Periodic solutions of planar systems with two delays

Shigui Ruan
Department of Mathematics and Statistics, Dalhousie University, Halifax, Nova Scotia, Canada B3H 3J5

Junjie Wei
Department of Mathematics, Northeast Normal University, Changchun, Jilin 130024, People's Republic of China

(MS received 16 December 1997; accepted 15 July 1998)

In this paper, we consider a planar system with two delays:

$$\begin{aligned}\dot{x}_1(t) &= -a_0x_1(t) + a_1F_1(x_1(t-\tau_1), x_2(t-\tau_2)),\\ \dot{x}_2(t) &= -b_0x_2(t) + b_1F_2(x_1(t-\tau_1), x_2(t-\tau_2)).\end{aligned}$$

Firstly, linearized stability and local Hopf bifurcations are studied. Then, existence conditions for non-constant periodic solutions are derived using degree theory methods. Finally, a simple neural network model with two delays is analysed as an example.

1. Introduction

In delay differential equations, periodic solutions can arise through the (local) Hopf bifurcation. However, these periodic solutions exist only locally since they are created when the bifurcation parameter, say α, passes through a critical value, say α_0, and exist only for α in a small neighbourhood of α_0. It is natural to ask if the non-constant periodic solutions exist globally, that is, if they exist for all parameter $\alpha \geqslant \alpha_0$.

In the last two decades, a great deal of research has been devoted to the global Hopf bifurcation of delay differential equations. The scalar delay differential equation,

$$\dot{x}(t) = -x(t) + f(x(t-\tau)), \tag{1.1}$$

which has been used to model a variety of biological and physical phenomena, has been studied by many researchers (cf. [2, 11, 17, 18] and references therein). The paper of Grafton [10] is the pioneering work in studying the global existence of periodic solutions of two-dimensional systems with delay (see also [13, 19, 20]), but his systems were derived from scalar second-order delay differential equations, so a delay appears only in the second equation of the systems. Similarly, Leung [16] and Zhao *et al.* [26] investigated the existence of non-constant periodic solutions for predator–prey systems with a delay in the predator equation. For the planar delay differential system with a single delay in both equations,

$$\left.\begin{aligned} \dot{x}_1(t) &= -x_1(t) + \alpha F_1(x_1(t-1), x_2(t-1)), \\ \dot{x}_2(t) &= -x_2(t) + \alpha F_2(x_1(t-1), x_2(t-1)), \end{aligned}\right\} \tag{1.2}$$

the first piece of research was done by Táboas [23], where $\alpha > 0$ is a constant, F_1 and F_2 are bounded C^3 functions on $\mathbb{R}^2$ satisfying

$$\frac{\partial F_1}{\partial x_2}(0,0) \neq 0, \qquad \frac{\partial F_2}{\partial x_1}(0,0) \neq 0,$$

and the *negative feedback conditions*:

$$\left.\begin{aligned} x_2 F_1(x_1, x_2) > 0, \quad x_2 \neq 0, \\ x_1 F_2(x_1, x_2) < 0, \quad x_1 \neq 0. \end{aligned}\right\} \tag{1.3}$$

Táboas showed that there is an $\alpha_0 > 0$ such that for any $\alpha > \alpha_0$, there is a non-constant periodic solution with period greater than 4. Further study on the global existence of periodic solutions to system (1.2) can be found in [1] and [8]. All together there are very few results on the global existence of periodic solutions of planar systems with a delay appearing in both equations, especially there are no results involving planar systems with two delays.

In this paper, we are interested in the following planar system with two delays

$$\left.\begin{aligned} \dot{x}_1(t) &= -a_0 x_1(t) + a_1 F_1(x_1(t-\tau_1), x_2(t-\tau_2)), \\ \dot{x}_2(t) &= -b_0 x_2(t) + b_1 F_2(x_1(t-\tau_1), x_2(t-\tau_2)), \end{aligned}\right\} \tag{1.4}$$

where $a_0 > 0$, $b_0 > 0$, a_1 and b_1 are constants, and F_1 and F_2 satisfy the following assumption:

$$\left.\begin{aligned} &F_j \in C^3(\mathbb{R}^2), \quad F_j(0,0) = 0, \quad \frac{\partial F_j}{\partial x_j}(0,0) = 0, \quad j = 1, 2, \\ &\frac{\partial F_1}{\partial x_2}(0,0) \neq 0, \quad \frac{\partial F_2}{\partial x_1}(0,0) \neq 0, \quad x_2 F_1(x_1, x_2) \neq 0, \quad \text{for } x_2 \neq 0 \\ &\qquad \text{and} \quad x_1 F_2(x_1, x_2) \neq 0, \quad \text{for } x_1 \neq 0. \end{aligned}\right\} \tag{H$_1$}$$

The method of showing the existence of non-constant periodic solutions used by the above-mentioned researchers came from a widely known idea of Jones [15]. In this seminal paper, Jones introduced the idea of finding a cone in the phase space that maps into itself under a certain operator defined by the flow. The fixed points of this operator are corresponding to periodic solutions of differential equations. The cone is easy to construct, but some other complications arise because most problems have zero as an equilibrium point, which corresponds to the trivial periodic solution. Thus, one needs to find non-zero fixed points of the flow operator knowing in advance that zero is a fixed point. This can be achieved by applying a theorem due to Nussbaum [20], which depends on the ejectivity of fixed points of the flow operator. For a more applicable form of Nussbaum's theorem, we refer to [3] and [12].

In this paper, instead of applying Nussbaum's theorem, we shall use a different approach, the degree theory, to study the global existence of periodic solutions of system (1.4). Degree theory has been employed to develop global Hopf bifurcation theory for delay differential equations since the work of [4]. Here, we shall use the global Hopf bifurcation theorem in [7], which was established using a purely topological argument. For more related results and references, we refer to the monograph of Wu [25].

We would like to make a few remarks about our system (1.4). First, deferring from the system of Táboas, the coefficients a_0 and b_0 in the non-delay terms are not necessarily 1. In fact, the Hopf bifurcation analysis in the paper of Táboas very much depended on the unitizing of these coefficients. Because of this, we have to analyse a general transcendental equation in the local Hopf bifurcation analysis. Second, we do not require the negative feedback conditions on F_1 and F_2. Third, there are two delays appearing in our system. This is significant since even scalar equations with two delays are difficult to analyse (cf. [21]). We believe that even our local analysis is new in the literature.

