))$ is an upper solution of (2.3). In view of Lemma 5.2, $\sigma(t) \ge h(t)$. Therefore

$$\limsup_{t\to\infty} \frac{h(t)}{t} \le \lim_{t\to\infty} \frac{\sigma(t)}{t} = c_{min}.$$

The proof is complete. □

6. Numerical simulations

In this section, we provide numerical computations of problem (2.3) by means of a finite difference scheme (Razvan and Gabriel [30]), using a Crank–Nicholson for time integration, and Adams–Bashforth scheme for the nonlinear operator.

In Takahashi et al. [16], the following parameter values of model (2.3) were given:

$$\gamma = 0.25, k = 6.66\times10^{-3}, \mu_1 = 1.33\times10^{-3}, \mu_2 = 3.33\times10^{-4}. \tag{6.1}$$

The values for the dimensional parameters are $D = 1.25\times10^{-2}$, $\tilde{\gamma} = 0.2$, $\tilde{k}_1 = 25$, $\tilde{\mu}_1 = 4\times10^{-2}$, $\tilde{r} = 30$, $\tilde{k}_2 = 100$, $\tilde{\mu}_2 = 1\times10^{-2}$. Here the time unit is day and the space unit is km. We can compute the unique endemic steady state and the basic reproduction number as follows:

$$(u^*, v^*) = (0.9991, 0.0259) \text{ and } R_0 = 750.8795. \tag{6.2}$$

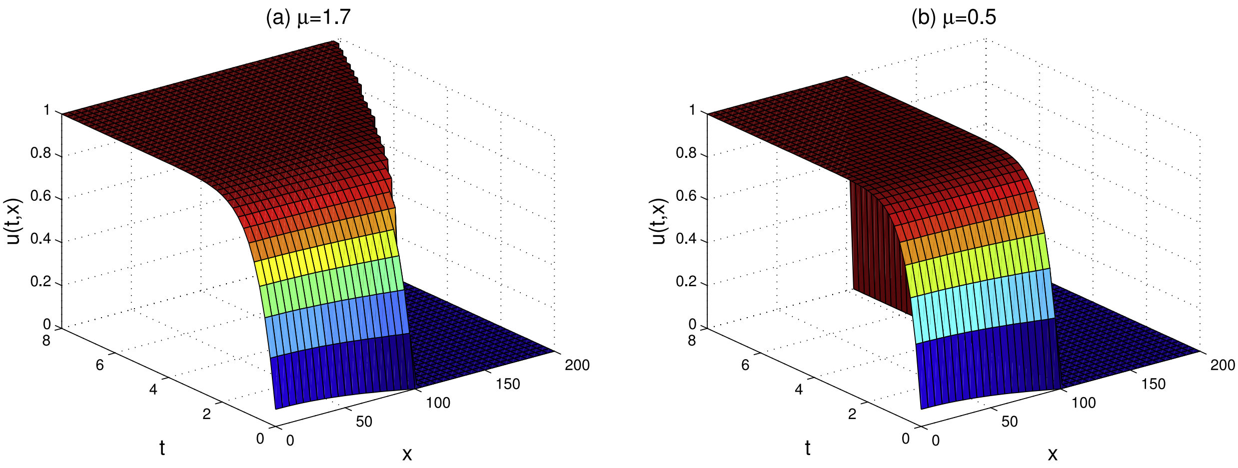

Fig. 1. A comparison densities of winged female mosquitoes between (a) large invasive ability and (b) small invasive ability. Here $\nu = 8.164 \times 10^{-2}$, and other parameters are given in (6.1).

Now we present details for numerical simulations of the terminal state of winged *Aedes aegypti* mosquitoes. Theorem 5.1 implies that the terminal state has a maximal asymptotic spreading speed regardless of the initial data. In other words, the initial data can only affect the immediate state, not the terminal state. Consider an initial data as follows:

$$u_0(x) = \begin{cases} 0.08 \cos \frac{\pi * x}{2h_0}, & x \in [0, h_0], \\ 0, & x > h_0, \end{cases} \quad \text{and } v_0(x) = \begin{cases} 0.0259 \cos \frac{\pi * x}{2h_0}, & x \in [0, h_0], \\ 0, & x > h_0. \end{cases} \tag{6.3}$$

It is easy to check that $R_0 > 1$. In a fixed region the advection–reaction–diffusion model means that the winged female mosquitoes will disperse to the whole space. However, in our free boundary model, the winged female mosquitoes do not necessarily disperse to the whole space; that is, h_∞ will be less than ∞ or equal to ∞. In the following, we will illustrate that the spreading of winged female mosquitoes depends on the invasive ability and the advection.