We should mention that we have to assume

$$\frac{\partial F_1}{\partial x_1}(0,0) = \frac{\partial F_2}{\partial x_2}(0,0) = 0.$$

Notice that, as pointed out in [8], this assumption was also made by Táboas [23] (see also [1]). The first example in which this assumption is satisfied is the example used by Táboas [23] and Baptistini and Táboas [1]:

$$\left.\begin{aligned} \dot{x}_1(t) &= -x_1(t) + \alpha \arctan x_2(t-1), \\ \dot{x}_2(t) &= -x_2(t) + \alpha \arctan x_1(t-1). \end{aligned}\right\} \tag{1.5}$$

The second example is a neural network model without self-connection,

$$\left.\begin{aligned} \dot{x}_1(t) &= -x_1(t) + a_{12} f(x_2(t-\tau_2)), \\ \dot{x}_2(t) &= -x_2(t) + a_{21} f(x_1(t-\tau_1)), \end{aligned}\right\} \tag{1.6}$$

which was studied in [9] and [22], where a_{12}, a_{22}, $\tau_1 > 0$ and $\tau_2 > 0$ are constants, $f(u) = \tanh u$ (see also [24]). The above-mentioned authors investigated the linearized stability and delay-induced oscillations in system (1.6). We shall discuss global existence of non-constant periodic solutions to system (1.6) with a general function f.

The paper is organized as follows. In § 2, local Hopf bifurcation analysis is carried out. The global existence of periodic solutions is discussed in § 3. As an example, system (1.6) is analysed in § 4.

2. Hopf bifurcation analysis

In this section, we first consider a general transcendental equation,

$$\lambda^2 + p\lambda + q(\mu)\mathrm{e}^{-\lambda\tau} + r = 0, \tag{2.1}$$

where $p > 0$, $\tau > 0$ and r are constants, $\mu \in \mathbb{R}$ is a parameter.

LEMMA 2.1. *Suppose that $q(\mu) \in C^1(\mathbb{R})$, $q'(\mu) > 0$ and $q(0) = 0$. Let*

$$\omega_j \in \left(\frac{2(j-1)\pi}{\tau}, \frac{(4j-1)\pi}{2\tau}\right), \quad j = 1, 2, \dots,$$

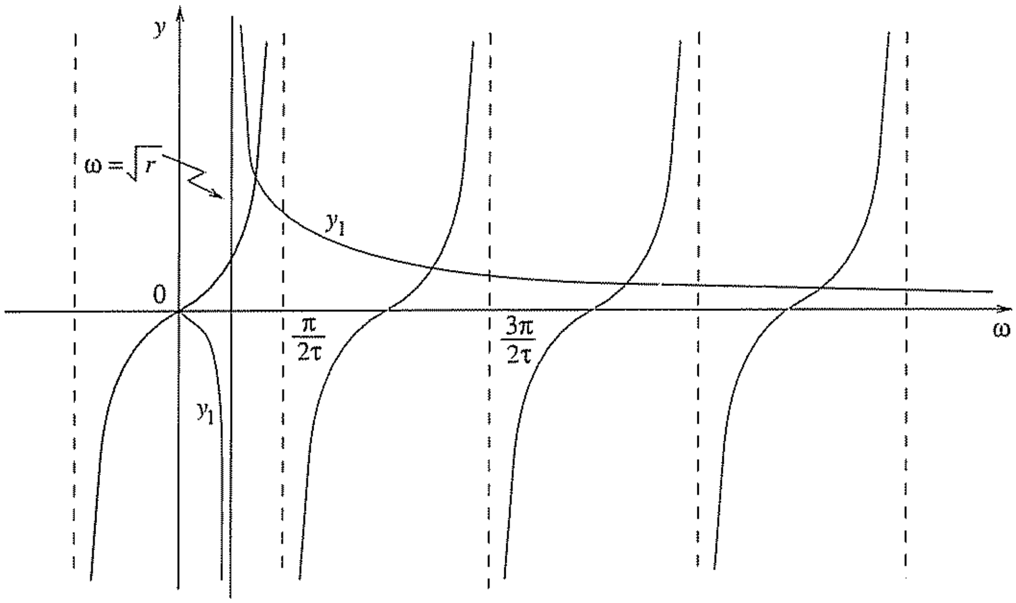

Figure 1. The points of intersection of $y_1 = [p\omega/(\omega^2 - r)]$ and $y_2 = \tan\omega\tau$ when $\sqrt{r} < [\pi/2\tau]$.

be the roots of the equation

$$\frac{p\omega}{\omega^2 - r} = \tan\omega\tau. \tag{2.2}$$

If there exists a sequence $\{\mu_j\}_{j\in N}$ *such that*

$$q(\mu_j) = [p\omega_j/\sin\omega_j\tau], \tag{2.3}$$

then equation (2.1) with $\mu = \mu_j$ *has a pair of purely imaginary roots* $\pm \mathrm{i}\omega_j$*, which are simple. Moreover, there exists a* $\mu_0 = \mu_{j_0} \in \{\mu_j\}$*, such that all other roots of equation (2.1) with* $\mu \in [0, \mu_0)$ *have strictly negative real parts provided that* $r \geqslant 0$*.*

In order to prove lemma 2.1, we give several claims. First, we know that $\pm\mathrm{i}\omega$ ($\omega > 0$) is a pair of purely imaginary roots of equation (2.1) if and only if ω satisfies

$$-\omega^2 + \mathrm{i}p\omega + q(\mu)[\cos\omega\tau - \mathrm{i}\sin\omega\tau] + r = 0.$$

Separating the real and imaginary parts, we have:

$$\left.\begin{aligned} \omega^2 - r &= q(\mu)\cos\omega\tau, \\ p\omega &= q(\mu)\sin\omega\tau. \end{aligned}\right\} \tag{2.4}$$

It then follows that ω must satisfy equation (2.2).

Claim 2.2. *Let* μ_j *be defined by (2.3).*

(i) *If* $r \geqslant 0$ *and* $\sqrt{r} < [\pi/2\tau]$*, then equation (2.2) has roots*

$$\omega_j \in \left(\frac{2(j-1)\pi}{\tau}, \frac{(4j-3)\pi}{2\tau}\right), \qquad j = 1, 2, \ldots,$$

satisfying equation (2.4) with $\mu = \mu_j$*.*

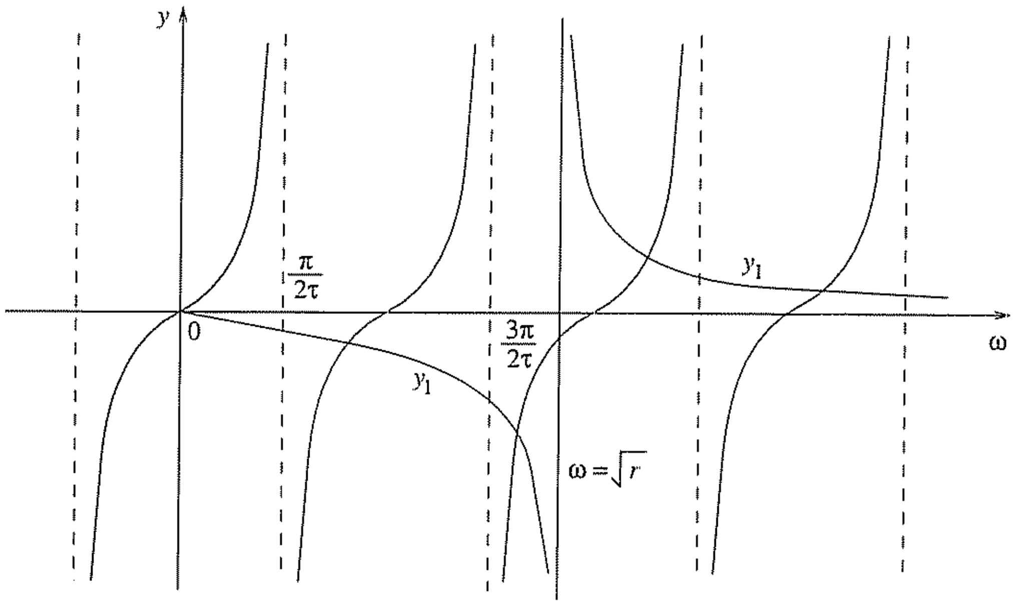

Figure 2. The points of intersection of $y_1 = [p\omega/(\omega^2 - r)]$ and $y_2 = \tan\omega\tau$ when $\sqrt{r} \geqslant [\pi/2\tau]$.

(ii) *If* $r > 0$ *and* $\sqrt{r} \geqslant [\pi/2\tau]$, *then there exists an integer* $k \geqslant 1$, *such that equation (2.2) has roots*

$$\omega_j \in \left(\frac{(4j-3)\pi}{2\tau}, \frac{(4j-1)\pi}{2\tau}\right) \qquad \text{for } 1 \leqslant j \leqslant k$$

and

$$\omega_j \in \left(\frac{(2(j-1)\pi}{\tau}, \frac{(4j-3)\pi}{2\tau}\right) \qquad \text{for } j > k,$$

satisfying equation (2.4) with $\mu = \mu_j$.

In fact, roots of equation (2.2) are the ω values at which the functions $y_1 = [p\omega/(\omega^2 - r)]$ and $y_2 = \tan\omega\tau$ intersect. It is clear that y_1 and y_2 intersect twice in the interval

$$\left[\frac{2(j-1)\pi}{\tau}, \frac{2j\pi}{\tau}\right], \qquad j = 1, 2, \dots, \quad \text{or} \quad j = 2, 3, \dots,$$

other than $\omega = 0$ (see figure 1). Meanwhile, the ω value of the intersection must satisfy equation (2.4), so it belongs to

$$\left(\frac{2(j-1)\pi}{\tau}, \frac{(4j-3)\pi}{2\tau}\right),$$

denoted by ω_j. Then, ω_j is a root of equation (2.4) with $\mu = \mu_j$. This proves (i).

Notice that there is an integer $k \geqslant 1$ such that

$$\frac{(2k-1)\pi}{\tau} \leqslant \sqrt{r} < \frac{(2k+3)\pi}{2\tau}$$

when $\sqrt{r} \geqslant [\pi/2\tau]$, (ii) can be proved similarly (see figure 2).

Proof of lemma 2.1. For any r, claim 2.2 implies that $\pm i\omega_j$ is a pair of purely imaginary roots of equation (2.1) with $\mu = \mu_j$. Since $q'(\mu) > 0$, there exists an integer $k \geqslant 1$, such that $q(\mu_{j+1}) > q(\mu_j)$ for $j \geqslant k$. Hence, we can find $j_0 \geqslant 0$ such that

$$q(\mu_{j_0}) = \min_{1\leqslant j\leqslant k}\{q(\mu_j)\}.$$

Denote

$$\mu_0 = \mu_{j_0}, \qquad \omega_0 = \omega_{j_0}.$$

Then, for any $\mu \in [0, \mu_0)$, equation (2.1) has no imaginary root.

Next, we prove that $\pm i\omega_j$ are simple. For a contradiction, assume that for some j, $i\omega_j$ is not simple, that means

$$\frac{d}{d\lambda}[\lambda^2 + p\lambda + q(\mu_j)e^{-\lambda\tau} + r]_{\lambda=i\omega_j} = 0.$$

We thus have

$$[2\lambda + p - \tau q(\mu_j)e^{-\lambda\tau}]_{\lambda=i\omega_j} = 0,$$

that is,

$$2i\omega_j + p - \tau q(\mu_j)[\cos\omega_j\tau - i\sin\omega_j\tau] = 0.$$

Separating the real and imaginary parts, we obtain

$$p = \tau q(\mu_j)\cos\omega_j\tau,$$
$$-2\omega_j = \tau q(\mu_j)\sin\omega_j\tau$$

From claim 2.2, we know that either

$$\omega_j \in \left(\frac{2(j-1)\pi}{\tau}, \frac{(4j-3)\pi}{2\tau}\right)$$

or

$$\omega_j \in \left(\frac{(4j-3)\pi}{2\tau}, \frac{(4j-1)\pi}{2\tau}\right).$$

The former contradicts $-2\omega_j = \tau q(\mu_j)\sin\omega_j\tau$, and the latter contradicts $p = \tau q(\mu_j)\cos\omega_j\tau$. Therefore, the roots $\pm i\omega_j$ are simple.

From the above discussion, we know that μ_0 is the first value of μ at which equation (2.1) has a pair of purely imaginary roots. Meanwhile, the roots of (2.1) with $\mu = 0$, i.e.

$$\lambda^2 + p\lambda + r = 0,$$

are

$$\lambda_{1,2} = \tfrac{1}{2}(-p \pm \sqrt{p^2 - 4r}).$$

Obviously, both λ_1 and λ_2 have negative real parts when $r > 0$. By Rouché's theorem [6, theorem 9.17.4], as μ varies the sum of the multiplicities of the roots of equation (2.1) in the open right half-plane can change only if a root appears on, or crosses, the imaginary axis. Since μ_0 is the minimal positive value of μ such that equation (2.1) has a pair of purely imaginary roots, we can see that all roots of equation (2.1) with $\mu \in [0, \mu_0)$ have strictly negative real parts.

If equation (2.1) with $\mu = \mu_0$ has a root with positive real part, say $\lambda(\mu) = \alpha(\mu) + i\omega(\mu)$, where $\alpha(\mu_0) > 0$. Since $\alpha(\mu)$ is continuous for $\mu \in \delta(\mu_0)$, a neighbourhood of

μ_0, we have $\alpha(\mu) > 0$ for $\mu < \mu_0$ and close to μ_0. It follows that equation (2.1) has a root with positive real part for $\mu < \mu_0$, $\mu \in \delta(\mu_0)$, which contradicts the above discussion. In the case that $r = 0$, we can apply [25, lemma 6.2.1] to obtain the conclusion. □

Let

$$\lambda_j(\mu) = \alpha_j(\mu) + \mathrm{i}\omega_j(\mu)$$

be the root of equation (2.1) with $\alpha_j(\mu_j) = 0$, $\omega_j(\mu_j) = \omega_j$. Then we have the following result.