6.1. Invasive ability of female mosquitoes driven boundary spreading

Firstly we simulate how the value of the boundary movement pressure μ affects the boundary spread. From Theorem 5.1, the minimal speed of traveling wave does not depend on μ. In this case we set $\nu = 8.164 \times 10^{-2}$, which corresponds to the dimensional parameter $\tilde{\nu} = 5 \times 10^{-2}$ km day^{-1}. The non-dimensional minimal speed of traveling waves is $c_{min} = 2.57$. In order to for the mosquitoes to expand, we set a large invasive ability $\mu = 1.7$, which corresponds to the dimensional parameter $\tilde{\mu} = 4.16 \times 10^{-2}$. From Fig. 1(a), we can see that the habitat of female mosquitoes prevails from the initial area [0, 100] to [0, 200] after $t = 8$. Meanwhile, the density of female mosquitoes will increase. Moreover, the density will tends to a steady state u^* confined in some finite domain. Hence, we illustrate the case of population expansion. When we set a small invasive ability $\mu = 0.5$, which corresponds to the dimensional parameter $\tilde{\mu} = 1.12 \times 10^{-2}$, the habitat of female mosquitoes almost does not increase (Fig. 1(b)). In this case the population does not expand. For this two cases, we illustrate the orbits of boundary in Fig. 2. For a large invasive ability, the length of boundary maintains a steady growth (Fig. 2(a)). For a small invasive ability, the length of boundary admits a finite upper bound (Fig. 2(b)). Moreover, when the mosquito population expands, we estimate that the speed of boundary is about 0.625, which is less than the minimal speed of traveling waves.

6.2. Advection driven boundary spreading

Secondly we simulate how boundary spreading depends on the advection. When we set a small $\mu = 0.5$ and a small $\nu = 8.164 \times 10^{-2}$, the disease does not spread (Fig. 3(a)). Once again in order to for the mosquitoes to expand, we set a large advection $\nu = 1.2$, which corresponds to the dimensional parameter $\tilde{\nu} = 2.94 \times 10^{-2}$. Theorem 5.1 indicates that when $\mu = 0.5$, the non-dimensional minimal speed of traveling waves is $c_{min} = 4.05$. From Fig. 3(b), we can see that the habitat of female mosquitoes prevails from a short initial area to a long final area after $t = 8$. Meanwhile, the density of female mosquitoes increases. Moreover, the density tends to a steady state u^* confined in some finite domain. We also illustrate the orbits of boundary in Fig. 4. For a small advection, the length of boundary admits a finite upper bound (Fig. 4(a)). For a large advection, the length of boundary maintains a steady growth (Fig. 4(b)). Moreover, when the mosquito population expands, we estimate that the speed of boundary is about 0.535, which is also less than the minimal speed of traveling waves.