LEMMA 2.3. *The following transversality condition holds:*

$$\left.\frac{\mathrm{d}\,\mathrm{Re}\,\lambda_j(\mu)}{\mathrm{d}\mu}\right|_{\mu=\mu_j} > 0.$$

Proof. From equation (2.1), we have

$$2\lambda_j(\mu)\frac{\mathrm{d}\lambda_j}{\mathrm{d}\mu} + p\frac{\mathrm{d}\lambda_j}{\mathrm{d}\mu} + q'(\mu)\mathrm{e}^{-\lambda_j\tau} - q(\mu)\tau\mathrm{e}^{-\lambda_j\tau}\frac{\mathrm{d}\lambda_j}{\mathrm{d}\mu} = 0,$$

that is,

$$\frac{\mathrm{d}\lambda_j(\mu)}{\mathrm{d}\mu} = -\frac{q'(\mu)\mathrm{e}^{-\lambda_j(\mu)\tau}}{2\lambda_j(\mu) + p - q(\mu)\tau\mathrm{e}^{-\lambda_j\tau}}.$$

Substituting $\mu = \mu_j$ in the above equation, we obtain

$$\begin{aligned}\frac{\mathrm{d}\lambda_j(\mu_j)}{\mathrm{d}\mu} &= -\frac{q'(\mu_j)[\cos\omega_j\tau - \mathrm{i}\sin\omega_j\tau]}{2\mathrm{i}\omega_j + p - q(\mu_j)\tau(\cos\omega_j\tau - \mathrm{i}\sin\omega_j\tau)}\\ &= -\frac{q'(\mu_j)}{\Delta}\{[(p - q(\mu_j)\tau\cos\omega_j\tau)\cos\omega_j\tau - (2\omega_j + q(\mu_j)\tau\sin\omega_j\tau)\sin\omega_j\tau]\\ &\qquad + \mathrm{i}[-(p - q(\mu_j)\tau\cos\omega_j\tau)\sin\omega_j\tau - (2\omega_j + q(\mu_j)\tau\sin\omega_j\tau)\cos\omega_j\tau]\}.\end{aligned}$$

Hence,

$$\frac{\mathrm{d}\,\mathrm{Re}\,\lambda_j(\mu_j)}{\mathrm{d}\mu} = -\frac{q'(\mu_j)}{\Delta}[(p - q(\mu_j)\tau\cos\omega_j\tau)\cos\omega_j\tau - (2\omega_j + q(\mu_j)\tau\sin\omega_j\tau)\sin\omega_j\tau],$$

where

$$\Delta = [p - q(\mu_j)\tau\cos\omega_j\tau]^2 + [2\omega_j + q(\mu_j)\tau\sin\omega_j\tau]^2.$$

Note that ω_j satisfies equation (2.4), so we have

$$\begin{aligned}\frac{\mathrm{d}\lambda_j(\mu_j)}{\mathrm{d}\mu} &= -\frac{q'(\mu_j)}{q(\mu_j)\Delta}[p(\omega_j^2 - r) - \tau(\omega_j^2 - r)^2 - 2p\omega_j^2 - \tau p^2\omega_j^2]\\ &= -\frac{q'(\mu_j)}{q(\mu_j)\Delta}[-pr - \tau(\omega_j^2 - r)^2 - p\omega_j^2 - \tau p^2\omega_j^2]\\ &= \frac{q'(\mu_j)}{q(\mu_j)\Delta}[p(\omega_j^2 + r) + \tau(\omega_j^2 - r)^2 + \tau p^2\omega_j^2]\\ &> 0.\end{aligned}$$

This completes the proof. □

By applying the above lemma and using a similar argument to the one in [5], we obtain the following lemma.

LEMMA 2.4. *If $\{\mu_j\}$ is reordered such that $\mu_{j+1} > \mu_j$, then for $\mu \in (\mu_j, \mu_{j+1})$, equation (2.1) has exactly $2j$ roots with positive real part.*

REMARK 2.5. Lemmas 2.1–2.4 generalize the results in [14] (see also [25]), where the condition $0 \leqslant r < [\pi/2\tau]$ is required. Here, we only require that $r \geqslant 0$. In the following, we shall apply lemmas 2.1–2.4 to system (1.4) and only require that $a_0 b_0 \geqslant 0$.

Now we consider the planar system (1.4). Under the hypothesis (H_1), $(x_1, x_2) = (0,0)$ is an isolated stationary point of (1.4), and the linearized system of (1.4) at $(0,0)$ has the following form:

$$\left.\begin{aligned} \dot{x}_1(t) &= -a_0 x_1(t) + a_1 \frac{\partial F_1}{\partial x_2}(0,0) x_2(t-\tau_2), \\ \dot{x}_2(t) &= -b_0 x_2(t) + b_1 \frac{\partial F_2}{\partial x_1}(0,0) x_1(t-\tau_1). \end{aligned}\right\} \tag{2.5}$$

Denote

$$\alpha_1 = a_1 \frac{\partial F_1}{\partial x_2}(0,0), \qquad \alpha_2 = b_1 \frac{\partial F_2}{\partial x_1}(0,0), \qquad \alpha = -\alpha_1 \alpha_2.$$

Then the associated characteristic equation is

$$\det \begin{pmatrix} -a_0 - \lambda & \alpha_1 e^{-\lambda\tau_1} \\ \alpha_2 e^{-\lambda\tau_2} & -b_0 - \lambda \end{pmatrix} = 0,$$

that is,

$$\lambda^2 + (a_0 + b_0)\lambda + \alpha e^{-\lambda(\tau_1+\tau_2)} + a_0 b_0 = 0. \tag{2.6}$$

Let ω_j be the roots of the equation

$$\frac{(a_0+b_0)\omega}{\omega^2 - a_0 b_0} = \tan \omega(\tau_1 + \tau_2),$$

in the interval

$$\left(\frac{2(j-1)\pi}{\tau_1+\tau_2}, \frac{(4j-3)\pi}{2(\tau_1+\tau_2)}\right)$$

for $j \geqslant k$, k an integer. Applying lemmas 2.1–2.4 to equation (2.6) and regarding α as the bifurcation parameter, we have the following lemma.