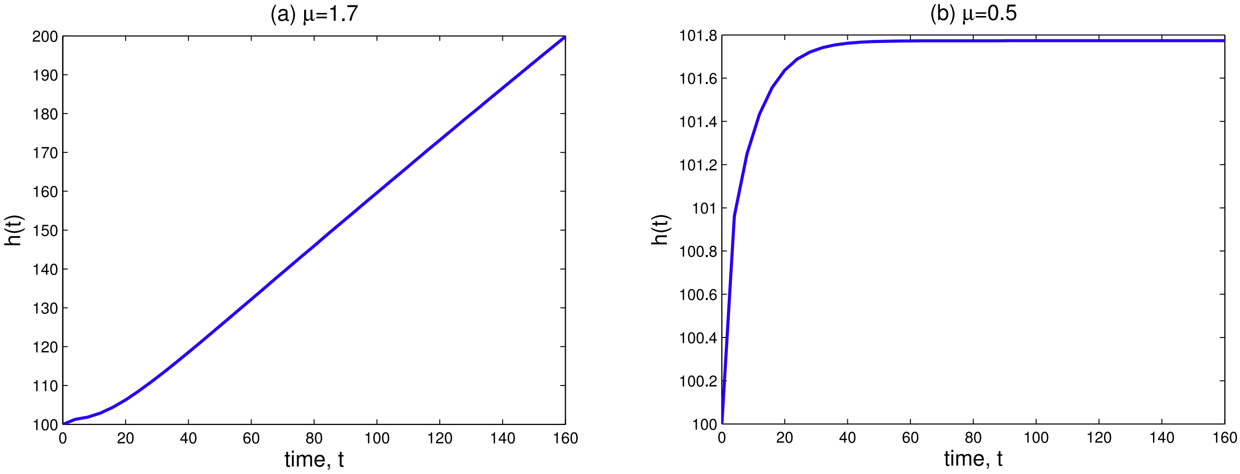

Fig. 2. A comparison of free boundary movement between (a) large invasive ability and (b) small invasive ability. Here $\nu = 8.164 \times 10^{-2}$, and other parameters are given in (6.1).

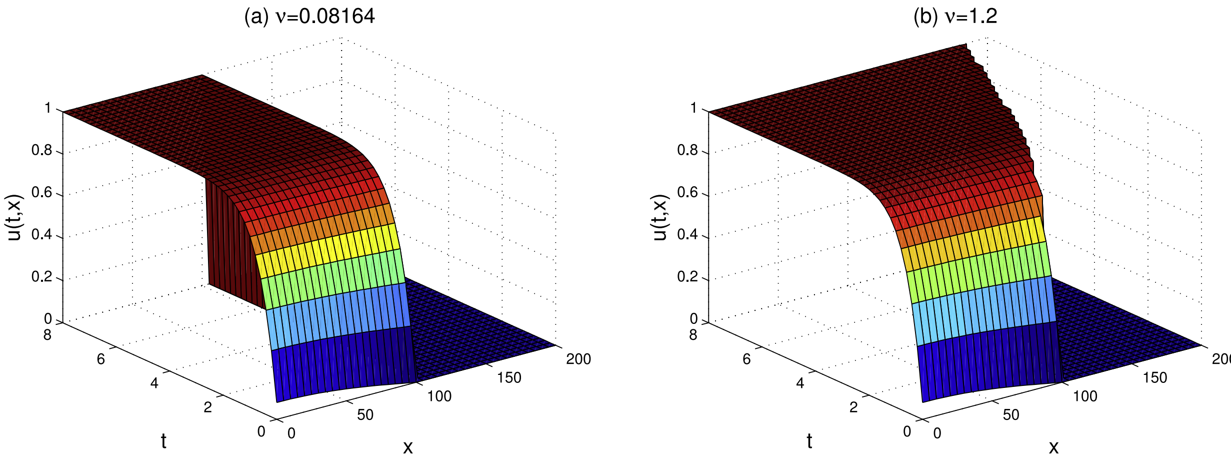

Fig. 3. A comparison of densities of winged female mosquitoes between (a) small advection and (b) large advection. Here $\mu = 0.5$, and other parameters are given in (6.1).