LEMMA 2.6. *If (H_1) is satisfied, then there exists a sequence $\{\alpha_j\}$, $j = 1, 2, \dots,$ satisfying $\alpha_{j+1} > \alpha_j$, and*

$$\alpha_j = \frac{(a_0+b_0)\omega_j}{\sin \omega_j(\tau_1+\tau_2)}, \qquad \alpha_0 = \min_{j \geqslant 1}\{\alpha_j\},$$

such that

(i) *all roots of equation (2.6) have strictly negative real parts for $\alpha \in [0, \alpha_0)$;*

(ii) *when $\alpha = \alpha_0$, equation (2.6) has a pair of purely imaginary roots $\pm i\omega_j$, which are simple, and all other roots have negative real parts;*

(iii) *when $\alpha > \alpha_0$, equation (2.6) has at least one root with a strictly positive real part.*

Applying lemma 2.6 to system (1.4), we obtain the following result on stability and bifurcation in the system.

THEOREM 2.7. *Suppose $\alpha = -\alpha_1\alpha_2 > 0$.*

(i) *If $\alpha \in [0, \alpha_0)$, then the zero solution of system (1.4) is asymptotically stable.*

(ii) *If $\alpha > \alpha_0$, then the zero solution of system (1.4) is unstable.*

(iii) *Every $\alpha_j \in \{\alpha_j\}$, $j = 1, 2, \dots$, is a Hopf bifurcation value for system (1.4).*

3. Global existence of periodic solutions

In this section, we shall study the global existence of periodic solutions of system (1.4). First, we state a general global bifurcation theorem due to Erbe *et al.* [7]. The form of the theorem we state here is from [25] (see [25] for unexplained notation).

Let X be a Banach space over $\mathbb{R}$ and $r > 0$ be a constant. Let $C = C([-r, 0]; X)$ denote the Banach space of continuous X-valued functions on $[-r, 0]$ with the supremum norm $\|\cdot\|$. For any real numbers $a \leqslant b$, $t \in [a, b]$ and any continuous function $u : [a - r, b] \to X$, u_t denotes the element of C given by $u_t(\theta) = u(t + \theta)$ for $\theta \in [-r, 0]$. Consider the abstract differential equation

$$\dot{u}(t) = A_T u(t) + f(\alpha, u_t), \tag{3.1}$$

where A_T generates an analytic semigroup $\{T(t)\}_{t \geqslant 0}$ on X, and $f : \mathbb{R} \times C \to X$ is a continuously differentiable mapping sending bounded sets into bounded sets. Let $P\sigma(A_T)$ denote the point spectrum of A_T, that is, the set of all $\lambda \in \mathbb{C}$ (where $\mathbb{C}$ is the set of all complex numbers) such that $A_T v = \lambda v$ has non-zero solution for $v \in \mathrm{Dom}(A_T)$. Assume that

(P1) there exist $u_0 \in \mathrm{Dom}(A_T)$ and $\alpha_0 \in \mathbb{R}$ such that $A_T u_0 + f(\alpha_0, \bar{u}_0) = 0$ and $0 \notin P\sigma(A_T + D_\phi f(\alpha_0, \bar{u}_0))$;

(P2) there exists $\beta_0 > 0$ such that $\pm \mathrm{i}\beta_0$ are characteristic values of the linearized system

$$\dot{u}(t) = A_T u(t) + D_\phi f(\alpha, \bar{\eta}(\alpha)) u_t, \tag{3.2}$$

with $\alpha = \alpha_0$, where $\eta : (\alpha_0 - \delta_0, \alpha_0 + \delta_0) \to \mathrm{Dom}(A_T)$ is a C^1-mapping such that $A_T \eta(\alpha) + f(\alpha, \bar{\eta}(\alpha)) = 0$ for each $\alpha \in (\alpha_0 - \delta_0, \alpha_0 + \delta_0)$, $\delta_0 > 0$;

(P3) for $(\alpha, \beta) \in D = [\alpha_0 - \delta_0, \alpha_0 + \delta_0] \times [\beta_0 - \varepsilon_0, \beta_0 + \varepsilon_0]$, where $\delta_0 > 0$ and $\varepsilon_0 > 0$ are sufficiently small, $\mathrm{i}\beta$ is a characteristic value of the linearized system (3.2) if and only if $\alpha = \alpha_0$ and $\beta = \beta_0$;

(P4) the set M^* of equilibria of system (3.1) is a complete one-dimensional smooth manifold such that (i) the assumption (P1) is satisfied for every $(u_0, \alpha_0) \in M^*$, and (ii) if $\pm \mathrm{i}\beta_0$ are characteristic values of (3.2), then (P3) is satisfied.

Define

$$\mu_k(u_0, \alpha_0, \beta_0) = \varepsilon(u_0, \alpha_0, \beta_0) \deg([\psi_k]),$$

where

$$\varepsilon(u_0, \alpha_0, \beta_0) = \begin{cases} 1, & \text{if } \psi_0(\alpha_0, \beta_0) \in GL^+(X), \\ -1, & \text{if } \psi_0(\alpha_0, \beta_0) \in GL^-(X), \end{cases}$$

$$\psi_0(\alpha, \beta) = Id + A_T^{-1} D\phi f(\alpha, \bar{\eta}(\alpha)),$$

$[\psi_k]$ is the homotopy class of $\psi_k : \partial D \to GL_c(E_k)$, and $\deg : [\partial D, GL_c(E_k)] \to Z$ is the bijection defined by the classical Brouwer degree.

The following theorem is theorem 5.5 from ch. 6 of [25].

THEOREM 3.1. *Suppose that (P1)–(P4) are satisfied. Let S denote the closure of the set $\{(z, \alpha, \beta) \in C(S^1, X) \times \mathbb{R} \times (0, \infty); u(t) = z(\beta t)$ is a non-trivial $2\pi/\beta$ periodic solution of (3.1)$\}$. Then, for each connected component C, at least one of the following holds:*

(i) *C is unbounded, i.e. $\sup\{\max_{t\in\mathbb{R}} |z(t)| + |\alpha| + \beta + \beta^{-1} : (z, \alpha, \beta) \in C\} = \infty$.*

(ii) *$C \cap (M^* \times (0, \infty))$ is finite, and for all $k \geqslant 1$, one has the equality*

$$\sum_{(u_0,\alpha_0,\beta_0)\in C\cap(M^*\times(0,\infty))} \mu_k(u_0, \alpha_0, \beta_0) = 0.$$

REMARK 3.2. The main step in applying theorem 3.1 is to compute $\deg([\psi_k])$. As pointed out in remark 6.5.7 in [25], this can be done under transversality conditions. If $i\beta_0$ is a simple characteristic value of system (3.2) with $\alpha = \alpha_0$ satisfying (P3), then $i\beta_0$ is a simple isolated eigenvalue of $A_T + D\phi f(\alpha_0, \bar{\eta}(\alpha_0))$ with eigenvector $\alpha_0 \in \mathrm{Dom}(A_T)$. In this case,

$$\deg([\psi_1]) = \deg_B(i\beta - \lambda(\alpha), \partial D),$$

where $\lambda : (\alpha_0 - \delta_0, \alpha_0 + \delta_0) \to \mathbb{C}$ satisfies $\lambda(\alpha_0) = i\beta_0$ and $(\alpha_0, \beta_0) \in \partial D$. Therefore, $\deg([\psi_1]) = 1$ if $\mathrm{Re}\,\lambda'(\alpha_0) > 0$ and $\deg([\psi_1]) = -1$ if $\mathrm{Re}\,\lambda'(\alpha_0) < 0$.

Now we consider system (1.4). We shall regard $\alpha = -\alpha_1\alpha_2$ as the bifurcation parameter. We first make the following assumptions.

(H_2) There exists a constant $L > 0$ such that $|F_j(x_1, x_2)| \leqslant L$, $j = 1, 2$, for all $(x_1, x_2) \in \mathbb{R}^2$.

(H_3) There exists an integer $m \geqslant 1$ such that either $\tau_1 = m\tau_2$ or $\tau_2 = m\tau_1$.

Without loss of generality, we assume that $\tau_2 = m\tau_1$. Clearly, by lemma 2.6, we have $\lim_{j\to+\infty} \omega_j = \infty$. Assume that $\omega_{j+1} > \omega_j$ for $j \geqslant 1$. Then, there exists an integer $j_0 \geqslant 1$ such that $[2\pi/\omega_{j_0}] \leqslant \tau_1$, $[2\pi/\omega_j] > \tau_1$ for $j < j_0$ and $[2\pi/\omega_j] < \tau_1$ for $j > j_0$.

(H_4) There exist constants $\bar{\alpha} > \alpha_{j_0}$ and

$$M \geqslant \max\left\{1, \frac{L(|\bar{a}_1| + |\bar{b}_1|)}{\bar{a}}\right\},$$

where $|\bar{a}_1| + |\bar{b}_1| = \max\{|a_1| + |b_1| : \alpha \in [0, \bar{\alpha}]\}$ and $\bar{a} = \min\{a_0, b_0\}$, such that

$$-(a_0 + b_0) + a_1 \frac{\partial F_1}{\partial x_1}(x_1, x_2) + b_1 \frac{\partial F_2}{\partial x_2}(x_1, x_2),$$

has definite sign for $(x_1, x_2) \in \{(x_1, x_2) : \sqrt{x_1^2 + x_2^2} \leqslant M\}$.

THEOREM 3.3. *Suppose (H$_1$)–(H$_4$) are satisfied and either $a_1 \neq 0$ or $b_1 \neq 0$. Then, for any $\alpha \in [\alpha_{j_0}, \bar{\alpha}]$, system (1.4) has at least one non-constant periodic solution.*

Proof. It is clear that for system (1.4), assumptions (P1)–(P4) are satisfied with $u_0 = 0$, $\alpha_0 = \alpha_j$, and $\beta_0 = \omega_j$, where α_j and ω_j are defined in lemma 2.6. Meanwhile, by lemma 2.3 and remark 3.2, we know that

$$\mu_1(0, \alpha_j, \omega_j) = -1.$$

Thus, the connected component C, which contains $(0, \alpha_j, \omega_j)$, is unbounded, that is,

$$\sup\{\max_{t \in \mathbb{R}} |x(t)| + |\alpha| + T + T^{-1} : (x, \alpha, T) \in C\} = \infty.$$

Firstly, we prove that for $\alpha \in [0, \bar{\alpha}]$, the periodic solution $x(t) = (x_1(t), x_2(t))$ of system (1.4) satisfies $|x(t)| < M$ for all t, where $\bar{\alpha}$ and M are given by (H$_4$).

Let

$$r(t) = \sqrt{x_1^2(t) + x_2^2(t)}.$$

Taking the derivative of $r(t)$ along solutions of system (1.4), we have

$$\begin{aligned}
\dot{r}(t) &= \frac{1}{r(t)}[x_1(t)\dot{x}_1(t) + x_2(t)\dot{x}_2(t)] \\
&= \frac{1}{r(t)}[-(a_0 x_1^2(t) + b_0 x_2^2(t)) + a_1 x_1(t) F_1(x_1(t - \tau_1), x_2(t - \tau_2)) \\
&\qquad + b_1 x_2(t) F_2(x_1(t - \tau_1), x_2(t - \tau_2))] \\
&\leqslant \frac{1}{r(t)}[-\bar{a}(x_1^2(t) + x_2^2(t)) + L(|a_1|\,|x_1(t)| + |b_1|\,|x_2(t)|)].
\end{aligned}$$

If there is a $t_0 > 0$ such that $|x(t_0)| = r(t_0) = A \geqslant M$, then, by (H$_4$), we have

$$\begin{aligned}
\dot{r}(t_0) &\leqslant (1/A)[-\bar{a}A^2 + AL(|a_1| + |b_1|)] \\
&= [-\bar{a}A + L(|a_1| + |b_1|)] \\
&< 0.
\end{aligned}$$

It follows that if $x(t)$ is a periodic solution of system (1.4), then either $r(t) < M$ or $r(t) > M$ for all t. If $r(t) > M$ for all t, by the above discussion, we have $\dot{r}(t) < 0$ for all t, which contradicts the fact that $x(t)$ is periodic in t. Therefore, for each periodic solution of system (1.4), $r(t) < M$ for all t.

Next, we prove that if a periodic solution $x(t)$ of system (1.4) with $\alpha \in [0, \bar{\alpha}_0]$ is on C, then its period T and T^{-1} are uniformly bounded.

In fact, since $\tau_2 = m\tau_1$, system (1.4) becomes

$$\left.\begin{aligned}
\dot{x}_1(t) &= -a_0 x_1(t) + a_1 F_1(x_1(t - \tau_1), x_2(t - m\tau_1)), \\
\dot{x}_2(t) &= -b_0 x_2(t) + b_1 F_2(x_1(t - \tau_1), x_2(t - m\tau_1)),
\end{aligned}\right\} \tag{3.3}$$

and it has no τ_1-periodic solution $x(t)$ satisfying $|x(t)| < M$. Otherwise, if system (3.3) has a τ_1-periodic solution, say $(u_1(t), u_2(t))$, then it satisfies the ordinary differential equations

$$\left.\begin{aligned}\dot{x}_1(t) &= -a_0x_1(t) + a_1F_1(x_1, x_2) = P(x_1, x_2),\\ \dot{x}_2(t) &= -b_0x_2(t) + b_1F_2(x_1, x_2) = Q(x_1, x_2),\end{aligned}\right\} \tag{3.4}$$

which means that system (3.4) has a non-constant periodic solution on $\{x : |x| < M\}$. On the other hand, from (H_4) we have that

$$\frac{\partial P(x_1, x_2)}{\partial x_1} + \frac{\partial Q(x_1, x_2)}{\partial x_2} = -(a_0 + b_0) + a_1\frac{\partial F_1(x_1, x_2)}{\partial x_1} + b_1\frac{\partial F_2(x_1, x_2)}{\partial x_2} \neq 0,$$

for all $x \in \{x : |x| \leqslant M\}$. By Bendixson's criterion, we know that system (3.4) has no non-constant periodic solutions on the region $x \in \{x : |x| \leqslant M\}$. Hence, system (3.3) has no $[\tau_1/n]$-periodic solution on $\{x : |x| \leqslant M\}$ for any $n \geqslant 1$.