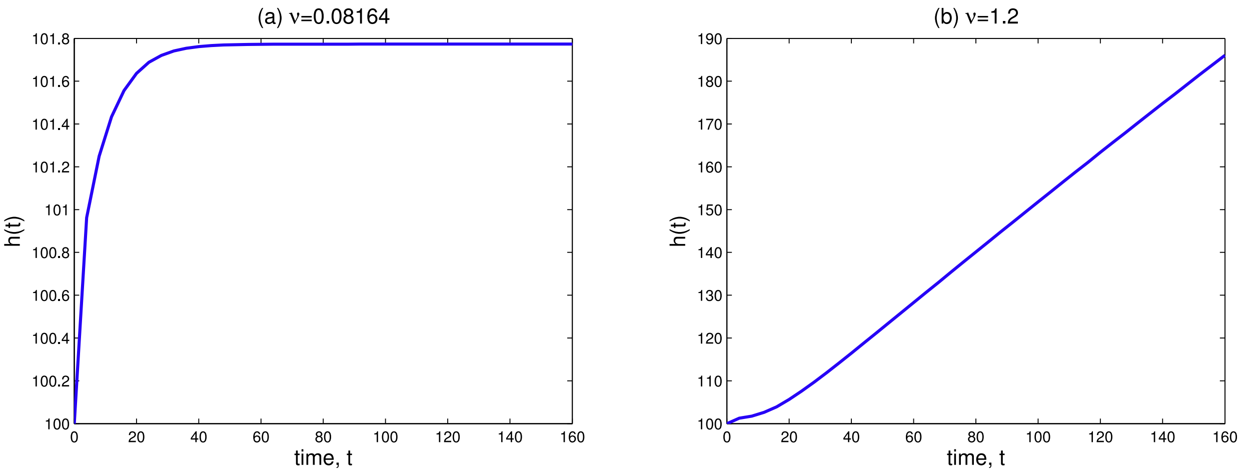

Fig. 4. A comparison of free boundary movement between (a) small advection and (b) large advection. Here $\mu = 0.5$, and other parameters are given in (6.1).

7. Discussion

In this paper, we have constructed an advection–reaction–diffusion model with free boundary to study the spatial dispersal dynamics of two *A. aegypti* mosquitoes sub-populations, where the free boundary describes the asymptotic behavior of spreading fronts. Our main attempt was to estimate the asymptotic spreading speed of the boundary, by which we can precisely predict the spatio-temporal domain of *A. aegypti* mosquitoes. It is worth mentioning that the spreading speed of the same advection–reaction–diffusion model described the *A. aegypti* mosquitoes has been estimated via traveling wave solutions (Takahashi et al. [16]). Comparing the methods of traveling wave solutions and free boundary, the asymptotic spreading speed of the free boundary cannot be faster than the minimal traveling speed provided that $R_0 > 1$ and spreading successes (Theorem 5.1). In the case of $R_0 < 1$, vanishing always happens and *A. aegypti* mosquitoes die out (Theorem 4.1). Moreover, in the case of $R_0 > 1$, the asymptotic spreading speed of the free boundary depends on the invasive ability of *A. aegypti* mosquitoes. We illustrated that strong invasive ability of *A. aegypti* mosquitoes and advection may induce the spreading of the boundary (Figs. 1 and 3). The biological meanings are as follows: when the rate of maturation is small and the mortality rates of winged and aquatic *A. aegypti* mosquitoes are large, the mosquitoes become extinct and the invasive boundary vanishes; when the rate of maturation is large and the mortality rates of winged and aquatic *A. aegypti* mosquitoes are small, the invasive boundary may spread under strong invasive ability of *A. aegypti* mosquitoes and large advection.

Dengue, chikungunya and Zika viruses are transmitted by *A. aegypti* mosquitoes, which are inhabiting in hot and humid climate regions (Cummings et al. [3]). Minimizing the mosquito population is an effective method to control these mosquito-borne viral diseases. In fact the Pan American Health Organization had carried out a programe for the eradication of *A. aegypti* in Americas. But with the growth of population and lack of public health services, *A. aegypti* mosquitoes have re-infested in some Latin American countries (Vasconcelos et al. [31]). Hence it is very important to understand the *A. aegypti* mosquito spreading dynamics for controlling mosquito-borne viral diseases. The results of our free boundary model conclude that when the rate of maturation is large and the mortality rates of winged and aquatic *A. aegypti* mosquitoes are small, the region is under a high risk of dengue, chikungunya and Zika epidemics. Recently as human transportation increases dramatically, the *Aedes aegypti* mosquitoes have the capability of the long distance movement (Oteroa et al. [32]). *Aedes aegypti* mosquito invasion has become a widely monitored scenario (Powell [33]). It has been reported that *Aedes aegypti* mosquitoes have increased their distributions in California (Gloria-Soria et al. [34]) and breed year-round in Washington D.C. (Lima et al. [35]). Our numerical simulations illustrate that the advection may contribute the mosquito population expansion.

Acknowledgments

The authors would like to thank the handling editor and three anonymous reviewers for their helpful comments and suggestions. Research of the first author (CT) was supported by the Overseas Study Program of Talent Young Teachers of Jiangsu Province. Research of the second author (SR) was partially supported by NSF grant DMS-1412454.

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