By the choice of α_{j_0}, there exists a $k \geqslant 1$ such that

$$\frac{\tau_1}{k+1} < \frac{2\pi}{\omega_{j_0}} < \frac{\tau_1}{k}.$$

Thus, on the connected component $\mathcal{C}$, which contains $(0, [2\pi/\omega_{j_0}], \alpha_{j_0})$, the period T of the periodic solution of system (3.4) with $\alpha \leqslant \bar{\alpha}$ satisfies

$$\frac{\tau_1}{k+1} < T < \frac{\tau_1}{k}.$$

Finally, we prove that system (3.3) with $\alpha = 0$ has no non-constant periodic solution.

If

$$\alpha = -a_1b_1\frac{\partial F_1(0,0)}{\partial x_2}\frac{\partial F_2(0,0)}{\partial x_1} = 0,$$

by (H_1), we have either $a_1 = 0$ or $b_1 = 0$. Without loss of generality, we assume $a_1 = 0$. Then system (3.3) becomes

$$\left.\begin{aligned}\dot{x}_1(t) &= -a_0x_1(t),\\ \dot{x}_2(t) &= -b_0x_2(t) + b_1F_2(x_1(t-\tau_1), x_2(t-m\tau_1)).\end{aligned}\right\} \tag{3.5}$$

For any $\phi = (\phi_1, \phi_2) \in C$, the solution of system (3.5) with the initial data $x(t) = \phi(t)$, $t \in [-m\tau_1, 0]$, can be expressed as

$$\begin{aligned}x_1(t) &= \phi_1(0)e^{-a_0t},\\ x_2(t) &= e^{-b_0t}\left[\phi_2(0) + b_1\int_0^t e^{b_0s}F_2(x_1(s-\tau_1), x_2(s-m\tau_1))\,ds\right], \quad t \geqslant 0.\end{aligned}$$

It is obvious that

$$\lim_{t\to\infty} x_1(t) = 0.$$

Meanwhile, from the expression of $x_2(t)$ and the fact that $|F_2(x_1, x_2)| \leqslant L$ for all $(x_1, x_2) \in \mathbb{R}^2$, we can see that $x_2(t)$ is bounded for all $t \geqslant -m\tau_1$. Hence, by the assumption that $x_1F_2(x_1, x_2) \neq 0$ for $x_1 \neq 0$ and the continuity of $F_2(x_1, x_2)$, we have

$$\lim_{t\to\infty} F_2(x_1(t-\tau_1), x_2(t-m\tau_1)) = 0.$$

Now, let us prove that

$$\lim_{t\to\infty}\left[\left(\int_0^t e^{b_0 s}F_2(x_1(s-\tau_1),x_2(s-m\tau_1))\,ds\right)\Big/e^{b_0 t}\right]=0. \tag{3.6}$$

For $t>0$, we know that

$$\left|\int_0^t e^{b_0 s}F_2(x_1(s-\tau_1),x_2(s-m\tau_1))\,ds\right|\leqslant\int_0^t e^{b_0 s}|F_2(x_1(s-\tau_1),x_2(s-m\tau_1))|\,ds.$$

Hence, we have either

$$\lim_{t\to\infty}\int_0^t e^{b_0 s}|F_2(x_1(s-\tau_1),x_2(s-m\tau_1))|\,ds<\infty, \tag{3.7}$$

or

$$\lim_{t\to\infty}\int_0^t e^{b_0 s}|F_2(x_1(s-\tau_1),x_2(s-m\tau_1))|\,ds=\infty. \tag{3.8}$$

If (3.7) holds, so does (3.6). If (3.8) holds, by L'Hospital's rule, we have

$$\begin{aligned}\lim_{t\to\infty}&\left[\left(\int_0^t e^{b_0 s}|F_2(x_1(s-\tau_1),x_2(s-m\tau_1))|\,ds\right)\Big/e^{b_0 t}\right]\\&=\lim_{t\to\infty}\frac{e^{b_0 t}|F_2(x_1(t-\tau_1),x_2(t-m\tau_1))|}{b_0e^{b_0 t}}\\&=0,\end{aligned}$$

which shows that (3.6) holds. Thus, by the expression of $x_2(t)$, we have

$$\lim_{t\to\infty}x_2(t)=0.$$

This implies that when $\alpha=0$, system (1.4) has no non-constant periodic solution.

By the above discussion, we can see that the projection of C on the α-axis includes $[\alpha_{j_0},\bar{\alpha}]$. This completes the proof. □

4. An example

Consider the neural network model without self-connection, i.e. the differential equations with two delays:

$$\left.\begin{aligned}\dot{x}_1(t)&=-x_1(t)+a_{12}f(x_2(t-\tau_2)),\\ \dot{x}_2(t)&=-x_2(t)+a_{21}f(x_1(t-\tau_1)).\end{aligned}\right\} \tag{4.1}$$

Related models have been studied by Gopalsamy and Leung [9], Olien and Bélair [22] and Wei and Ruan [24]. We shall use the results in §§ 2 and 3 to study the stability of the zero solutions and the global existence of periodic solutions of system (4.1).

First of all, we make the following assumptions.

(H_1') $f\in C^3(\mathbb{R})$, $xf(x)\neq 0$ for $x\neq 0$, and $f'(0)\neq 0$.

(H_2') There exists a constant $L>0$ such that $|f(t)|\leqslant L$ for all $x\in\mathbb{R}$.

(H_3') There exists an integer $m\geqslant 1$ such that either $\tau_1=m\tau_2$ or $\tau_2=m\tau_1$.

Clearly, under the assumption (H_1'), $(0,0)$ is an isolated stationary point of system (4.1) and the linearization of system (4.1) at $(0,0)$ is

$$\left.\begin{aligned} \dot{x}_1(t) &= -x_1(t) + a_{12}f'(0)x_2(t-\tau_2), \\ \dot{x}_2(t) &= -x_2(t) + a_{21}f'(0)x_1(t-\tau_1). \end{aligned}\right\} \tag{4.2}$$

For convenience, we denote

$$\alpha = -a_{12}a_{21}[f'(0)]^2, \tag{4.3}$$

which will be regarded as the bifurcation parameter, and the associated characteristic equation is

$$\lambda^2 + 2\lambda + \alpha e^{-\lambda(\tau_1+\tau_2)} + 1 = 0. \tag{4.4}$$

Denote

$$\alpha_j = \frac{2\omega_j}{\sin\omega_j(\tau_1+\tau_2)}, \qquad \alpha_0 = \min_{j\geqslant 1}\{\alpha_j\}. \tag{4.5}$$

Let

$$\omega_j \in \left(\frac{2(j-1)\pi}{\tau_1+\tau_2}, \frac{(4j-3)\pi}{2(\tau_1+\tau_2)}\right)$$

for $j \geqslant k$, k an integer, be the roots of the equation

$$[2\omega/(\omega^2-1)] = \tan\omega(\tau_1+\tau_2). \tag{4.6}$$

By applying theorem 2.7, we have the following result.

THEOREM 4.1. *Suppose that $\alpha > 0$ and (H_1') holds.*

(i) *For $\alpha \in [0, \alpha_0)$, the zero solution of system (4.1) is asymptotically stable.*

(ii) *For $\alpha > \alpha_0$, the zero solution of system (4.1) is unstable.*

(iii) *Every $\alpha_j \in \{\alpha_j\}_{j=1,2,\dots}$ is a Hopf bifurcation value of system (4.1).*

Applying theorem 3.3 to system (4.1), we obtain the global existence of periodic solutions.

THEOREM 4.2. *Suppose that (H_1')–(H_3') are satisfied and either $a_{12} \neq 0$ or $a_{21} \neq 0$ is fixed. Let α_{j_0} be the value such that $[2\pi/\omega_{j_0}] \leqslant \tau_1$ and $[2\pi/\omega_j] > \tau_1$ for $j < j_0$ and $[2\pi/\omega_j] < \tau_1$ for $j > j_0$. Then, for any $\alpha \geqslant \alpha_{j_0}$, system (4.1) has at least one non-constant periodic solution.*

Without loss of generality, we assume that $\tau_2 = m\tau_1$. Clearly, under (H_1')–(H_3'), the system satisfies the assumptions (H_1)–(H_3). Notice that

$$\frac{\partial P(x_1,x_2)}{\partial x_1} + \frac{\partial Q(x_1,x_2)}{\partial x_2} = -2 + a_{12}\frac{\partial f(x_2)}{\partial x_1} + a_{21}\frac{\partial f(x_1)}{\partial x_2} = -2 \neq 0,$$

for all $(x_1,x_2) \in \mathbb{R}^2$. Thus, assumption ($H_4$) is also satisfied. By theorem 3.3, we obtain our conclusion.

Acknowledgments

The research of S.R. was supported by the Natural Science and Engineering Research Council of Canada and the Petro-Canada Young Innovator Award, and the research of J.W. was supported by the National Natural Science Foundations of China.

References

1 M. Baptistini and P. Táboas. On the existence and global bifurcation of periodic solutions to planar differential delay equations. *J. Diff. Eqns* **127** (1996), 391–425.

2 S.-N. Chow. Existence of periodic solutions of autonomous functional differential equations. *J. Diff. Eqns* **15** (1974), 350–378.

3 S.-N. Chow and J. K. Hale. Periodic solutions of autonomous equations. *J. Math. Analysis Appl.* **66** (1978), 495–506.

4 S.-N. Chow and J. Mallet-Paret. The Fuller index and global Hopf bifurcation. *J. Diff. Eqns* **29** (1978), 66–85.

5 K. L. Cooke and Z. Grossman. Discrete delay, distributed delay and stability switches. *J. Math. Analysis Appl.* **86** (1982), 592–627.

6 J. Dieudonné. *Foundations of modern analysis* (New York: Academic Press, 1960).

7 L. H. Erbe, W. Krawcewicz, K. Gęba and J. Wu. S^1-degree and global Hopf bifurcation theory of functional-differential equations. *J. Diff. Eqns* **98** (1992), 277–298.

8 S. M. S. Godoy and J. G. dos Reis. Stability and existence of periodic solutions of a functional-differential equation. *J. Math. Analysis Appl.* **198** (1996), 381–398.

9 K. Gopalsamy and I. Leung. Delay induced periodicity in a neural netlet of excitation and inhibition. *Physica* D **89** (1996), 395–426.

10 R. Grafton. A periodicity theorem for autonomous functional differential equations. *J. Diff. Eqns* **6** (1969), 87–109.

11 K. P. Hadeler and J. Tomiuk. Periodic solutions of differential-difference equations. *Arch. Ration. Mech. Analysis* **65** (1977), 87–95.

12 J. K. Hale and S. M. V. Lunel. *Introduction to functional differential equations* (New York: Springer 1993).

13 U. an der Heiden. Periodic solutions of a nonlinear second order differential equation with delay. *J. Math. Analysis Appl.* **70** (1979), 599–609.

14 Q. Huang, J. Wei, J. Wu and X. Zou. Direction and stability of bifurcating periodic solutions in predator–prey systems with discrete delay. In *Differential equations and control theory* (ed. Z. Deng, Z. Liang, G. Lu and S. Ruan), pp. 107–119 (New York: Marcel Dekker, 1996).

15 G. Jones. The existence of periodic solutions of $f'(t) = -\alpha f(x-1)[1+f(x)]$, *J. Math. Analysis Appl.* **5** (1962), 435–450.

16 A. Leung. Periodic solutions for a prey–predator differential delay equation. *J. Diff. Eqns* **26** (1977), 391–403.

17 J. Mallet-Paret and R. D. Nussbaum. Global continuation and asymptotic behavior for periodic solutions of a differential-delay equation. *Ann. Math. Pura Appl.* **145** (1986), 33–128.

18 J. Mallet-Paret and R. D. Nussbaum. A differential-delay equation arising in optics and physiology. *SIAM J. Math. Analysis* **20** (1989), 249–292.

19 R. D. Nussbaum. Periodic solutions of some nonlinear autonomous functional differential equations II. *J. Diff. Eqns* **14** (1973), 360–394.

20 R. D. Nussbaum. Periodic solutions of some nonlinear autonomous functional differential equations. *Ann. Math. Pura Appl.* **101** (1974), 263–306.

21 R. D. Nussbaum. Differential delay equations with two delays. *Mem. Am. Math. Soc.* **16** (1978), 1–62.

22 L. Olien and J. Bélair. Bifurcations, stability, and monotonicity properties of a delayed neural network model. *Physica* D **102** (1997), 349–363.

23 P. Táboas. Periodic solutions of a planar delay equation. *Proc. R. Soc. Edinb.* A **116** (1990), 85–101.

24 J. Wei and S. Ruan. Stability and bifurcation in a neural network model with two delays. *Physica* D **130** (1999), 255–272.

25 J. Wu. *Theory and applications of partial functional differential equations* (New York: Springer, 1996).

26 T. Zhao, Y. Kuang and H. L. Smith. Global existence of periodic solutions in a class of delayed Gause-type predator–prey systems. *Nonlinear Analysis* **28** (1997), 1373–1394.

(*Issued 15 October 1999*)